INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

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Let $\mathcal{P}_n$ denote the linear space of polynomials $p(z) = \sum_{k=0}^{n} a_k z^k$ of degree at most $n$. There are various ways in which we can introduce norm $|| \cdot ||$ in $\mathcal{P}_n$. Given $\beta$ let $\mathcal{P}_{n, \beta}$ denote the subspace consisting of those polynomials which vanish at $\beta$. Then how large can $|| p(z)/(z-\beta) ||$ be if $p(z) \in \mathcal{P}_{n, \beta}$ and $|| p(z) || = 1$? This general question does not seem to have received much attention. Here the problem is investigated when (i) $|| p(z) || = \max_{|z| \leq 1} |p(z)|$, (ii) $|| p(z) || = (1/2^n) \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta^{1/2}$.

It was shown by Rahman and Mohammad [1] that if $p(z) \in \mathcal{P}_{n, 1}$ and $\max_{|z| \leq 1} |p(z)| \leq 1$ then

$$\max_{|z| \leq 1} |p(z)/(z-1)| \leq n/2.$$  

We observe that if $p(z) \in \mathcal{P}_{n, \beta}$ and $\max_{|z| \leq 1} |p(z)| = 1$ then $\max_{|z| \leq 1} |p(z)/(z-\beta)|$ can be greater than $n/2$ if $\beta$ is arbitrary. For $n = 1$ we may simply take $p(z) = z$. When $n > 1$ we consider the polynomial

$$p(z) = (n/2)(n^2 - 1)^{-1/2}(1 + z + z^2 + \cdots + z^{n-1})(z-1+2n^{-2}).$$

If $z = e^{i\theta}$ then for $\cos \theta \leq 1 - 2n^{-2}$

$$|p(z)| \leq (1/2) |(1 + z + z^2 + \cdots + z^{n-1})(z-1)| \leq 1,$$

and also for $\cos \theta \geq 1 - 2n^{-2}$

$$|p(z)| \leq n(n^2-1)^{-1/2}(n/2) |z-1+2n^{-2}| \leq 1$$

while

$$\max_{|z|=1} |p(z)/(z-1+2n^{-2})| = (n/2)(n^2 - 1)^{-1/2} > \frac{n}{2}.$$  

We note however that if $p(z) \in \mathcal{P}_{n, \beta}$ and $\max_{|z| \leq 1} |p(z)| \leq 1$, then

$$\max_{|z|=1} |p(z)/(z-\beta)| \leq (n+1)/2.$$  

Proof of inequality (2). Without loss of generality we may assume $\beta$ to be real and nonnegative. Put $p(z) = (z-\beta)q(z)$ and write

$$p(z)/(z-\beta) = q(z).$$

If $\beta$ is real, $q(z)$ is an even function of $z$ and $q(z)$ is an odd function of $z$. Therefore

$$\max_{|z| \leq 1} |q(z)| = \max_{|z| \leq 1} |p(z)/(z-\beta)| \leq (n+1)/2.$$  

If $\beta$ is even, $p(z)$ is an even function of $z$ and $q(z)$ is an odd function of $z$. Therefore

$$\max_{|z| \leq 1} |q(z)| = \max_{|z| \leq 1} |p(z)/(z-\beta)| \leq (n+1)/2.$$  

If $\beta$ is odd, $p(z)$ is an odd function of $z$ and $q(z)$ is an even function of $z$. Therefore

$$\max_{|z| \leq 1} |q(z)| = \max_{|z| \leq 1} |p(z)/(z-\beta)| \leq (n+1)/2.$$  

Therefore

$$\max_{|z| \leq 1} |q(z)| = \max_{|z| \leq 1} |p(z)/(z-\beta)| \leq (n+1)/2.$$  

This completes the proof of inequality (2).
\( p^*(z) = (z-1)q(z) \). Then

\[
|p^*(e^{i\theta})| = |(e^{i\theta} - 1)/(e^{i\theta} - \beta)| \leq 2/(1+\beta)
\]
which gives us

\[
\max_{|z|=1} |p^*(z)| \leq 2(1+\beta)^{-1} \max_{|z|=1} |p(z)| .
\]

From inequalities (1) and (4) we obtain

\[
\max_{|z|=1} |q(z)| \leq (n/2) \max_{|z|=1} |p^*(z)| \leq n(1+\beta)^{-1} \max_{|z|=1} |p(z)| 
\]
\[
\leq \frac{n+1}{2} \max_{|z|=1} |p(z)|
\]
provided \( \beta \geq (n-1)/(n+1) \).

For \( \beta \leq (n-1)/(n+1) \) we have

\[
|q(e^{i\theta})| = |p(e^{i\theta})/(e^{i\theta} - \beta)| \leq (1-\beta)^{-1} |p(e^{i\theta})| \leq \frac{n+1}{2} |p(e^{i\theta})|
\]
and hence

\[
\max_{|z|=1} |q(z)| \leq \frac{n+1}{2} \max_{|z|=1} |p(z)| .
\]

This completes the proof of inequality (2). Unfortunately, with the exception of \( n = 1 \) the bound \((n+1)/2\) does not appear to be sharp.

We now examine the \( L^2 \) analogue of the above problem. We prove the following theorem.

**Theorem.** If \( p(z) \) is a polynomial of degree \( n \) such that \( p(\beta) = 0 \) where \( \beta \) is an arbitrary nonnegative number then

\[
\int_0^{2\pi} |p(e^{i\theta})/(e^{i\theta} - \beta)|^2 \, d\theta \leq \left(1+\beta^2 - 2\beta \cos\left(\frac{\pi}{n+1}\right)\right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta .
\]

**Proof of the theorem.** Let us write

\[
p(z)/(z-\beta) = \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_{n-1} \neq 0 .
\]

Then

\[
p(z) = \alpha_{n-1} z^n + (\alpha_{n-2}-\beta\alpha_{n-1}) z^{n-1} + \cdots + (\alpha_0 - \beta \alpha_1) z - \beta \alpha_0 .
\]

We therefore have to consider the ratio

\[
R \equiv \left(\sum_{a=0}^{n-1} |\alpha_a|^2\right) / \left( |\alpha_{n-1}|^2 + \sum_{a=0}^{n-1} |\alpha_{a-1} - \beta \alpha_a|^2 + |\alpha_0|^2 \right) .
\]

Now
\[ R \leq \left( \sum_{s=0}^{n-1} |\alpha_s|^2 \right) \left/ \left( (1+\beta^2) \sum_{s=0}^{n-1} |\alpha_s|^2 - 2\beta \sum_{s=1}^{n-1} |\alpha_s| |\alpha_{s-1}| \right) \right. \]
\[ = \frac{1}{1+\beta^2 - 2\beta \left( \sum_{s=1}^{n-1} |\alpha_s| |\alpha_{s-1}| \right) / \left( \sum_{s=0}^{n-1} |\alpha_s|^2 \right)}. \]

Thus we require the maximum of the function
\[ f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|) = \left( \sum_{s=1}^{n-1} |\alpha_s|^2 \right)^{-1} \left( \sum_{s=1}^{n-1} |\alpha_s| |\alpha_{s-1}| \right) \]
with respect to \(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|\). It is clear that the maximum is less than 1.

If for some \(v, \alpha_v = 0\) and \(j\) is the smallest positive integer such that \(\alpha_{-j}, \alpha_{+j}\) are not both zero (\(\alpha_{-1}, \alpha_{-2}, \text{etc...}\) are to be interpreted as zero) then
\[ f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|, 0, |\alpha_{+1}|, \cdots, |\alpha_{n-1}|) \leq f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|, |\alpha_v|, |\alpha_{+1}|, \cdots, |\alpha_{n-1}|) \]
provided
\[ |\alpha_v| \leq (|\alpha_{-j}| + |\alpha_{+j}|) / f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|, 0, |\alpha_{+1}|, \cdots, |\alpha_{n-1}|). \]

This implies that the maximum is not attained when one or more of the numbers \(|\alpha_v|\) are zero.

On the other hand if one or more of the numbers \(|\alpha_v|\) are allowed to be arbitrarily large the function \(f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|)\) is bounded above by \((n-1)/n\).

Consider now the partial derivatives of \(f\) with respect to the variables \(|\alpha_v|\). For a local maximum we have to find \(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|\) such that
\[ \left( \sum_{s=0}^{n-1} |\alpha_s|^2 \right) \left( \frac{\partial f}{\partial |\alpha_0|} \right) = |\alpha_1| - 2f |\alpha_0| = 0, \]
\[ \left( \sum_{s=0}^{n-1} |\alpha_s|^2 \right) \left( \frac{\partial f}{\partial |\alpha_{s+1}|} \right) = |\alpha_{s+1}| + |\alpha_{s-1}| - 2f |\alpha_s| = 0, \quad \mu = 1, 2, \cdots, n-2, \]
\[ \left( \sum_{s=0}^{n-1} |\alpha_s|^2 \right) \left( \frac{\partial f}{\partial |\alpha_{s-1}|} \right) = |\alpha_{s-1}| - 2f |\alpha_{s+1}| = 0. \]

Let us suppose that the required local maximum is \(\lambda\). Since \(\lambda < 1\) we write \(\lambda = \cos \gamma (\gamma \neq 0)\). Then from the first \(n-1\) equations of the system (14) we obtain
\[ |\alpha_{\mu}| = U_{\mu}(\lambda) |\alpha_0|, \quad \mu = 1, 2, \cdots, n-1 \]
where \(U_{\mu}(\lambda) = (\sin (\mu+1)\gamma)/(\sin \gamma)\) is the Chebyshev polynomial of the second kind of degree \(\mu\) in \(\lambda\). Using equations (15) the last equation of the system (14) gives us
The only solution of (16) which is consistent with all the numbers $|\alpha_\tau|$ being nonnegative is $\gamma = \pi/(n+1)$. Hence

$$\lambda = \cos \left( \frac{\pi}{n+1} \right).$$

Since $\cos (\pi/(n+1)) \geq (n-1)/n$ the required maximum of the function $f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|)$ is $\cos (\pi/(n+1))$. This immediately leads to the inequality (8).

We note that the polynomial

$$p(z) = (z-\beta) \sum_{s=0}^{n-1} U_s \left( \cos \left( \frac{\pi}{n+1} \right) \right) z^s$$

is extremal.

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