Pacific Journal of Mathematics

INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

JOHN D. DONALDSON AND QAZI IBADUR RAHMAN

Vol. 41, No. 2 December 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 2, 1972

INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

J. D. DONALDSON AND Q. I. RAHMAN

Let $\mathscr{P}_n$ denote the linear space of polynomials $p(z) = \sum_{k=0}^{n} a_k z^k$ of degree at most n. There are various ways in which we can introduce norm ($\|\ \|$) in $\mathscr{P}_n$. Given β let $\mathscr{P}_{n,\beta}$ denote the subspace consisting of those polynomials which vanish at β. Then how large can $\| p(z)/(z-\beta) \|$ be if $p(z) \in \mathscr{P}_{n,\beta}$ and $\| p(z) \| = 1$? This general question does not seem to have received much attention. Here the problem is investigated when (i) $\|p(z)\| = \max_{|z| \leqslant 1} |p(z)|$, (ii) $\|p(z)\| = (1/2\pi \int_0^{2\pi} | p(e^{i\theta}) |^2 d\theta)^{1/2}$.

It was shown by Rahman and Mohammad [1] that if $p(z) \in \mathscr{P}_{n,1}$ and $\max_{|z|\leqslant 1} | p(z) | \leqq 1$ then

$$(1) \qquad \max_{|z|\leqslant 1} | p(z)/(z-1) | \leqq n/2 .$$

We observe that if $p(z) \in \mathscr{P}_{n,\beta}$ and $\max_{|z|\leqslant 1} | p(z) | = 1$ then $\max_{|z|\leqslant 1} | p(z)/(z-\beta) |$ can be greater than $n/2$ if β is arbitrary. For $n = 1$ we may simply take $p(z) = z$. When $n > 1$ we consider the polynomial

$$p(z) = (n/2)(n^2-1)^{-1/2}(1+z+z^2+\cdots+z^{n-1})(z-1+2n^{-2}) .$$

If $z = e^{i\theta}$ then for $\cos\theta \leqq 1 - 2n^{-2}$

$$| p(z) | \leqq (1/2) | (1+z+z^2+\cdots+z^{n-1})(z-1) | \leqq 1 ,$$

and also for $\cos\theta \geqq 1 - 2n^{-2}$

$$| p(z)) | \leqq n(n^2-1)^{-1/2}(n/2) | z-1+2n^{-2} | \leqq 1$$

while

$$\max_{|z|=1} | p(z)/(z-1+2n^{-2}) | = (n^2/2)(n^2-1)^{-1/2} > \frac{n}{2} .$$

We note howevever that if $p(z) \in \mathscr{P}_{n,\beta}$ and $\max_{|z|\leqslant 1} | p(z) | \leqslant 1$, then

$$(2) \qquad \max_{|z|=1} | p(z)/(z-\beta) | \leqq (n+1)/2 .$$

Proof of inequality (2). Without loss of generality we may assume β to be real and nonnegative. Put $p(z) = (z-\beta)q(z)$ and write

$p^*(z) = (z-1)q(z)$. Then

$$|p^*(e^{i\theta})/p(e^{i\theta})| = |(e^{i\theta}-1)/(e^{i\theta}-\beta)| \leqq 2/(1+\beta) \tag{3}$$

which gives us

$$\max_{|z|=1} |p^*(z)| \leqq 2(1+\beta)^{-1} \max_{|z|=1} |p(z)| . \tag{4}$$

From inequalities (1) and (4) we obtain

$$\begin{aligned}\max_{|z|=1} |q(z)| \leqq (n/2) \max_{|z|=1} |p^*(z)| \leqq n(1+\beta)^{-1} \max_{|z|=1} |p(z)| \\ \leqq \frac{n+1}{2} \max_{|z|=1} |p(z)|\end{aligned} \tag{5}$$

provided $\beta \geqq (n-1)/(n+1)$.

For $\beta \leqq (n-1)/(n+1)$ we have

$$|q(e^{i\theta})| = |p(e^{i\theta})/(e^{i\theta}-\beta)| \leqq (1-\beta)^{-1} |p(e^{i\theta})| \leqq \frac{n+1}{2} |p(e^{i\theta})| \tag{6}$$

and hence

$$\max_{|z|=1} |q(z)| \leqq \frac{n+1}{2} \max_{|z|=1} |p(z)| . \tag{7}$$

This completes the proof of inequality (2). Unfortunately, with the exception of $n = 1$ the bound $(n+1)/2$ does not appear to be sharp.

We now examine the L^2 analogue of the above problem. We prove the following theorem.

THEOREM. *If $p(z)$ is a polynomial of degree n such that $p(\beta) = 0$ where β is an arbitrary nonnegative number then*

$$\int_0^{2\pi} |p(e^{i\theta})/(e^{i\theta}-\beta)|^2 \, d\theta \leqq \left(1+\beta^2-2\beta\cos\left(\frac{\pi}{n+1}\right)\right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta . \tag{8}$$

Proof of the theorem. Let us write

$$p(z)/(z-\beta) = \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_1 z + \alpha_0, \ \alpha_{n-1} \neq 0 . \tag{9}$$

Then

$$p(z) = \alpha_{n-1} z^n + (\alpha_{n-2}-\beta\alpha_{n-1})z^{n-1} + \cdots + (\alpha_0-\beta\alpha_1)z - \beta\alpha_0 . \tag{10}$$

We therefore have to consider the ratio

$$R \equiv \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \Big/ \left(|\alpha_{n-1}|^2 + \sum_{\nu=0}^{n-1} |\alpha_{\nu-1} - \beta\alpha_\nu|^2 + \beta|\alpha_0|^2\right) . \tag{11}$$

Now

$$R \leqq \left(\sum_{\nu=1}^{n-1} |\alpha_\nu|^2\right)\Big/\left((1+\beta^2)\sum_{\nu=0}^{n-1} |\alpha_\nu|^2 - 2\beta\sum_{\nu=1}^{n-1} |\alpha_\nu||\alpha_{\nu-1}|\right)$$
$$= 1/\left(1+\beta^2-2\beta\left(\sum_{\nu=1}^{n-1} |\alpha_\nu||\alpha_{\nu-1}|\right)\Big/\left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right)\right).$$

Thus we require the maximum of the function

$$f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|) = \left(\sum_{\nu=1}^{n-1} |\alpha_\nu|^2\right)^{-1}\left(\sum_{\nu=1}^{n-1} |\alpha_\nu||\alpha_{\nu-1}|\right) \tag{12}$$

with respect to $|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|$. It is clear that the maximum is less than 1.

If for some ν, $\alpha_\nu = 0$ and j is the smallest positive integer such that $\alpha_{\nu-j}$, $\alpha_{\nu+j}$ are not both zero (α_{-1}, α_{-2}, etc... are to be interpreted as zero) then

$$\begin{aligned} &f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \cdots, |\alpha_{n-1}|) \\ &\leqq f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{\nu-1}|, |\alpha'_\nu|, |\alpha_{\nu+1}|, \cdots, |\alpha_{n-1}|) \end{aligned} \tag{13}$$

provided

$$|\alpha'_\nu| \leqq (|\alpha_{\nu-j}|+|\alpha_{\nu+j}|)/f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \cdots, |\alpha_{n-1}|).$$

This implies that the maximum is not attained when one or more of the numbers $|\alpha_\nu|$ are zero.

On the other hand if one or more of the numbers $|\alpha_\nu|$ are allowed to be arbitrarily large the function $f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|)$ is bounded above by $(n-1)/n$.

Consider now the partial derivatives of f with respect to the variables $|\alpha_\nu|$. For a local maximum we have to find $|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|$ such that

$$\begin{cases} \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \dfrac{\partial f}{\partial |\alpha_0|} = |\alpha_1| - 2f|\alpha_0| = 0\,, \\ \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \dfrac{\partial f}{\partial |\alpha_\mu|} = |\alpha_{\mu+1}| + |\alpha_{\mu-1}| - 2f|\alpha_\mu| = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu = 1, 2, \cdots, n-2, \\ \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) \dfrac{\partial f}{\partial |\alpha_{n-1}|} = |\alpha_{n-2}| - 2f|\alpha_{n-1}| = 0\,. \end{cases} \tag{14}$$

Let us suppose that the required local maximum is λ. Since $\lambda < 1$ we write $\lambda = \cos\gamma$ ($\gamma \neq 0$). Then from the first $n-1$ equations of the system (14) we obtain

$$|\alpha_\mu| = U_\mu(\lambda)|\alpha_0|\,, \qquad \mu = 1, 2, \cdots, n-1 \tag{15}$$

where $U_\mu(\lambda) = (\sin(\mu+1)\gamma)/(\sin\gamma)$ is the Chebyshev polynomial of the second kind of degree μ in λ. Using equations (15) the last equation of the system (14) gives us

$$\sin(n+1)\gamma = 0 \,. \tag{16}$$

The only solution of (16) which is consistent with all the numbers $|\alpha_\nu|$ being nonnegative is $\gamma = \pi/(n+1)$. Hence

$$\lambda = \cos\left(\frac{\pi}{n+1}\right).$$

Since $\cos(\pi/(n+1)) \geqq (n-1)/n$ the required maximum of the function $f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|)$ is $\cos(\pi/(n+1))$. This immediately leads to the inequality (8).

We note that the polynomial

$$p(z) = (z-\beta)\sum_{\nu=0}^{n-1} U_\nu\left(\cos\left(\frac{\pi}{n+1}\right)\right) z^\nu$$

is extremal.

REFERENCES

1. Q. I. Rahman and Q. G. Mohammad, *Remarks on Schwarz's lemma*, Pacific J. Math. **23** (1967), 139-142.

Received January 19, 1971.

UNIVERSITY OF MONTREAL

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 41, No. 2 December, 1972