Pacific Journal of Mathematics

DIFFERENTIABLE POWER-ASSOCIATIVE GROUPOIDS

JOHN P. HOLMES

Vol. 41, No. 2 December 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 2, 1972

DIFFERENTIABLE POWER-ASSOCIATIVE GROUPOIDS

JOHN P. HOLMES

Suppose H is a Banach space, D is an open set of H containing 0, and V is a function from $D \times D$ to H satisfying $V(0, x) = V(x, 0) = x$ for each x in D. If n is an integer greater than 1, denote by x^n the product of $n - x$'s associated as follows whenever the product exists.

$$x^n = V(x, V(x, \cdots V(x, x) \cdots)) .$$

Define $x^0 = 0$ and $x^1 = x$. V is said to be power associative if and only if $V(x^n, x^m) = x^{n+m}$ whenever each of n and m is a nonnegative integer and x^{n+m} exists.

THEOREM A. **If H and V are as above, V is power associative and continuously differentiable in the sense of Frechet on $D \times D$ then there are positive numbers a and c such that if x is in H and $||x|| < a$ there is a unique continuous function T_x from $[0, 1]$ to the ball of radius c centered at 0 satisfying $V(T_x(s), T_x(t)) = T_x(s + t)$ whenever each of s, t, and $s + t$ is in $[0, 1]$, $T_x(0) = 0$, and $T_x(1) = x$.**

Theorem A is similar to a result in [1] of Birkhoff. He showed that if H and V are as above, V is associative, V is Frechet differentiable on a neighborhood of (0, 0), and V' is continuous at (0, 0) then some neighborhood of 0 is covered by partial homomorphic images of the additive group of real numbers.

To see that Theorem A is not a special case of this result of Birkhoff, we offer the following example. Denote by E the 2-dimensional Euclidean space and define V from $E \times E$ to E by $V((x, y), (z, w)) = (x + [1 + (xw - yz)]z, y + [1 + (xw - yz)]w)$. If S is a 1-dimensional linear subspace of E and each of p and q is in S then $V(p, q) = p + q$. Thus V is power associative and 0 is an identity for V. V is not associative but V is continuously differentiable on $E \times E$.

We will now prove Theorem A. Regard $H \times H$ as a Banach space in the usual way, defining the norm of a member (x, y) of $H \times H$ by $||(x, y)|| = \max\{||x||, ||y||\}$. If c is a positive number, denote by $R(c)$ the set to which x belongs if and only if x is in H and $||x|| < c$. Finally, if B is a bounded linear transformation from $H \times H$ to H or from H to H, denote the norm of B by $|B|$.

Define f from D to H by $f(x) = V(x, x) = x^2$ for each x in D. Note f is continuously differentiable on D and if x is in D, $f'(x)(y) = V'(x, x)(y, y)$ for each y in H. Moreover, $V'(0, 0)(z, w) = z + w$ for each pair (z, w) in $H \times H$ so $f'(0) = 2I$ where I is the identity transformation on H.

Employing the inverse function theorem (for instance [2] page 268) we see that there is a positive number b and an open set U of H such that $(f \mid U)$ is a homeomorphism of U onto $R(b)$ and $g = (f \mid U)^{-1}$ is continuously differentiable on $R(b)$ with $g'(y) = [f'(g(y))]^{-1}$ for each y in $R(b)$. Hence $g'(0) = 1/2\, I$.

By continuity of g' and V' there is a positive number d and a number M such that if p is in $R(d) \times R(d)$ and x is in $R(d)$ then $|V'(p)| < M$ and $|g'(x)| < 2/3$.

Suppose each of x, y, z, and w is in $R(d)$. Then $\|V(x, y) - V(z, w)\| = \|\int_0^1 dt\ V'((z, w) + t(x - z, y - w))(x - z, y - w)\| < M\|(x - z, y - w)\|$. As special cases of this inequality we obtain

1. $\|V(x, y)\| < M\|(x, y)\|$ and
2. $\|V(x, y) - y\| < M\|x\|$.

Similarly, if each of x and y is in $R(d)$ we have $\|g(x) - g(y)\| = \|\int_0^1 dt\ g'(y + t(x - y))(x - y)\| < 2/3\ \|x - y\|$. Hence g is Lipschitz on $R(d)$ and has Lipschitz norm less than 2/3. In particular, for each x in $R(d)$ and each positive integer m we have $\|g^m(x)\| < (2/3)^m\|x\|$ where g^m is g composed with itself m times.

LEMMA 1. *Let $r = d/3M$. If x is in $R(r)$, m is a positive integer, and n is an integer in $[0, 2^m]$ then $[g^m(x)]^n$ exists and has norm less than $M\|x\| \sum_1^m (2/3)^i$.*

Proof. Note $|V'(0, 0)| = 2$ so $M > 3/2$. If x is in $R(r)$, it is clear, using inequality 1, that $g^1(x)^i$ exists for each $i = 0, 1,$ or 2 and has norm less than $M\|x\|(2/3)$.

Suppose m is an integer greater than 1 and assume that for each integer k in $[1, m)$ that $g^k(x)^s$ exists for each integer s in $[0, 2^k]$ and has norm less than $M\|x\| \sum_1^k (2/3)^i$.

As has been observed before, $g^m(x)$ exists and $\|g^m(x)^0\| = 0$. Suppose n is an integer in $(0, 2^m]$ and assume for each integer c in $[0, n)$ that $g^m(x)^c$ exists and has norm less than $M\|x\| \sum_1^m (2/3)^i$.

Then $g^m(x)^{n-1}$ exists and $\|g^m(x)^{n-1}\| < M\|x\| \sum_1^m (2/3)^i < 2M\|x\| < 2Mr = 2M\, d/3M < d$. Thus $g^m(x)^{n-1}$ is in D and $g^m(x)^n = V(g^m(x), g^m(x)^{n-1})$ exists.

If n is even, we may use power associativity and the equality $g^m(x)^2 = g^{m-1}(x)$ to obtain $g^m(x)^n = g^{m-1}(x)^{n/2}$. Hence, by the first inductive hypothesis, $\|g^m(x)^n\| < M\|x\| \sum_1^m (2|3)_i$.

If n is odd then $g^m(x)^n = V(g^m(x), g^{(m-1)}(x)^{(n-1)/2})$. Using the triangle inequality, inequality 2, and the first inductive hypothesis we obtain $||g^m(x)^n|| \leqq ||V(g^m(x), g^{m-1}(x)^{(n-1)/2}) - g^{m-1}(x)^{(n-1)}|| + ||g^{m-1}(x)^{(n-1)/2}|| < M||x||\sum_1^m (2/3)^i$.

Thus we have Lemma 1.

Suppose x is in $R(r)$. Denote by E the set of dyadic rational numbers in $[0, 1]$ and define T from E to H by $T(n/2^m) = g^m(x)^n$. T exists by Lemma 1 and is well defined by power associativity. Moreover, by power associativity, $V(T(h), T(k)) = T(h+k)$ whenever each of h, k, and $h+k$ is in E. Lemma 2 will show that T has a continuous extension to all of $[0, 1]$.

LEMMA 2. *If x and T are as above, each of h and k is in E, and $|h-k| < 1/2^m$ for some positive integer m then* $||T(h) - T(k)|| < 9M||x||(2/3)^{m+1}$.

Proof. Suppose $h = s/2^{m+n}$ for some nonnegative integers s and n, and u is an integer with each of $u/2^m$ and $(u+1)/2^m$ in E so that h is in $[u/2^m, (u+1)/2^m]$. There is a sequence $a_1, \cdots, a_n$ such that $h = u/2^m + a_1/2^{m+1} + \cdots + a_n/2^{m+n}$ and each a_i is in the set $\{0, 1\}$. Thus $T(h) = V(T(u/2^m), V(T(a_1/2^{m+1}), \cdots, V(T(a_{n-1}/2^{m+n-1}), T(a_n/2^{m+n})) \cdots))$.

Let w be defined from $\{0, 1, \cdots, n\}$ by $w_i = u/2^m + \sum_1^i a_j/2^{m+j}$. Then $w_i = w_{i-1} + a_i/2^{m+i}$ for each i in $\{1, \cdots, n\}$. Now, using the triangle inequality, we have $||T(h) - T(u/2^m)|| \leqq \sum_1^n ||T(w_i) - T(w_{i-1})||$. But, using inequality 2 we obtain $||T(w_i) - T(w_{i-1})|| \leqq M||T(a_i/2^{m+i})|| < M||x||(2/3)^{m+i}$. Hence $||T(h) - T(u/2^m)|| < M||x||\sum_1^m (2/3)^{m+i} < 3M||x||(2/3)^{m+1}$.

There is an integer u such that each of $(u-1)/2^m$ and $(u+1)/2^m$ is in E and each of h and k is in $[(u-1)/2^m, (u+1)/2^m]$. Hence, by using the triangle inequality and the inequality just proved, we obtain Lemma 2.

From Lemma 2 it is clear that T has a continuous extension to all of $[0, 1]$. If each of s, t, and $s+t$ is in $[0, 1]$, choose sequences $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ in E converging to s and t respectively so that for each positive integer n, $d_n = a_n + b_n$ is in E. By continuity of V and T, we have $V(T(s), T(t)) = \lim_n V(T(a_n), T(b_n)) = \lim_n T(d_n) = T(s+t)$.

Choose c positive and less than r so that $R(c)$ is contained in $g(R(d))$. Let $a = c/3M$. If x is in $R(a)$ then, by Lemma 1, T_x maps into $R(c)$. Suppose F satisfies the conclusion of theorem A for x in $R(a)$. $F(1/2)$ is in $R(c)$ and hence in $g(R(d))$. $F(1/2)^2 = x$ and x is in $R(d)$ so $g(x) = F(1/2)$. Similarly $g^m(x) = F(1/2^m)$ for each positive integer m, and hence F agrees with T_x on E. Since each of F and T_x is continuous, the proof is complete.

References

1. Garrett Birkhoff, *Analytical groups*, Trans. Amer. Math. Soc., **43** (1938), 61-101.

2. J. Dieudonne, *Foundations of Modern Analysis*, New York and London: Academic Press, 1960.

Received August 9, 1971 and in revised form December 20, 1971. This research was supported in part by an NDEA Title IV Graduate Fellowship.

EMORY UNIVERSITY
AND
UNIVERSITY OF FLORIDA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 41, No. 2   December, 1972