

# Pacific Journal of Mathematics

## **INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO**

JOHN D. DONALDSON AND QAZI IBADUR RAHMAN

## INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

J. D. DONALDSON AND Q. I. RAHMAN

Let  $\mathcal{P}_n$  denote the linear space of polynomials  $p(z) = \sum_{k=0}^n a_k z^k$  of degree at most  $n$ . There are various ways in which we can introduce norm ( $\| \cdot \|$ ) in  $\mathcal{P}_n$ . Given  $\beta$  let  $\mathcal{P}_{n,\beta}$  denote the subspace consisting of those polynomials which vanish at  $\beta$ . Then how large can  $\|p(z)/(z-\beta)\|$  be if  $p(z) \in \mathcal{P}_{n,\beta}$  and  $\|p(z)\| = 1$ ? This general question does not seem to have received much attention. Here the problem is investigated when (i)  $\|p(z)\| = \max_{|z| \leq 1} |p(z)|$ , (ii)  $\|p(z)\| = (1/2\pi \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta)^{1/2}$ .

It was shown by Rahman and Mohammad [1] that if  $p(z) \in \mathcal{P}_{n,1}$  and  $\max_{|z| \leq 1} |p(z)| \leq 1$  then

$$(1) \quad \max_{|z| \leq 1} |p(z)/(z-1)| \leq n/2.$$

We observe that if  $p(z) \in \mathcal{P}_{n,\beta}$  and  $\max_{|z| \leq 1} |p(z)| = 1$  then  $\max_{|z| \leq 1} |p(z)/(z-\beta)|$  can be greater than  $n/2$  if  $\beta$  is arbitrary. For  $n = 1$  we may simply take  $p(z) = z$ . When  $n > 1$  we consider the polynomial

$$p(z) = (n/2)(n^2-1)^{-1/2} (1+z+z^2+\dots+z^{n-1})(z-1+2n^{-2}).$$

If  $z = e^{i\theta}$  then for  $\cos \theta \leq 1 - 2n^{-2}$

$$|p(z)| \leq (1/2) |1+z+z^2+\dots+z^{n-1}| |z-1| \leq 1,$$

and also for  $\cos \theta \geq 1 - 2n^{-2}$

$$|p(z)| \leq n(n^2-1)^{-1/2} (n/2) |z-1+2n^{-2}| \leq 1$$

while

$$\max_{|z|=1} |p(z)/(z-1+2n^{-2})| = (n^2/2)(n^2-1)^{-1/2} > \frac{n}{2}.$$

We note however that if  $p(z) \in \mathcal{P}_{n,\beta}$  and  $\max_{|z| \leq 1} |p(z)| \leq 1$ , then

$$(2) \quad \max_{|z|=1} |p(z)/(z-\beta)| \leq (n+1)/2.$$

*Proof of inequality (2).* Without loss of generality we may assume  $\beta$  to be real and nonnegative. Put  $p(z) = (z-\beta)q(z)$  and write

$p^*(z) = (z-1)q(z)$ . Then

$$(3) \quad |p^*(e^{i\theta})/p(e^{i\theta})| = |(e^{i\theta}-1)/(e^{i\theta}-\beta)| \leq 2/(1+\beta)$$

which gives us

$$(4) \quad \max_{|z|=1} |p^*(z)| \leq 2(1+\beta)^{-1} \max_{|z|=1} |p(z)|.$$

From inequalities (1) and (4) we obtain

$$(5) \quad \begin{aligned} \max_{|z|=1} |q(z)| &\leq (n/2) \max_{|z|=1} |p^*(z)| \leq n(1+\beta)^{-1} \max_{|z|=1} |p(z)| \\ &\leq \frac{n+1}{2} \max_{|z|=1} |p(z)| \end{aligned}$$

provided  $\beta \geq (n-1)/(n+1)$ .

For  $\beta \leq (n-1)/(n+1)$  we have

$$(6) \quad |q(e^{i\theta})| = |p(e^{i\theta})/(e^{i\theta}-\beta)| \leq (1-\beta)^{-1} |p(e^{i\theta})| \leq \frac{n+1}{2} |p(e^{i\theta})|$$

and hence

$$(7) \quad \max_{|z|=1} |q(z)| \leq \frac{n+1}{2} \max_{|z|=1} |p(z)|.$$

This completes the proof of inequality (2). Unfortunately, with the exception of  $n=1$  the bound  $(n+1)/2$  does not appear to be sharp.

We now examine the  $L^2$  analogue of the above problem. We prove the following theorem.

**THEOREM.** *If  $p(z)$  is a polynomial of degree  $n$  such that  $p(\beta) = 0$  where  $\beta$  is an arbitrary nonnegative number then*

$$(8) \quad \int_0^{2\pi} |p(e^{i\theta})/(e^{i\theta}-\beta)|^2 d\theta \leq \left(1+\beta^2-2\beta \cos\left(\frac{\pi}{n+1}\right)\right)^{-1} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

*Proof of the theorem.* Let us write

$$(9) \quad p(z)/(z-\beta) = \alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2} + \dots + \alpha_1z + \alpha_0, \alpha_{n-1} \neq 0.$$

Then

$$(10) \quad p(z) = \alpha_{n-1}z^n + (\alpha_{n-2}-\beta\alpha_{n-1})z^{n-1} + \dots + (\alpha_0-\beta\alpha_1)z - \beta\alpha_0.$$

We therefore have to consider the ratio

$$(11) \quad R \equiv \left(\sum_{\nu=0}^{n-1} |\alpha_\nu|^2\right) / \left(|\alpha_{n-1}|^2 + \sum_{\nu=0}^{n-1} |\alpha_{\nu-1} - \beta\alpha_\nu|^2 + \beta|\alpha_0|^2\right).$$

Now

$$R \leq \left( \sum_{\nu=1}^{n-1} |\alpha_\nu|^2 \right) / \left( (1 + \beta^2) \sum_{\nu=0}^{n-1} |\alpha_\nu|^2 - 2\beta \sum_{\nu=1}^{n-1} |\alpha_\nu| |\alpha_{\nu-1}| \right) \\ = 1 / \left( 1 + \beta^2 - 2\beta \left( \sum_{\nu=1}^{n-1} |\alpha_\nu| |\alpha_{\nu-1}| \right) / \left( \sum_{\nu=0}^{n-1} |\alpha_\nu|^2 \right) \right).$$

Thus we require the maximum of the function

$$(12) \quad f(|\alpha_0|, |\alpha_1|, \dots, |\alpha_{n-1}|) = \left( \sum_{\nu=1}^{n-1} |\alpha_\nu|^2 \right)^{-1} \left( \sum_{\nu=1}^{n-1} |\alpha_\nu| |\alpha_{\nu-1}| \right)$$

with respect to  $|\alpha_0|, |\alpha_1|, \dots, |\alpha_{n-1}|$ . It is clear that the maximum is less than 1.

If for some  $\nu, \alpha_\nu = 0$  and  $j$  is the smallest positive integer such that  $\alpha_{\nu-j}, \alpha_{\nu+j}$  are not both zero ( $\alpha_{-1}, \alpha_{-2},$  etc... are to be interpreted as zero) then

$$(13) \quad f(|\alpha_0|, |\alpha_1|, \dots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \dots, |\alpha_{n-1}|) \\ \leq f(|\alpha_0|, |\alpha_1|, \dots, |\alpha_{\nu-1}|, |\alpha'_\nu|, |\alpha_{\nu+1}|, \dots, |\alpha_{n-1}|)$$

provided

$$|\alpha'_\nu| \leq (|\alpha_{\nu-j}| + |\alpha_{\nu+j}|) / f(|\alpha_0|, |\alpha_1|, \dots, |\alpha_{\nu-1}|, 0, |\alpha_{\nu+1}|, \dots, |\alpha_{n-1}|).$$

This implies that the maximum is not attained when one or more of the numbers  $|\alpha_\nu|$  are zero.

On the other hand if one or more of the numbers  $|\alpha_\nu|$  are allowed to be arbitrarily large the function  $f(|\alpha_0|, |\alpha_1|, \dots, |\alpha_{n-1}|)$  is bounded above by  $(n-1)/n$ .

Consider now the partial derivatives of  $f$  with respect to the variables  $|\alpha_\nu|$ . For a local maximum we have to find  $|\alpha_0|, |\alpha_1|, \dots, |\alpha_{n-1}|$  such that

$$(14) \quad \begin{cases} \left( \sum_{\nu=0}^{n-1} |\alpha_\nu|^2 \right) \frac{\partial f}{\partial |\alpha_0|} = |\alpha_1| - 2f |\alpha_0| = 0, \\ \left( \sum_{\nu=0}^{n-1} |\alpha_\nu|^2 \right) \frac{\partial f}{\partial |\alpha_\mu|} = |\alpha_{\mu+1}| + |\alpha_{\mu-1}| - 2f |\alpha_\mu| = 0, \\ \hspace{20em} \mu = 1, 2, \dots, n-2, \\ \left( \sum_{\nu=0}^{n-1} |\alpha_\nu|^2 \right) \frac{\partial f}{\partial |\alpha_{n-1}|} = |\alpha_{n-2}| - 2f |\alpha_{n-1}| = 0. \end{cases}$$

Let us suppose that the required local maximum is  $\lambda$ . Since  $\lambda < 1$  we write  $\lambda = \cos \gamma$  ( $\gamma \neq 0$ ). Then from the first  $n-1$  equations of the system (14) we obtain

$$(15) \quad |\alpha_\mu| = U_\mu(\lambda) |\alpha_0|, \quad \mu = 1, 2, \dots, n-1$$

where  $U_\mu(\lambda) = (\sin(\mu+1)\gamma) / (\sin \gamma)$  is the Chebyshev polynomial of the second kind of degree  $\mu$  in  $\lambda$ . Using equations (15) the last equation of the system (14) gives us

$$(16) \quad \sin (n+1)\gamma = 0 .$$

The only solution of (16) which is consistent with all the numbers  $|\alpha_\nu|$  being nonnegative is  $\gamma = \pi/(n+1)$ . Hence

$$\lambda = \cos \left( \frac{\pi}{n+1} \right) .$$

Since  $\cos (\pi/(n+1)) \geq (n-1)/n$  the required maximum of the function  $f(|\alpha_0|, |\alpha_1|, \dots, |\alpha_{n-1}|)$  is  $\cos (\pi/(n+1))$ . This immediately leads to the inequality (8).

We note that the polynomial

$$p(z) = (z-\beta) \sum_{\nu=0}^{n-1} U_\nu \left( \cos \left( \frac{\pi}{n+1} \right) \right) z^\nu$$

is extremal.

#### REFERENCES

1. Q. I. Rahman and Q. G. Mohammad, *Remarks on Schwarz's lemma*, Pacific J. Math. **23** (1967), 139-142.

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