

# Pacific Journal of Mathematics

**DIFFERENTIABLE POWER-ASSOCIATIVE GROUPOIDS**

JOHN P. HOLMES

## DIFFERENTIABLE POWER-ASSOCIATIVE GROUPOIDS

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**Suppose  $H$  is a Banach space,  $D$  is an open set of  $H$  containing  $0$ , and  $V$  is a function from  $D \times D$  to  $H$  satisfying  $V(0, x) = V(x, 0) = x$  for each  $x$  in  $D$ . If  $n$  is an integer greater than  $1$ , denote by  $x^n$  the product of  $n - x$ 's associated as follows whenever the product exists.**

$$x^n = V(x, V(x, \cdots V(x, x) \cdots)).$$

**Define  $x^0 = 0$  and  $x^1 = x$ .  $V$  is said to be power associative if and only if  $V(x^n, x^m) = x^{n+m}$  whenever each of  $n$  and  $m$  is a nonnegative integer and  $x^{n+m}$  exists.**

**THEOREM A. If  $H$  and  $V$  are as above,  $V$  is power associative and continuously differentiable in the sense of Frechet on  $D \times D$  then there are positive numbers  $a$  and  $c$  such that if  $x$  is in  $H$  and  $\|x\| < a$  there is a unique continuous function  $T_x$  from  $[0, 1]$  to the ball of radius  $c$  centered at  $0$  satisfying  $V(T_x(s), T_x(t)) = T_x(s+t)$  whenever each of  $s, t$ , and  $s+t$  is in  $[0, 1]$ ,  $T_x(0) = 0$ , and  $T_x(1) = x$ .**

Theorem A is similar to a result in [1] of Birkhoff. He showed that if  $H$  and  $V$  are as above,  $V$  is associative,  $V$  is Frechet differentiable on a neighborhood of  $(0, 0)$ , and  $V'$  is continuous at  $(0, 0)$  then some neighborhood of  $0$  is covered by partial homomorphic images of the additive group of real numbers.

To see that Theorem A is not a special case of this result of Birkhoff, we offer the following example. Denote by  $E$  the 2-dimensional Euclidean space and define  $V$  from  $E \times E$  to  $E$  by  $V((x, y), (z, w)) = (x + [1 + (xw - yz)]z, y + [1 + (xw - yz)]w)$ . If  $S$  is a 1-dimensional linear subspace of  $E$  and each of  $p$  and  $q$  is in  $S$  then  $V(p, q) = p + q$ . Thus  $V$  is power associative and  $0$  is an identity for  $V$ .  $V$  is not associative but  $V$  is continuously differentiable on  $E \times E$ .

We will now prove Theorem A. Regard  $H \times H$  as a Banach space in the usual way, defining the norm of a member  $(x, y)$  of  $H \times H$  by  $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ . If  $c$  is a positive number, denote by  $R(c)$  the set to which  $x$  belongs if and only if  $x$  is in  $H$  and  $\|x\| < c$ . Finally, if  $B$  is a bounded linear transformation from  $H \times H$  to  $H$  or from  $H$  to  $H$ , denote the norm of  $B$  by  $|B|$ .

Define  $f$  from  $D$  to  $H$  by  $f(x) = V(x, x) = x^2$  for each  $x$  in  $D$ . Note  $f$  is continuously differentiable on  $D$  and if  $x$  is in  $D$ ,  $f'(x)(y) = V'(x, x)(y, y)$  for each  $y$  in  $H$ . Moreover,  $V'(0, 0)(z, w) = z + w$  for each pair  $(z, w)$  in  $H \times H$  so  $f'(0) = 2I$  where  $I$  is the identity transfor-

mation on  $H$ .

Employing the inverse function theorem (for instance [2] page 268) we see that there is a positive number  $b$  and an open set  $U$  of  $H$  such that  $(f|U)$  is a homeomorphism of  $U$  onto  $R(b)$  and  $g = (f|U)^{-1}$  is continuously differentiable on  $R(b)$  with  $g'(y) = [f'(g(y))]^{-1}$  for each  $y$  in  $R(b)$ . Hence  $g'(0) = 1/2 I$ .

By continuity of  $g'$  and  $V'$  there is a positive number  $d$  and a number  $M$  such that if  $p$  is in  $R(d) \times R(d)$  and  $x$  is in  $R(d)$  then  $|V'(p)| < M$  and  $|g'(x)| < 2/3$ .

Suppose each of  $x, y, z$ , and  $w$  is in  $R(d)$ . Then  $\|V(x, y) - V(z, w)\| = \left\| \int_0^1 dt V'((z, w) + t(x - z, y - w))(x - z, y - w) \right\| < M \|(x - z, y - w)\|$ . As special cases of this inequality we obtain

1.  $\|V(x, y)\| < M \|(x, y)\|$  and
2.  $\|V(x, y) - y\| < M \|x\|$ .

Similarly, if each of  $x$  and  $y$  is in  $R(d)$  we have  $\|g(x) - g(y)\| = \left\| \int_0^1 dt g'(y + t(x - y))(x - y) \right\| < 2/3 \|x - y\|$ . Hence  $g$  is Lipschitz on  $R(d)$  and has Lipschitz norm less than  $2/3$ . In particular, for each  $x$  in  $R(d)$  and each positive integer  $m$  we have  $\|g^m(x)\| < (2/3)^m \|x\|$  where  $g^m$  is  $g$  composed with itself  $m$  times.

**LEMMA 1.** *Let  $r = d/3M$ . If  $x$  is in  $R(r)$ ,  $m$  is a positive integer, and  $n$  is an integer in  $[0, 2^m]$  then  $[g^m(x)]^n$  exists and has norm less than  $M \|x\| \sum_1^m (2/3)^i$ .*

*Proof.* Note  $|V'(0, 0)| = 2$  so  $M > 3/2$ . If  $x$  is in  $R(r)$ , it is clear, using inequality 1, that  $g^i(x)^i$  exists for each  $i = 0, 1$ , or  $2$  and has norm less than  $M \|x\| (2/3)$ .

Suppose  $m$  is an integer greater than 1 and assume that for each integer  $k$  in  $[1, m)$  that  $g^k(x)^s$  exists for each integer  $s$  in  $[0, 2^k]$  and has norm less than  $M \|x\| \sum_1^k (2/3)^i$ .

As has been observed before,  $g^m(x)$  exists and  $\|g^m(x)^0\| = 0$ . Suppose  $n$  is an integer in  $(0, 2^m]$  and assume for each integer  $c$  in  $[0, n)$  that  $g^m(x)^c$  exists and has norm less than  $M \|x\| \sum_1^m (2/3)^i$ .

Then  $g^m(x)^{n-1}$  exists and  $\|g^m(x)^{n-1}\| < M \|x\| \sum_1^m (2/3)^i < 2M \|x\| < 2Mr = 2M d/3M < d$ . Thus  $g^m(x)^{n-1}$  is in  $D$  and  $g^m(x)^n = V(g^m(x), g^m(x)^{n-1})$  exists.

If  $n$  is even, we may use power associativity and the equality  $g^m(x)^2 = g^{m-1}(x)$  to obtain  $g^m(x)^n = g^{m-1}(x)^{n/2}$ . Hence, by the first inductive hypothesis,  $\|g^m(x)^n\| < M \|x\| \sum_1^m (2/3)^i$ .

If  $n$  is odd then  $g^m(x)^n = V(g^m(x), g^{(m-1)}(x)^{(n-1)/2})$ . Using the triangle

inequality, inequality 2, and the first inductive hypothesis we obtain  $\|g^m(x)^n\| \leq \|V(g^m(x), g^{m-1}(x)^{(n-1)/2}) - g^{m-1}(x)^{(n-1)}\| + \|g^{m-1}(x)^{(n-1)/2}\| < M \|x\| \sum_1^m (2/3)^i$ .

Thus we have Lemma 1.

Suppose  $x$  is in  $R(r)$ . Denote by  $E$  the set of dyadic rational numbers in  $[0, 1]$  and define  $T$  from  $E$  to  $H$  by  $T(n/2^m) = g^m(x)^n$ .  $T$  exists by Lemma 1 and is well defined by power associativity. Moreover, by power associativity,  $V(T(h), T(k)) = T(h + k)$  whenever each of  $h, k$ , and  $h + k$  is in  $E$ . Lemma 2 will show that  $T$  has a continuous extension to all of  $[0, 1]$ .

LEMMA 2. *If  $x$  and  $T$  are as above, each of  $h$  and  $k$  is in  $E$ , and  $|h - k| < 1/2^m$  for some positive integer  $m$  then  $\|T(h) - T(k)\| < 9M \|x\| (2/3)^{m+1}$ .*

*Proof.* Suppose  $h = s/2^{m+n}$  for some nonnegative integers  $s$  and  $n$ , and  $u$  is an integer with each of  $u/2^m$  and  $(u + 1)/2^m$  in  $E$  so that  $h$  is in  $[u/2^m, (u + 1)/2^m]$ . There is a sequence  $a_1, \dots, a_n$  such that  $h = u/2^m + a_1/2^{m+1} + \dots + a_n/2^{m+n}$  and each  $a_i$  is in the set  $\{0, 1\}$ . Thus  $T(h) = V(T(u/2^m), V(T(a_1/2^{m+1}), \dots, V(T(a_{n-1}/2^{m+n-1}), T(a_n/2^{m+n})) \dots))$ .

Let  $w$  be defined from  $\{0, 1, \dots, n\}$  by  $w_i = u/2^m + \sum_1^i a_j/2^{m+j}$ . Then  $w_i = w_{i-1} + a_i/2^{m+i}$  for each  $i$  in  $\{1, \dots, n\}$ . Now, using the triangle inequality, we have  $\|T(h) - T(u/2^m)\| \leq \sum_1^n \|T(w_i) - T(w_{i-1})\|$ . But, using inequality 2 we obtain  $\|T(w_i) - T(w_{i-1})\| \leq M \|T(a_i/2^{m+i})\| < M \|x\| (2/3)^{m+i}$ . Hence  $\|T(h) - T(u/2^m)\| < M \|x\| \sum_1^n (2/3)^{m+i} < 3M \|x\| (2/3)^{m+1}$ .

There is an integer  $u$  such that each of  $(u - 1)/2^m$  and  $(u + 1)/2^m$  is in  $E$  and each of  $h$  and  $k$  is in  $[(u - 1)/2^m, (u + 1)/2^m]$ . Hence, by using the triangle inequality and the inequality just proved, we obtain Lemma 2.

From Lemma 2 it is clear that  $T$  has a continuous extension to all of  $[0, 1]$ . If each of  $s, t$ , and  $s + t$  is in  $[0, 1]$ , choose sequences  $\{a_n\}_1^\infty$  and  $\{b_n\}_1^\infty$  in  $E$  converging to  $s$  and  $t$  respectively so that for each positive integer  $n, d_n = a_n + b_n$  is in  $E$ . By continuity of  $V$  and  $T$ , we have  $V(T(s), T(t)) = \lim_n V(T(a_n), T(b_n)) = \lim_n T(d_n) = T(s + t)$ .

Choose  $c$  positive and less than  $r$  so that  $R(c)$  is contained in  $g(R(d))$ . Let  $a = c/3M$ . If  $x$  is in  $R(a)$  then, by Lemma 1,  $T_x$  maps into  $R(c)$ . Suppose  $F$  satisfies the conclusion of theorem A for  $x$  in  $R(a)$ .  $F(1/2)$  is in  $R(c)$  and hence in  $g(R(d))$ .  $F(1/2)^2 = x$  and  $x$  is in  $R(d)$  so  $g(x) = F(1/2)$ . Similarly  $g^m(x) = F(1/2^m)$  for each positive integer  $m$ , and hence  $F$  agrees with  $T_x$  on  $E$ . Since each of  $F$  and  $T_x$  is continuous, the proof is complete.

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