INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

JOHN D. DONALDSON AND QAZI IBADUR RAHMAN
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Let $\mathcal{P}_n$ denote the linear space of polynomials $p(z) = \sum_{k=0}^{n} a_k z^k$ of degree at most $n$. There are various ways in which we can introduce norm $\| \|_r$ in $\mathcal{P}_n$. Given $\beta$ let $\mathcal{P}_{n,\beta}$ denote the subspace consisting of those polynomials which vanish at $\beta$. Then how large can $\| p(z)/(z-\beta) \|$ be if $p(z) \in \mathcal{P}_{n,\beta}$ and $\| p(z) \| = 1$? This general question does not seem to have received much attention. Here the problem is investigated when (i) $\| p(z) \| = \max_{|z| < 1} |p(z)|$, (ii) $\| p(z) \| = (1/2\pi) \int_{0}^{2\pi} |p(e^{i\theta})|^{2} d\theta)^{1/2}$.

It was shown by Rahman and Mohammad [1] that if $p(z) \in \mathcal{P}_{n,1}$ and $\max_{|z| < 1} |p(z)| \leq 1$ then

\[(1) \quad \max_{|z| < 1} |p(z)/(z-1)| \leq n/2.\]

We observe that if $p(z) \in \mathcal{P}_{n,\beta}$ and $\max_{|z| < 1} |p(z)| = 1$ then $\max_{|z| < 1} |p(z)/(z-\beta)|$ can be greater than $n/2$ if $\beta$ is arbitrary. For $n = 1$ we may simply take $p(z) = z$. When $n > 1$ we consider the polynomial

$$p(z) = (n/2) (n^2-1)^{-1/2} (1+z+z^2+\cdots+z^{n-1}) (z-1+2n^{-2}).$$

If $z = e^{i\theta}$ then for $\cos \theta \leq 1 - 2n^{-2}$

$$|p(z)| \leq (1/2) \left| (1+z+z^2+\cdots+z^{n-1}) (z-1) \right| \leq 1,$$

and also for $\cos \theta \geq 1 - 2n^{-2}$

$$|p(z)| \leq n(n^2-1)^{-1/2} (n/2) |z-1+2n^{-2}| \leq 1$$

while

$$\max_{|z|=1} |p(z)/(z-1+2n^{-2})| = (n^2/2) (n^2-1)^{-1/2} > \frac{n}{2}.$$

We note however that if $p(z) \in \mathcal{P}_{n,\beta}$ and $\max_{|z| < 1} |p(z)| \leq 1$, then

\[(2) \quad \max_{|z|=1} |p(z)/(z-\beta)| \leq (n+1)/2.\]

Proof of inequality (2). Without loss of generality we may assume $\beta$ to be real and nonnegative. Put $p(z) = (z-\beta)q(z)$ and write
\( p^*(z) = (z-1)q(z) \). Then

\[
\left| p^*(e^{i\theta})/p(e^{i\theta}) \right| = \left| (e^{i\theta} - 1)/(e^{i\theta} - \beta) \right| \leq 2/(1 + \beta)
\]

which gives us

\[
\max_{|z|=1} |p^*(z)| \leq 2 \left( 1 + \beta \right)^{-1} \max_{|z|=1} |p(z)|.
\]

From inequalities (1) and (4) we obtain

\[
\max_{|z|=1} |q(z)| \leq (n/2) \max_{|z|=1} |p^*(z)| \leq n(1 + \beta)^{-1} \max_{|z|=1} |p(z)|
\]

\[
\leq \frac{n + 1}{2} \max_{|z|=1} |p(z)|
\]

provided \( \beta \geq (n - 1)/(n + 1) \).

For \( \beta \leq (n - 1)/(n + 1) \) we have

\[
\max_{|z|=1} |q(z)| \leq \frac{n + 1}{2} \max_{|z|=1} |p(z)|.
\]

This completes the proof of inequality (2). Unfortunately, with the exception of \( n = 1 \) the bound \((n + 1)/2\) does not appear to be sharp.

We now examine the \( L^2 \) analogue of the above problem. We prove the following theorem.

**Theorem.** If \( p(z) \) is a polynomial of degree \( n \) such that \( p(\beta) = 0 \) where \( \beta \) is an arbitrary nonnegative number then

\[
\int_0^{2\pi} |p(e^{i\theta})/(e^{i\theta} - \beta)|^2 \, d\theta \leq \left( 1 + \beta^2 - 2\beta \cos\left( \frac{\pi}{n+1} \right) \right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta.
\]

**Proof of the theorem.** Let us write

\[
p(z)/(z - \beta) = \alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2} + \cdots + \alpha_z + \alpha_o, \alpha_{n-1} \neq 0.
\]

Then

\[
p(z) = \alpha_{n-1}z^n + (\alpha_{n-2} - \beta \alpha_{n-1})z^{n-1} + \cdots + (\alpha_o - \beta \alpha_1)z - \beta \alpha_o.
\]

We therefore have to consider the ratio

\[
R = \left( \sum_{r=0}^{n-1} |\alpha_r|^2 \right) / \left( |\alpha_{n-1}|^2 + \sum_{r=0}^{n-1} |\alpha_{r-1} - \beta \alpha_r|^2 + \beta |\alpha_o|^2 \right).
\]

Now
\[ R \leq \left( \sum_{\nu=1}^{n-1} |\alpha_{\nu}|^2 \right) \left( (1+\beta^2) \sum_{\nu=1}^{n-1} |\alpha_{\nu}|^2 - 2\beta \sum_{\nu=1}^{n-1} |\alpha_{\nu}| \right) \left( \sum_{\nu=1}^{n-1} |\alpha_{\nu-1}|^2 \right) \]

Thus we require the maximum of the function

\[ f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{n-1}|) = \left( \sum_{\nu=1}^{n-1} |\alpha_{\nu}|^2 \right)^{-1} \left( \sum_{\nu=1}^{n-1} |\alpha_{\nu}| \right) \]

with respect to $|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{n-1}|$. It is clear that the maximum is less than 1.

If for some $\nu$, $\alpha_\nu = 0$ and $j$ is the smallest positive integer such that $\alpha_{\nu-j}, \alpha_{\nu+j}$ are not both zero ($\alpha_{-1}, \alpha_{-2}, \text{etc...}$ are to be interpreted as zero) then

\[ f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \ldots, |\alpha_{n-1}|) \leq f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{\nu-1}|, |\alpha'_\nu|, |\alpha_{\nu+1}|, \ldots, |\alpha_{n-1}|) \]

provided

\[ |\alpha'_\nu| \leq (|\alpha_{\nu-j}| + |\alpha_{\nu+j}|)/f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \ldots, |\alpha_{n-1}|) . \]

This implies that the maximum is not attained when one or more of the numbers $|\alpha_{\nu}|$ are zero.

On the other hand if one or more of the numbers $|\alpha_{\nu}|$ are allowed to be arbitrarily large the function $f(|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{n-1}|)$ is bounded above by $(n-1)/n$.

Consider now the partial derivatives of $f$ with respect to the variables $|\alpha_{\nu}|$. For a local maximum we have to find $|\alpha_0|, |\alpha_1|, \ldots, |\alpha_{n-1}|$ such that

\[
\begin{align*}
\left( \sum_{\nu=0}^{n-1} |\alpha_{\nu}|^2 \right) \frac{\partial f}{\partial |\alpha_\nu|} &= |\alpha_\nu| - 2f |\alpha_\nu| = 0 , \\
\left( \sum_{\nu=0}^{n-1} |\alpha_{\nu}|^2 \right) \frac{\partial f}{\partial |\alpha_\mu|} &= |\alpha_{\mu+1}| + |\alpha_{\mu-1}| - 2f |\alpha_\mu| = 0 , \\
\left( \sum_{\nu=0}^{n-1} |\alpha_{\nu}|^2 \right) \frac{\partial f}{\partial |\alpha_{\mu-1}|} &= |\alpha_{\mu-2}| - 2f |\alpha_{\mu-1}| = 0 .
\end{align*}
\]

Let us suppose that the required local maximum is $\lambda$. Since $\lambda < 1$ we write $\lambda = \cos \gamma (\gamma \neq 0)$. Then from the first $n-1$ equations of the system (14) we obtain

\[ |\alpha_{\mu}| = U_{\mu}(\lambda) |\alpha_0|, \quad \mu = 1, 2, \ldots, n-1 \]

where $U_{\mu}(\lambda) = (\sin (\mu+1)\gamma)/(\sin \gamma)$ is the Chebyshev polynomial of the second kind of degree $\mu$ in $\lambda$. Using equations (15) the last equation of the system (14) gives us
The only solution of (16) which is consistent with all the numbers $|\alpha_\nu|$ being nonnegative is $\gamma = \pi / (n+1)$. Hence

$$\lambda = \cos \left( \frac{\pi}{n+1} \right).$$

Since $\cos \left( \frac{\pi}{n+1} \right) \geq \frac{1}{n}$ the required maximum of the function $f(|\alpha_0|, |\alpha_1|, \cdots, |\alpha_{n-1}|)$ is $\cos \left( \frac{\pi}{n+1} \right)$. This immediately leads to the inequality (8).

We note that the polynomial

$$p(z) = (z - \beta) \sum_{\nu=0}^{n-1} U_{\nu} \left( \cos \left( \frac{\pi}{n+1} \right) \right) z^\nu$$

is extremal.

**References**


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