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## **THE EMBEDDING OF HOMEOMORPHISMS OF THE PLANE IN CONTINUOUS FLOWS**

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**A study of fundamental regions of the plane under an orientation preserving, fixed point free, self-homeomorphism of the plane is made under the conditions that there are finitely many fundamental regions  $R_i$  under  $f$ , if  $x \in R_i - \text{Int } R_i$ , then  $x \in C \subset R_i - \text{Int } R_i$  where  $C$  is a proper flowline, and if  $x_1$  and  $x_2$  are in  $\text{Int } R_i$ , then  $x_1 \sim x_2 \text{ mod } \text{Int } R_i$ . The topological structure of the fundamental regions is determined. Using these results, it is shown that in certain cases the embedding problem can be reduced to a problem of extending a continuous flow defined on an open set to the closure of the set.**

**In the last section, sufficient conditions for self-homeomorphisms of the plane and the closed unit disc with one fixed point to be embedded in continuous flows are given.**

Let  $X$  be a topological space and let  $G$  denote a topological group. The ordered triple  $(X, G, \pi)$  is a *dynamical system* if (1)  $\pi: X \times G \rightarrow X$  is continuous, (2)  $\pi(\pi(x, g_1), g_2) = \pi(x, g_1 + g_2)$ , and (3)  $\pi(x, e) = x$  for every  $x \in X$ . If  $G$  is the additive group of real numbers, then  $(X, G, \pi)$  is called a continuous flow. If  $G$  is the additive group of integers, then  $(X, G, \pi)$  is a discrete flow.

This paper is concerned with the following problem. If  $G$  is a given topological group and  $(X, G^*, \pi^*)$  is a given dynamical system, where  $G^*$  is a subgroup of  $G$  with the relative topology, then find a dynamical system  $(Y, G, \pi)$  such that (1)  $\pi(y, g^*)$  for  $g^* \in G^*$  is invariant on a subset  $Z$  of  $Y$ , (2)  $Z$  is homeomorphic to  $X$ , and (3)  $\pi$  on  $Z \times G^*$  is topologically equivalent to  $\pi^*$  on  $X \times G^*$ , i.e. there is a homeomorphism  $h: X \rightarrow Z$  such that  $h^{-1}(\pi(h(x), g^*)) = \pi^*(x, g^*)$ . If  $G$  is the additive group of real numbers and  $G^*$  is the additive group of integers, then the problem is that of embedding a self-homeomorphism in a continuous flow.

If the space  $X$  is allowed to be enlarged in order to accommodate  $\pi$ , the problem is referred to as the *unrestricted problem*. If  $X = Y = Z$  the problem is referred to as the *restricted problem* [5].

If  $G$  is the additive group of real numbers and  $G^*$  is the integers, then the unrestricted problem is easily solved [5], [7], [8] for any topological space  $X$  and any self-homeomorphism.

The restricted problem is only partially solved. Fine and Schweigert [2] and Fort [6] proved that a self-homeomorphism  $T$  of an interval can be embedded in a continuous flow if and only if it is order pre-

serving. Utz [11] gave a constructive proof of this theorem.

Foland and Utz [5] solved the restricted problem where  $T$  is an orientation preserving self-homeomorphism of a circle. Foland [3] has shown that if  $T$  is an almost periodic, orientation preserving self-homeomorphism of a closed 2-cell, then  $T$  can be embedded in a continuous flow on the 2-cell. Moreover, if  $T$  is almost periodic but not periodic, then the embedding is unique.

Andrea [1] has given sufficient conditions for a fixed point free self-homeomorphism of the Euclidean plane to be embedded in a continuous flow.

**2. Fundamental regions of the plane.** In this section numerous properties of fundamental regions of the plane will be given, and using these properties it will be possible to give a description of the plane when it consists of  $n$  fundamental regions.

In this section  $f$  will be a self-homeomorphism of the plane which is orientation preserving and which has no fixed points.

**DEFINITION 1.** [1]. If  $p$  and  $q$  are points in the plane, then  $p \sim q$  if  $p$  and  $q$  are endpoints of some curve segment  $C$  (the homeomorph of the closed unit interval) for which  $f^n(C) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ .

**THEOREM 1.** [1] *The relation  $\sim$  is an equivalence relation.*

**DEFINITION 2.** [1] The fundamental regions of the plane under  $f$  are the equivalence classes of the equivalence relation  $\sim$ .

**DEFINITION 3.** If  $A$  is a subset of the plane, then  $x \sim y \bmod A$  if  $x$  and  $y$  are endpoints of some curve segment  $C \subset A$  such that for every compact set  $K$  in the plane there is a natural number  $N$  such that  $f^n(C) \cap K = \emptyset$  when  $|n| > N$ .

**DEFINITION 4.** [1]. A proper flowline for  $f$  is a subset  $F$  of the plane such that

- (a)  $f(F) = F$ ,
- (b)  $F$  is a one-to-one continuous image of the real line, and
- (c)  $F \cup \{\infty\}$  is a Jordan curve on the sphere  $R^2 \cup \{\infty\}$ .

The following three conditions will be assumed in this section and in §3.

- (1) There exists exactly  $n$  fundamental regions  $R_i$  under  $f$ .
- (2) If  $x \in R_i - \text{Int } R_i$ , then  $x \in C \subset R_i - \text{Int } R_i$  where  $C$  is a proper flowline.
- (3) If  $x_1, x_2 \in \text{Int } R_i$ , then  $x_1 \sim x_2 \bmod \text{Int } R_i$ .

We will let  $R$  or  $R$  with a subscript denote fundamental regions of

the plane under  $f$ .

It follows from the following remark that each fundamental region is  $f$ -invariant.

REMARK 1. [1]. Suppose that  $\{E_i\}$  is a finite collection of disjoint arcwise connected sets. If each  $f(E_i) = E_j$ , then  $f(E_i) = E_i$  for every  $i$ .

REMARK 2.

- (a)  $R$  is arcwise connected,
- (b)  $R$  is unbounded, and
- (c) if  $C$  is a simple closed curve contained in  $R$ , then  $\text{Int } (C) \subset R$ .

*Proof.* (a) and (b) follow directly from the definitions. Suppose that (c) does not hold. Then there is  $x \in \text{Int } C$  such that  $x \notin R$ . Suppose  $x \in S$  where  $S$  is a fundamental region under  $f$ . Since  $f^n(x) \rightarrow \infty$ , as  $n \rightarrow \infty$ ,  $f^n(x) \notin C \cup \text{Int } C$  for some  $n$ . But  $f^n(x) \in S$  and  $S$  is arcwise connected. Therefore, there is an arc  $L$  joining  $x$  and  $f^n(x)$  where  $L \subset S$ . But that is not possible since  $L \cap C \neq \emptyset$ .

REMARK 3. If there exists  $x \in R_i - \text{Int } R_i$ , then  $x \in C$ , where  $C$  is defined by (2), and  $C$  divides the plane into two open half planes  $H_1$  and  $H_2$  such that

- (a)  $x_1, x_2 \in \text{Int } R_i$  implies  $x_1$  and  $x_2$  are in the same half plane,
- (b) if  $\text{Int } R_i \neq \emptyset$  and if  $x \in C$ , then there is a neighborhood  $U$  of  $x$  such that  $U \cap H_1 \subset \text{Int } R_i$  where  $\text{Int } R_i \subset H_1$ , and (c) if  $\text{Int } R_i = \emptyset$ , then  $R_i = C$ .

*Proof.* (a) Let  $x_1$  and  $x_2 \in \text{Int } R_i$ . Let  $x_1 \in H_1$  and  $x_2 \in H_2$ . By (3), there is an arc  $A$  joining  $x_1$  and  $x_2$  such that  $A \subset \text{Int } R_i$ . Therefore,  $A \cap C \neq \emptyset$ , which is impossible since  $C \subset R_i - \text{Int } R_i$ .

(b) Let  $x \in C$ . Let  $x_1 \in \text{Int } R_i$ . Then there is a curve  $L \subset R_i$  such that  $x_1$  and  $y$  are endpoints of  $L$ , where  $y \in C$ , and  $L \cap C = \{y\}$ . Since  $f(C) = C$  and  $f$  is fixed point free, there are integers  $m$  and  $n$  such that  $f^n(y)$  and  $f^m(y)$  are separated on  $C$  by  $x$ .

If  $f^n(L) \cap f^m(L) \neq \emptyset$ , let  $L_1$  be the smallest closed subsegment of  $f^n(L)$  that contains  $f^n(y)$  such that  $L_1 \cap f^m(L) \neq \emptyset$ . Let  $L_2$  be the smallest closed subsegment of  $f^m(L)$  that contains  $f^m(y)$  such that  $L_2 \cap L_1 \neq \emptyset$ . If  $C_1$  is the closed subsegment of  $C$  from  $f^m(y)$  to  $f^n(y)$ , then  $C_1 \cup L_1 \cup L_2$  forms a simple closed curve. Let  $\varepsilon = \min \{d(x, L_1), d(x, L_2)\}$ . Then if  $S = \{z: d(z, x) < \varepsilon\}$ ,  $S \cap H_1$  is contained in the simple closed curve  $C_1 \cup L_1 \cup L_2$ , which, by Remark 2, is contained in  $R_i$ .

There is an arc  $K \subset \text{Int } R_i$  such that  $f^n(x_1)$  and  $f^m(x_1)$  are its endpoints. If  $f^n(L) \cap f^m(L) = \emptyset$ , let  $L_1$  be the smallest closed subsegment of  $f^n(L)$  containing  $f^n(y)$  such that  $L_1 \cap K = \{z_1\}$ . Let  $L_2$  be the

smallest closed subsegment of  $f^m(L)$  containing  $f^m(y)$  such that  $L_2 \cap K = \{z_2\}$ . Let  $K_1$  be the subsegment of  $K$  such that  $z_1$  and  $z_2$  are its endpoints. Now  $C_1 \cup K_1 \cup L_1 \cup L_2$  is a simple closed curve. Let  $S = \{u: d(x, u) < \varepsilon\}$  where  $\varepsilon = \min \{d(x, K_1), d(x, L_1), d(x, L_2)\}$ . Then  $S \cap H_1$  is contained in the simple closed curve  $C_1 \cup L_1 \cup L_2 \cup K_1$  which is contained in  $R_i$ .

(c) If  $x \in R_i$ , then by (2) there is a proper flowline  $C$  such that  $x \in C$ . If  $y \in R_i - C$  then there is a proper flowline  $C'$  such that  $y \in C'$ .

If  $C \cap C' \neq \phi$ , let  $L_1^+$  be the closed half line from  $y$  containing  $f(y)$  and  $L_1^-$  the closed half line from  $y$  containing  $f^{-1}(y)$ . Let  $C_1$  and  $C_2$  be the smallest closed subsegments of  $L_1^-$  and  $L_1^+$  respectively such that  $C_i \cap C = \{x_i\}$  for  $i = 1, 2$ . Such segments do exist because  $C' \cap C \neq \phi$ ,  $y \notin C$ , and the properties of  $f$ . Thus the segment from  $x_1$  to  $x_2$  on  $C$ ,  $C_1$  and  $C_2$  form a simple closed curve and every point in its interior must be in  $R_i$ , which is not possible.

If  $C \cap C' = \phi$ , we can join  $C$  to  $C'$  by a segment  $A \subset R_i$  such that  $A \cap C = \{x_1\}$  and  $A \cap C' = \{x_2\}$ . Let  $C_i$  be the closed subsegments of  $C$  and  $C'$  respectively from  $x_i$  to  $f(x_i)$  for  $i = 1, 2$ . Let  $A_1$  be the smallest closed subsegment of  $f(A)$  such that  $A_1 \cap (A \cup C_2) \neq \phi$ . Now  $C_1$ ,  $A_1$ , and  $(A \cup C_2)$  from  $x_1$  to  $A_1 \cap (A \cup C_2)$  is a simple closed curve and the result follows.

REMARK 4. If  $C_1$  and  $C_2$  contained in  $R_i$  are curves defined as in (2) such that  $C_1 \neq C_2$ , then  $C_1 \cap C_2 = \phi$ . Also, if  $C_1$  is a curve defined as in (2), then  $R_i$  is contained in the closed half plane defined by  $C_1$  containing  $\text{Int } R_i$ .

*Proof.* Suppose  $C_1 \neq C_2$ , but  $C_1 \cap C_2 \neq \phi$ . Suppose  $y \in C_1 - C_2$ . Let  $A$  be a closed subsegment of  $C_1$  containing  $y$  such that  $x_1$  and  $x_2$  are endpoints of  $A$ ,  $x_1$  and  $x_2 \in C_1 \cap C_2$ , and  $(A - \{x_1, x_2\}) \cap C_2 = \phi$  for  $i = 1, 2$ . Now  $A$  together with the subsegment of  $C_2$  from  $x_1$  to  $x_2$  forms a simple closed curve  $K$ . Let  $z \in$  the interior of  $K$ . Now  $f^n(z) \rightarrow \infty$  as  $n \rightarrow \infty$ , thus  $f^n(z) \in$  the interior of  $f^n(K)$  but  $f^n(z) \notin$  interior  $K$  for some integer  $n$ . Now  $z$  and  $f^n(z) \in \text{Int } R_i$  by Remark 2, but  $z$  and  $f^n(z)$  can not be connected by an arc in  $\text{Int } R_i$  which is contrary to (3).

Suppose there is  $x \in R_i$  but in the opposite open half plane, determined by  $C_1$ , from  $\text{Int } R_i$ . Then by the above paragraph and (2), there is a proper flowline  $C_2$  such that  $x \in C_2$  and  $C_1 \cap C_2 = \phi$ . Now the proof of Remark 3c can be applied to show that there must be a point in the same half plane as  $x$  which is in  $\text{Int } R_i$ , which is not possible.

THEOREM 2. [1]. *The self-homeomorphism  $f$  is equivalent to a*

*translation if and only if it has exactly one fundamental region.*

REMARK 5. Let  $C$  be a proper flowline of  $f$ . Then there is a homeomorphism  $g$  such that  $g|_C = f|_C$  and such that  $g$  is topologically equivalent to a translation.

*Proof.* Since  $C$  is a proper flowline of  $f$  there is a selfhomeomorphism  $h$  of the plane where  $h(C) = L$  and where  $L$  is the  $y$ -axis. Define  $T: P \rightarrow P$ , where  $P$  is the plane, by

$$\begin{aligned} T(0, y) &= hfh^{-1}(0, y), \text{ where } (0, y) \in L \\ T(x, y) &= (x, (hfh^{-1}(0, y))_y) \text{ where } (x, y) \notin L \end{aligned}$$

and where  $(hfh^{-1}(0, y))_y$  denotes the  $y$  coordinate of  $hfh^{-1}(0, y)$ . Then  $T$  is an orientation preserving self-homeomorphism of the plane with no fixed points and with exactly one fundamental region. Therefore,  $g = h^{-1}Th$  has this property,  $g|_C = f|_C$ , and  $g$  is equivalent to a translation by Theorem 2.

REMARK 6. If  $R - \text{Int } R$  contains two distinct proper flow lines  $C_1$  and  $C_2$ , then  $R$  is the closed strip bounded by  $C_1$  and  $C_2$ .

*Proof.* By Remark 3, if  $C_1$  and  $C_2$  are distinct proper flow lines in  $R - \text{Int } R$ , then  $\text{Int } R \neq \emptyset$ . Let  $H_j^1$  and  $H_j^2$  be the two closed half planes defined by  $C_j$ . By Remark 4, it can be assumed that  $R \subset H_1^2$ . Therefore,  $C_2 \subset H_1^2 - C_1$ . Let  $H_2^1$  be the half plane containing  $H_1^1$ . Then,  $R \subset H_2^1$  and therefore  $R \subset H_2^1 \cap H_1^2$ . Let  $x_3 \in H_2^1 \cap H_1^2 - (C_1 \cup C_2)$ . Let  $B$  be a closed segment joining  $C_1$  and  $C_2$  such that  $x_3 \in B$ ,  $B \cap C_1 = \{y\}$ , and  $B \cap C_2 = \{z\}$ . Since  $y \sim z$ , there is an arc  $A \subset R$  joining  $y$  and  $z$  such that  $f^n(A) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . Let  $A_1$  be a closed subsegment of  $A$  such that  $A_1 \cap C_1 = \{y_1\}$  and  $A_1 \cap C_2 = \{z_1\}$ . There is an integer  $N > 0$  such that if  $n > N$ ,  $f^n(y_1)$  and  $f^{-n}(y_1)$  are separated on  $C_1$  by  $y$ , and  $f^n(z_1)$  and  $f^{-n}(z_1)$  are separated on  $C_2$  by  $z$ . Since  $f^n(A_1) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ , there is an  $M > N$  such that  $f^M(A_1)$ ,  $f^{-M}(A_1)$ , and the closed subsegments of  $C_1$  and  $C_2$  from  $f^M(y_1)$  to  $f^{-M}(y_1)$  and  $f^M(z_1)$  to  $f^{-M}(z_1)$  respectively form a simple closed curve which contains  $x_3$  in its interior. But that implies  $x_3 \in \text{Int } R$ .

REMARK 7. If  $x \in (R_i - \text{Int } R_i) \cap \bar{R}_k$ , then there is an arc  $A$  joining a point of  $R_k$  to  $x$ , where  $A - \{x\} \subset R_k$ . Further, if  $x \in C$ , where  $C$  is defined as in (2), then for every  $w \in C$ , there is a neighborhood  $U$  of  $w$  such that  $U \cap (H - C) \subset R_k$ , where  $H$  is the closed half plane defined by  $C$  which contains  $R_k$ .

*Proof.* Let  $x \in C_i$ , where  $C_i$  is defined by (2). Let the closed half planes defined by  $C_i$  be  $H_i^1$  and  $H_i^2$ . Suppose  $R_k \subset H_i^2$ . By Remark 6, there are at most  $2n$  distinct curves  $C_j$  defined as in (2). Let  $\alpha = \min_{C_j \subset (H_i^2 - C_i)} \{d(x, C_j)\}$ . Consider the open disc  $S$  with center at  $x$  and radius  $\alpha/2$ . Since  $x \in \bar{R}_k$ , there exists  $y \in R_k \cap S$ . Suppose  $z \in S \cap (H_i^2 - C_i)$ , where  $z \notin R_k$ . Since  $(\cup_j R_j) \cap S = S$ , there exists  $z_0 \in S \cap (H_i^2 - C_i)$  where  $z_0 \in R_{k_0} - \text{Int } R_{k_0}$ . But then there would exist a curve  $C_{k_0}$  defined by (2), where  $C_{k_0} \cap S \cap (H_i^2 - C_i) \neq \emptyset$ , which is not possible. The proof of the last part is similar to the proof of Remark 3b.

REMARK 8. If  $C$  is an arcwise connected component of  $\bar{R} - \text{Int } R$ , then  $C$  is a proper flow line contained in exactly one fundamental region.

*Proof.* Let  $x \in C$ . Then  $x \in (R_j - \text{Int } R_j) \cap \bar{R}_k$  for some fundamental regions  $R_j$  and  $R_k$ . If  $\text{Int } R_j \neq \emptyset$ , by Remarks 3 and 7, there is a neighborhood  $U$  of  $x$  such that  $U \subset R_j \cup R_k$ . Thus,  $R$  is  $R_j$  or  $R_k$ . Let  $x \in C_1$ , where  $C_1$  is given by (2). Remarks 3 and 7 imply  $C_1 \subset C$ . If  $w \in C - C_1$ , there must be an arc in  $\bar{R} - \text{Int } R$  from  $w$  to  $C_1$ , which is not possible by Remarks 3 and 7. If  $\text{Int } R_j = \emptyset$  and if  $R_j = R$  the result follows from Remark 3. Otherwise if  $x \in C$ , then  $x \in R_j \cap \bar{R}$ . Now applying Remark 7, it follows that  $C_1 = C$ .

THEOREM 3. [1] *The plane under  $f$  can not have exactly two fundamental regions.*

The notation  $\#(R)$  will be used to represent the number of arcwise connected components of  $\bar{R} - \text{Int } R$ .

REMARK 9. If  $\#(R) = 1$ , then  $R$  is a proper flow line or a closed half plane.

*Proof.* If  $\text{Int } R = \emptyset$ , then  $R$  is a proper line by Remark 2. If  $\text{Int } R \neq \emptyset$ , then by Remarks 3, 7 and 8,  $R$  is an open half plane or a closed half plane.

Suppose  $R$  is not closed. Then  $\bar{R} - \text{Int } R \subset R_2$ . Thus, there exists  $x \in (R_2 - \text{Int } R_2) \cap \bar{R}$ . Let  $x \in C$ , where  $C$  is defined as in (2). Let  $H_1^2$  and  $H_1^1$  be the closed half planes defined by  $C$ . By Remark 2, it can be assumed that  $R \subset H_1^1$ , and then, by Remarks 3 and 4,  $R_2 \subset H_1^2$ . Using Remark 5, define  $g$  such that  $g|_e = f|_e$ . Define  $F: P \rightarrow P$ , where  $P$  is the plane, by

$$F(x) = \begin{cases} f(x) & \text{if } x \in H_1^1 \\ g(x) & \text{if } x \in H_1^2. \end{cases}$$

Then  $F$  is an orientation preserving self-homeomorphism of the plane with no fixed points, and the plane under  $F$  has exactly one funda-

mental region, since, by Theorem 3, it can not have two fundamental regions. Let  $A$  be any arc joining  $y \in R$  to  $x$ , where  $A \subset H_1^+$ . Then  $f^n(A) = F^n(A) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . Therefore,  $x \in R$ , and it follows that  $R$  is closed.

REMARK 10. If  $\#(R) = 2$ , then  $R$  is either an open strip or a closed strip.

*Proof.* Let  $C_1$  and  $C_2$  be the two arcwise connected components of  $\bar{R} - \text{Int } R$ . It follows that the interior of the strip bounded by  $C_1$  and  $C_2$  is contained in  $R$ , and that  $R$  is contained in the closed strip bounded by  $C_1$  and  $C_2$ .

Suppose  $x_1 \in R - \text{Int } R$ . Then  $x_1 \in C_1$  or  $x_1 \in C_2$ . Suppose  $x_1 \in C_1$ . By Remark 8,  $C_1 \subset R$ .

Using Remark 5, define  $g_i$  where  $g_i$  agrees with  $f$  on  $C_i$ ,  $i = 1, 2$ . Define  $F: P \rightarrow P$ , where  $P$  is the plane, by

$$F(x) = \begin{cases} g_1(x) & \text{if } x \in \text{closed half plane defined by } C_1 \text{ not containing } \text{Int } R \\ f(x) & \text{if } x \in \text{closed strip bounded by } C_1 \text{ and } C_2 \\ g_2(x) & \text{if } x \in \text{closed half plane defined by } C_2 \text{ not containing } \text{Int } R. \end{cases}$$

Then  $F$  is an orientation preserving self-homeomorphism of the plane, and the plane has at most two fundamental regions under  $F$ . Therefore, by Theorem 3, it has exactly one. It now follows that  $C_2 \subset R$ .

REMARK 11. If  $\#(R) \geq 3$ , then either  $R$  is open or else  $R$  contains just one of the components of  $\bar{R} - \text{Int } R$ .

*Proof.* If  $R$  contains at least two of the components of  $\bar{R} - \text{Int } R$ , then by Remark 6,  $\#(R) = 2$ .

Consider the homeomorphism defined in Figure 1. Each point  $x$  lies on a curve. Let  $f(x)$  be in the direction of the arrow along the curve one unit. The homeomorphism  $f$  satisfies (1), (2), and (3), and  $\#(R_4) = 3$  but  $R_4$  is not open.

REMARK 12. If  $R_1$  and  $R_2$  are not separated and  $R_2$  and  $R_3$  are not separated, then  $R_1$  and  $R_3$  are separated by  $R_2$ .

*Proof.* Since  $R_1$  and  $R_2$  are not separated, either there exists  $x_2 \in (R_2 - \text{Int } R_2) \cap \bar{R}_1$ , or there exists  $x_2 \in (R_1 - \text{Int } R_1) \cap \bar{R}_2$ .

Case I. Suppose  $x_2 \in (R_2 - \text{Int } R_2) \cap \bar{R}_1$ . Let  $x_2 \in C_2$ , where  $C_2$  is defined by (2). Let  $H_2^1$  and  $H_2^2$  be the closed half planes defined by



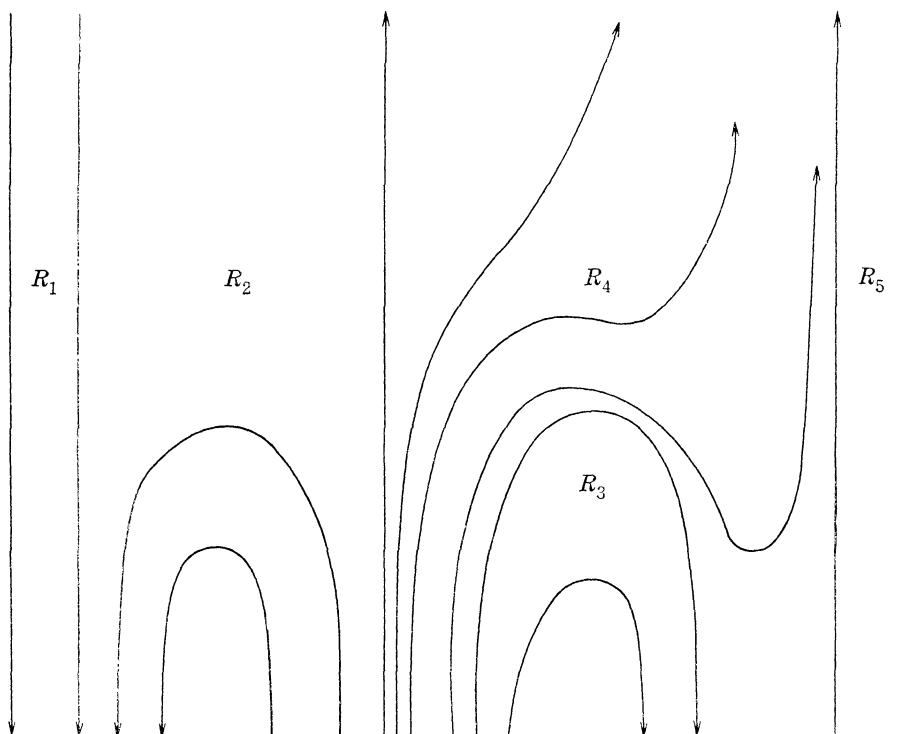


FIGURE 1

$C_2$ . Suppose for definiteness that  $R_1 \subset H_2^1 - C_2$  and then by Remark 3,  $R_2 \subset H_2^2$ . By Remark 7, for every  $w \in C_2$ , there is a neighborhood  $U$  of  $w$  such that  $U \cap (H_2^1 - C_2) \subset R_1$ . Since  $R_3$  and  $R_2$  are not separated,  $R_3 \subset H_2^2 - C_2$ . Therefore,  $R_1$  and  $R_3$  are separated by  $R_2$ .

*Case II.* The proof is similar to Case I.

**THEOREM 4.** Suppose  $R_i$  and  $R_{i+1}$  are not separated for  $i = 1, 2, \dots, n-1$ . Then

- (a)  $R_i$  and  $R_k$  are separated for  $k \neq i-1, i+1$ ,
- (b)  $R_1$  and  $R_n$  are closed half planes,
- (c)  $R_{2k-1}$  is closed,  $R_{2k}$  is open for  $k = 1, 2, \dots$ , and
- (d)  $n$  is odd.

The proof of this theorem follows easily from the above remarks.

The above remarks can be applied to describe the plane if there are  $n$  fundamental regions.

For example, suppose the plane  $P = R_1 \cup R_2 \cup R_3$ . Since  $P$  is connected, suppose  $R_1$  and  $R_2$  are not separated. Then either  $R_1$  and  $R_3$  or  $R_2$  and  $R_3$  are separated. Suppose  $R_2$  and  $R_3$  are not separated.

By Remark 12,  $R_1$  and  $R_3$  are separated by  $R_2$ . Thus, by Remark 9,  $R_1$  and  $R_3$  are closed and  $R_i - \text{Int } R_i = C_i$ , where  $C_i$  is a proper flowline, for  $i = 1, 3$ . Therefore,  $R_2$  is open.

In the case  $n = 4, 5$ , or  $6$ , there are respectively 1, 3, or 4 possible arrangements of the fundamental regions.

**3. Self-homeomorphism of the plane with no fixed points.** In this section it will be shown that in certain cases, the embedding problem can be reduced to a problem of extending a continuous flow defined on an open set to the closure of the set. Sufficient conditions for this extension will be given. Some properties of the homeomorphisms on the fundamental regions will also be given.

In order to show that an open connected subset  $B$  of the plane is homeomorphic to the plane, it is enough to show that every simple closed curve in  $B$  separates  $B$  and at least one of the components has compact closure in  $B$  [12]. Thus, Theorem 5 follows.

**THEOREM 5.** *If  $R_i$  and  $R_{i+1}$  are not separated for  $i = 1, 2, \dots, k-1$ , then  $\text{Int}(\cup_{j=1}^k R_j)$  is homeomorphic to the plane.*

*Proof.* By the earlier remarks, we can assume that  $R_1$  and  $R_k$  have nonempty interior, for otherwise  $R_1$  or  $R_k \subseteq \text{Int}(\cup_{j=1}^k R_j)$ . By (3),  $\text{Int } R_j$  is connected. Since  $R_i$  and  $R_{i+1}$  are not separated, suppose for definiteness that there exists  $x \in (R_i - \text{Int } R_i) \cap \bar{R}_{i+1}$ . By Remarks 3 and 7, there is a neighborhood  $U$  of such that  $U \subset R_{i-1} \cup R_i \cup R_{i+1}$  if  $i \neq 1$  or  $R_i \cup R_{i+1}$  if  $i = 1$ . Therefore,  $x \in \text{Int}(\cup_{j=1}^k R_j)$ . Thus,  $\text{Int}(\cup_{j=1}^k R_j)$  is connected.

Let  $B$  be a simple closed curve such that  $B \subset \text{Int}(\cup_{j=1}^k R_j)$ . Let  $x \in \text{Int } B$ . Since  $f^n(x) \rightarrow \infty$  as  $n \rightarrow \infty$ , it follows that there is an integer  $n$  such  $f^n(x) \notin B \cup \text{Int } B$ . But, since  $x \sim f^n(x)$ , there is an arc  $A$ , where  $A \subset R_i$  if  $x \in R_i$  and where  $A$  joins  $x$  and  $f^n(x)$ . Therefore,  $\{x\}$  and  $A \cap B$  are in the same fundamental region. Thus,  $B \cup \text{Int } B \subset \text{Int}(\cup_{j=1}^k R_j)$ . It follows that  $B \cup \text{Int } B$  is compact in  $\text{Int}(\cup_{j=1}^k R_j)$ . Therefore,  $\text{Int}(\cup_{j=1}^k R_j)$  is homeomorphic to the plane.

**THEOREM 6.** *Suppose  $R_i$  and  $R_{i+1}$  are not separated for  $i = 1, \dots, k-1$ . Then, there is a homeomorphism  $g: \text{Int}(\cup_{i=1}^k R_i) \rightarrow P_2$ , where  $P_2$  is a plane, and  $P_2$  under  $gfg^{-1}$  has at most  $k$  fundamental regions.*

*Proof.* The homeomorphism  $g$  exists by Theorem 5.

Let  $x_1, x_2 \in R_j \cap \text{Int}(\cup_{i=1}^k R_i)$ . If  $R_j \subset \text{Int}(\cup_{i=1}^k R_i)$ , then by definition of  $R_i$ , there is a curve segment  $L \subset \text{Int}(\cup_{i=1}^k R_i)$  joining  $x_1$  and  $x_2$  for which  $f^n(L) \rightarrow \infty$  as  $\pm n \rightarrow \infty$ . Since  $\text{Int } R_j \subset \text{Int}(\cup_{i=1}^k R_i)$ , if  $x_1, x_2 \in$

$\text{Int } R_j$ , then by (3), there is a curve segment  $L \subset \text{Int}(\cup_{i=1}^k R_i)$  for which  $f^n(L) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . If  $x_1 \in R_j - \text{Int } R_j$ , then by (2),  $x_1 \in C_1 \subset R_j - \text{Int } R_j$ . By Remarks 3 and 7,  $C_1 \subset \text{Int}(\cup_{i=1}^k R_i)$ . Let  $y \in R_j$  such that  $y \notin C_1$ . Since  $x_1 \sim y$ , there is an arc  $L_1$  joining  $x_1$  and  $y$  such that  $f^n(L_1) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . By Remark 3, there is a subarc  $L$  of  $L_1$  such that  $L$  joins  $x_1$  to an interior point of  $R_j$  and such that  $L \subset \text{Int}(\cup_{i=1}^k R_i)$ . Therefore, if  $x_1, x_2 \in R_j \cap \text{Int}(\cup_{i=1}^k R_i)$ , then there is a curve  $L \subset \text{Int}(\cup_{i=1}^k R_i)$  which  $f^n(L) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . Let  $C$  be any compact set in  $P_2$ . Then,  $g^{-1}(C)$  is a compact set in  $P_1$ . Thus, there is a natural number  $N$  such that if  $|n| > N$ , then  $f^n(L) \cap g^{-1}(C) = \phi$ . Now,  $g(L)$  is a curve segment joining  $g(x_1)$  and  $g(x_2)$  in plane  $P_2$ . Since  $f^n(L) \cap g^{-1}(C) = \phi$ ,  $gf^n g^{-1}g(L) \cap C = \phi$ . Therefore,  $g(x_1)$  and  $g(x_2)$  are in the same fundamental region.

**COROLLARY.** *If  $k = 2$ , then  $P_2$  has exactly one fundamental region under  $gf g^{-1}$ .*

The following two embedding theorems can be considered as a continuation of the work started by Andrea [1].

**THEOREM 7.** *If the plane under  $f$  has exactly one fundamental region and if  $C$  is a proper flowline, then  $f$  can be embedded in a continuous flow  $\Pi$ , such that  $\Pi(C, R) = C$ .*

*Proof.* The homeomorphism  $f$  is topologically equivalent to a translation. Hence, there is a curve  $K$  such that  $K$  separates the plane,  $f^n(K) \cap f^m(K) = \phi$  for  $n \neq m$ , if  $z$  is a point of the plane  $f^n(z)$  is in the strip bounded by  $K$  and  $f(K)$  for some unique integer  $n$ , and  $K \cap C = \{x_0\}$ . The plane,  $P_1$ , is homeomorphic to another plane  $P_2$  under a homeomorphism  $h$  such that  $h(K)$  and  $(hf)K$  are parallel straight lines and  $h(C)$  from  $h(x_0)$  to  $(hf)x_0$  is a straight line. Now if  $x \in K$  and if  $0 \leq t < 1$  define

$$\Pi_1(x, t) = y$$

where  $h(y) = (1 - t)h(x) + t(hf)x$ . If  $x \in K$  and  $t$  is any real number, define

$$\Pi_2(x, t) = f^n(\Pi_1(x, s)),$$

where  $n + s = t$ ,  $n$  is an integer, and  $0 \leq s < 1$ . If  $x$  is in the interior of the strip bounded by  $K$  and  $f(K)$  or if  $x \in K$ , and if  $t_1$  is any real number, define

$$\Pi_3(x, t_1) = \Pi_2(y, t_2 + t_1)$$

where  $h(x) = (1 - t_2)(hf)y + t_2 h(y)$  and  $y \in K$ . If  $x \in P_1$  and  $t$  is any

real number, define

$$\Pi(x, t) = \Pi_3(f^{-m}(x), t + m),$$

where  $f^{-m}(x)$  is in the strip bounded by  $K$  and  $f(K)$ . It follows that  $\Pi$  is a continuous flow and that  $\Pi(x, n) = f^n(x)$  for every  $x \in P_1$  and every integer  $n$ .

**THEOREM 8.** *Suppose  $R_1$  and  $R_i$  are not separated for  $i = 2, \dots, n$ . Then*

- (a)  $R_1$  is open
- (b)  $f|_{R_1}$  can be embedded in a continuous flow, and
- (c) if  $f|_{\bar{R}_1}$  can be embedded in a continuous flow  $\Pi_1$ , then  $f$  can be embedded in a continuous flow  $\Pi$ , where  $\Pi(x, r) = \Pi_1(x, r)$  if  $(x, r) \in \bar{R}_1 \times \mathbb{R}$ .

(d) If  $f|_{R_1}$  can be embedded in a continuous flow  $\Pi_1$ , such that  $\Pi_1$  restricted to  $R_1 \times [0, 1]$  is uniformly continuous, then  $f$  can be embedded in a continuous flow  $\Pi$ .

*Proof.* (a) By Remark 12,  $R_j$  and  $R_k$  are separated for  $j \neq k$  and  $j, k \neq 1$ . Thus by Remark 9,  $R_j$  is closed for  $j \neq 1$ . Thus,  $R_1 = \text{Plane} - (\cup_{j=2}^n R_j)$  is open.

(b) By Theorem 5, there is a homeomorphism  $g: R_1 \rightarrow P_2$  where  $P_2$  is a plane. By Theorem 6,  $P_2$  has exactly one fundamental region under  $gfg^{-1}$ . Thus,  $gfg^{-1}$  is topologically a translation and thus can be embedded in a continuous flow. Therefore  $f|_{R_1}$  can be embedded in a continuous flow.

(c) If  $x \in (R_2 - \text{Int } R_2) \cap \bar{R}_1$  then  $x \in C_2 \subset R_2 - \text{Int } R_2$ , where  $C_2$  is a proper flowline. By Remarks 3 and 7,  $C_2 \subset \text{Int } (R_1 \cup R_2)$ . It follows that  $\text{Int } (R_1 \cup R_2) = R_1 \cup R_2$ , and from the proof of Remark 9 that  $R_2 - \text{Int } R_2 = C_2$ . From Theorem 5,  $R_1 \cup R_2$  and plane  $P_2$  are homeomorphic, under a homeomorphism  $g$ . By the corollary to Theorem 6,  $P_2$  has exactly one fundamental region under  $gfg^{-1}$ . Therefore,  $gfg^{-1}$  can be embedded in a continuous flow  $\sigma$  such that  $\sigma(g(C), R) = g(C)$ , by Theorem 7. It also can be shown that we can choose  $\sigma$  such that  $\Pi_1(x, r) = g^{-1}(\sigma(g(z), r))$  for  $g(z) = x \in C$ . Now define  $\Pi'_2: (R_1 \cup R_2) \times \mathbb{R} \rightarrow (R_1 \cup R_2)$  by

$$\Pi'_2(z, r) = g^{-1}(\sigma(g(z), r)).$$

Then  $\Pi'_2$  is a continuous flow and  $\Pi'_2(z, n) = f^n(z)$ , where  $z \in R_1 \cup R_2$ . Define  $\Pi_2: (R_1 \cup R_2) \times \mathbb{R} \rightarrow (R_1 \cup R_2)$  by

$$\Pi_2(z, r) = \begin{cases} \Pi'_2(z, r) & \text{if } z \in R_2 \\ \Pi_1(z, r) & \text{if } z \in \bar{R}_1. \end{cases}$$

Then  $\Pi_2$  is a continuous flow on  $R_1 \cup R_2$  since  $\Pi'_2$  and  $\Pi_1$  agree on  $C_2$ . In the same way, define  $\Pi_i: (R_1 \cup R_i) \times R \rightarrow (R_1 \cup R_i)$  for  $i = 2, 3, \dots, n$ . Define  $\Pi: P_1 \times R \rightarrow P_1$  by

$$\Pi(z, r) = \Pi_i(z, r) \text{ if } z \in R_i \text{ for } i = 1, 2, \dots, n.$$

The fact that  $\Pi$  is a continuous flow follows from the fact that  $R_i$  and  $R_j$  are closed and separated for  $i \neq j$ ;  $i, j \neq 1$ .

(d) It is enough to show that  $f|_{\bar{R}_1}$  can be embedded in a continuous flow. Define  $\Pi: \bar{R}_1 \times \text{Reals} \rightarrow \bar{R}_1$  by

$$\Pi(x, r) = \lim_{n \rightarrow \infty} f^k(\Pi_1(x_n, s)),$$

where  $x_n \rightarrow x$  as  $n \rightarrow \infty$ ,  $x_n \in R_1$ , and  $r = k + s$ , where  $k$  is an integer and  $0 \leq s < 1$ .

Let  $(x, r)$  and  $\varepsilon > 0$  be given. Then  $r = k + s$ , where  $k$  is an integer and  $0 \leq s < 1$ . Let  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , where  $x_n \in R_1$ . By the uniform continuity of  $\Pi$ , on  $R_1 \times [0, 1]$ , there is a  $\delta > 0$  such that if  $d(x_n, x_m) < \delta$ , then  $d(\Pi_1(x_n, s), \Pi_1(x_m, s)) < \varepsilon$ . Since  $x_n \rightarrow x$ , there is a natural number  $N$  such that if  $n, m > N$ , then  $d(x_n, x_m) < \delta$ . Therefore,  $\lim_{n \rightarrow \infty} f^k \Pi_1(x_n, s) = f^k(\lim_{n \rightarrow \infty} \Pi_1(x_n, s))$  exists.

Suppose  $x_n \rightarrow x$  and  $\bar{x}_m \rightarrow x$ , where  $x_n, \bar{x}_m \in R_1$  for every  $n, m$ . Let  $\lim_{n \rightarrow \infty} \Pi_1(x_n, s) = A$ , and  $\lim_{n \rightarrow \infty} \Pi_1(\bar{x}_m, s) = B$ . By the uniform continuity of  $\Pi_1$  on  $R_1 \times [0, 1]$ , there is a  $\delta > 0$  such that if  $d(y, \bar{y}) < \delta$  then  $d(\Pi_1(y, s), \Pi_1(\bar{y}, s)) < \varepsilon/3$ . Since  $x_n \rightarrow x$  and  $\bar{x}_m \rightarrow x$  as  $n, m \rightarrow \infty$ , there is a natural number  $N$  such that if  $n, m > N$ , then  $d(x_n, x) < \delta/2$  and  $d(\bar{x}_m, x) < \delta/2$ . Therefore  $d(x_n, \bar{x}_m) \leq d(x_n, x) + d(\bar{x}_m, x) < \delta$  if  $n, m > N$ . Thus,  $d(A, B) \leq d(A, \Pi_1(x_n, s)) + d(\Pi_1(x_n, s), \Pi_1(\bar{x}_m, s)) + d(\Pi_1(\bar{x}_m, s), B) < \varepsilon$  for large  $m$  and  $n$ . Since  $r = k + s$ , where  $k$  is an integer and  $0 \leq s < 1$ , is a unique representation,  $\Pi$  is well defined.

Since  $\Pi_1(x_n, 0) = x_n$  if  $x_n \in R_1$ ,  $\Pi(x, 0) = x$  for all  $x \in \bar{R}_1$ .

Let  $x_n \rightarrow x$  as  $n \rightarrow \infty$ ,  $r_1 = k_1 + s_1$ ,  $r_2 = k_2 + s_2$ , and  $r_1 + r_2 = k_3 + s_3$ , where  $k_i$  are integers and  $0 \leq s_i < 1$  for  $i = 1, 2, 3$ , and where  $x_n \in R_1$ . Then,  $\Pi(x, r_1 + r_2) = \lim_{n \rightarrow \infty} f^{k_3}(\Pi_1(x_n, s_3)) = \lim_{n \rightarrow \infty} f^{k_2}(\Pi_1(x_n, s_3 + r_1 - s_3 + s_2)) = \lim_{n \rightarrow \infty} f^{k_2}(\Pi_1(x_n, r_1 + r_2)) = \lim_{n \rightarrow \infty} f^{k_2}(\Pi_1(x_n, k_1 + s_1 + s_2)) = \lim_{n \rightarrow \infty} f^{k_2}(\Pi_1(\Pi_1(x_n, k_1 + s_1), s_2)) = \lim_{n \rightarrow \infty} f^{k_2}(\Pi_1(f^{k_1}(\Pi_1(x_n, s_1)), s_2)) = \Pi(\Pi(x, r_1), r_2)$ .

Let  $(x, g) \in \bar{R}_1 \times [0, 1]$ . Let  $\varepsilon > 0$  be given. Choose a neighborhood  $N \times U$  of  $(x, g)$  in  $\bar{R}_1 \times [0, 1]$  so that if  $(x_1, g_1), (x_2, g_2) \in (N \times U) \cap (R_1 \times [0, 1])$  then  $d(\Pi_1(x_1, g_1), \Pi_1(x_2, g_2)) < \varepsilon$ . By the definition of  $\Pi$ , there exists  $x_1 \in R_1 \cap N$  such that  $d(\Pi_1(x_1, g), \Pi(x, g)) < \varepsilon$ . Suppose  $(\bar{x}, \bar{g}) \in N \times U$ . Then there exists  $x_2 \in R_1 \cap N$  so that  $d(\Pi_1(x_2, \bar{g}), \Pi(\bar{x}, \bar{g})) < \varepsilon$ . Thus,  $d(\Pi(x, g), \Pi(\bar{x}, \bar{g})) \leq d(\Pi(x, g), \Pi_1(x_1, g)) + d(\Pi_1(x_1, g), \Pi_1(x_1, \bar{g})) + d(\Pi_1(x_1, \bar{g}), \Pi_1(x_2, \bar{g})) + d(\Pi_1(x_2, \bar{g}), \Pi(\bar{x}, \bar{g})) < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$ . Thus  $\Pi$  is continuous on  $\bar{R}_1 \times [0, 1]$ . Therefore, by continuity of  $f^n$  for every

integer  $n$ ,  $\Pi$  is continuous on  $\bar{R}_1 \times \text{Reals}$ .

It follows from Theorem 8 and the remarks §2, that if the plane has either three or four fundamental regions under  $f$ , then the embedding problem is reduced to the problem of embedding  $f|_{R_1}$  in a flow that can be extended to  $\bar{R}_1$ .

REMARK 13. If  $B \subset R_k$  is a proper flowline and  $B$  separates  $R_j$  and  $R_i$ ,  $i \neq k \neq j$ , then each closed half plane defined by  $B$  contains at least two fundamental regions other than  $R_k$ .

*Proof.* Use Remark 5 to define a homeomorphism  $g$  such that  $g(x) = f(x)$  if  $x \in B$  and such that  $g$  is topologically equivalent to a translation. Let  $H^1$  be the half plane defined by  $B$  that contains  $R_j$ , and let  $H^2$  be the half plane defined by  $B$  that contains  $R_i$ . Define  $F: P_1 \rightarrow P_1$ , where  $P_1$  is the plane, by

$$F(x) = \begin{cases} g(x) & \text{if } x \in H^1 \\ f(x) & \text{if } x \in H^2. \end{cases}$$

Suppose that  $R_i$  is the only fundamental region other than  $R_k$  that has a point in  $H^2$ . Let  $z \in H^1 - B$ , and let  $x \in B$ . Join  $z$  and  $x$  by a curve  $J \subset H^1$ . Then  $F^n(J) = g^n(J) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . Therefore, the plane has at most two fundamental regions under  $F$ , and thus it has exactly one. Now join  $y \in R_i$  to  $x \in B$  by a curve  $K \subset H^2$ . Then  $f^n(K) = F^n(K) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . Thus,  $x$  and  $y$  are in the same fundamental region under  $f$ .

COROLLARY. If  $n = 3$  or  $4$ , then  $B$  does not separate any two fundamental regions if neither contains  $B$ .

REMARK 14. Suppose there are exactly three fundamental regions under  $f$ , and  $R_i$  and  $R_{i+1}$  are not separated for  $n = 1, 2$ . If  $y \in R_2$ ,  $z \in R_1$ , and  $U$  is any open set such that  $R_3 \subset U$ , then for every arc  $C$  connecting  $y$  and  $z$ , there is an integer  $N$  such that  $f^N(C) \cap U \neq \emptyset$ .

*Proof.* By Theorem 4,  $R_2$  separates  $R_1$  and  $R_3$ , and  $R_2$  is open. Thus,  $\text{Int}(R_1 \cup R_2) = R_1 \cup R_2$ . By Theorem 5,  $R_1 \cup R_2$  is homeomorphic to the plane. Suppose  $g: R_1 \cup R_2 \rightarrow P_2$ , where  $P_2$  is the plane and  $g$  is a homeomorphism. By Theorem 6,  $P_2$  has exactly one fundamental region under  $gf g^{-1}$ . Suppose that for some  $y \in R_2$ ,  $z \in R_1$ , and for some open set  $U \supset R_3$ , there is an arc  $C$  connecting  $y$  and  $z$  such that  $f^n(C) \cap U = \emptyset$  for every integer  $n$ . Let  $K$  be any compact set in  $P_1$ . Then  $(P_1 - U) \cap K$  is compact. Since  $f^n(C) \cap U = \emptyset$  for every integer  $n$ , it follows that  $f^n(C) \cap (P_1 - U) \cap K = \emptyset$  implies  $f^n(C) \cap K = \emptyset$ . Now,  $gf^n g^{-1}(g(C)) \rightarrow \infty$  as  $n \rightarrow \pm \infty$ . Since  $(P_1 - U) \cap K = K_1 \subset R_1 \cup R_2$ ,

$g(K_1)$  is compact in  $P_2$ . Thus, there is a natural number  $N$  such that if  $|n| > N$ , then  $gf^n(C) \cap g(K_1) = \phi$ . But that implies  $f^n(C) \cap K_1 = \phi$ . Therefore,  $y$  and  $z$  are in the same fundamental region, which is a contradiction.

REMARK 15. Suppose  $C_j \subset R_1$ , where  $C_j$  is a proper flowline for  $j = 1, 2$ ;  $H_j^k$ ,  $k = 1, 2$ , are the two closed half planes defined by  $C_j$ ; and  $C_1 \subset \text{Int } H_2^1$  and  $C_2 \subset \text{Int } H_1^2$ . Then  $H_2^1 \cap H_1^2 \subset R_1$ .

*Proof.* The proof is similar to the proof of Remark 6.

REMARK 16. Suppose  $C_i \subset R_2$ , where  $C_i$  is a proper flowline for  $i = 1, 2$ . If  $C_1$  separates  $R_1$  and  $R_3$ , then  $C_2$  separates  $R_1$  and  $R_3$ .

*Proof.* Suppose  $C_1$  separates  $R_1$  and  $R_3$ , but  $C_2$  does not. Let  $H_i^k$ ,  $k = 1, 2$ , be the closed half plane defined by  $C_i$ . Suppose  $R_1 \subset H_2^1$  and  $R_3 \subset H_1^2$ . Since  $C_2$  does not separate  $R_1$  and  $R_3$ , and since each  $R_i$  is arcwise connected, suppose  $R_1$  and  $R_3$  are contained in  $H_2^1$ . If  $C_1 \cap C_2 = \phi$ , suppose  $C_2 \subset H_1^2$ . Then  $R_3 \subset H_2^1 \cap H_1^2$ . By Remark 15,  $R_3 \subset H_2^1 \cap H_1^2 \subset R_2$ . If  $C_2 \subset H_1^1$ , then  $R_1 \subset H_1^1 \cap H_2^1$ . By Remark 15,  $R_1 \subset H_1^1 \cap H_2^1 \subset R_2$ . If  $C_1 \cap C_2 \neq \phi$ , let  $y \in C_1 \cap C_2$ . Then  $f^n(y) \in C_1 \cap C_2$  for every integer  $n$ . In this case there is a point  $x \in R_1 \cup R_3$ , where  $x$  is in the region bounded by  $C_1$  and  $C_2$ . Since that is not possible,  $C_2$  separates  $R_1$  and  $R_3$ .

The following Remark easily follows from the above results.

REMARK 17. Suppose  $R_i$  and  $R_{i+1}$  are not separated for  $i = 1, 2$ . Then, there is a homeomorphism  $g: \text{Int}(\cup_{i=1}^3 R_i) \rightarrow P_2$ , where  $P_2$  is a plane. If  $P_2$  has exactly three fundamental regions under  $gfg^{-1}$  if  $y \in R_2 \cap \text{Int}(\cup_{i=1}^3 R_i)$ , if  $z \in R_1 \cap \text{Int}(\cup_{i=1}^3 R_i)$ , and if  $U$  is any open set such that  $R_3 \subset U$ , then for every arc  $B \subset \text{Int}(\cup_{i=1}^3 R_i)$  connection  $y$  and  $z$  there is an integer  $N$  such that  $f^N(B) \cap U \neq \phi$ .

The above results gives us some information about the possible behavior of a self-homeomorphism of the plane with  $n$  fundamental regions. Consider, for example, the case where the plane  $P_1 = \cup_{i=1}^4 R_i$ , where  $R_1$  and  $R_i$  are not separated for  $i = 2, 3, 4$ . By Corollary to Remark 13,  $R_1$  does not contain a proper flowline  $B$  such that  $B$  separates  $R_j$  and  $R_k$  for  $j \neq k$ ,  $j, k \neq 1$ . By Theorem 5,  $\text{Int}(R_1 \cup R_2 \cup R_i) = R_1 \cup R_2 \cup R_i$  is homeomorphic to the plane for  $i = 3, 4$ . By Remark 17, if  $y \in R_2 - \text{Int } R_2$ , if  $U$  is any neighborhood of  $y$ , and if  $V_i$  is an open set such that  $V_i \supset R_i$ , then there is an integer  $N$  such that  $f^N(U) \cap V_i \neq \phi$ .

4. Self-homeomorphisms of the plane and the closed disc with one fixed point. In this section sufficient conditions to embed self-homeomorphisms of the plane and the closed disc will be given.

**THEOREM 9.** *Let  $f$  be an orientation preserving self-homeomorphism of the plane with one fixed point,  $x_0$ . Suppose  $f^n(x) \rightarrow \infty$  as  $n \rightarrow -\infty$  and  $f^n(x) \rightarrow x_0$  as  $n \rightarrow \infty$  for  $x \neq x_0$ . Then  $f$  can be embedded in a continuous flow.*

*Proof.* The proof is an immediate consequence of the theorem by Homma and Kinoshita [10] that says that such a homeomorphism is topologically equivalent to  $x' = (1/2)x$ ,  $y' = (1/2)y$ .

We conclude this paper with the following embedding theorem.

**THEOREM 10.** *Let  $f$  be an orientation preserving self-homeomorphism of the closed unit disc  $D$ . Suppose*

(1) *if  $x \in D$ , then  $f^{\pm n}(x) \rightarrow x_0$  as  $n \rightarrow \infty$ , where  $x_0 \in D - \text{Int } D$  and  $f(x_0) = x_0$ , and*

(2) *if  $x_1, x_2 \in D - \{x_0\}$ , there is an arc  $A \subset D$  such that  $A$  joins  $x_1$  and  $x_2$  and  $f^{\pm n}(A) \rightarrow x_0$  as  $n \rightarrow \infty$ .*

*Then  $f$  can be embedded in a continuous flow.*

*Proof.* Let  $L$  be a straight line tangent to the circle  $C = D - \text{Int } D$  at  $x_0$ . Let  $L_1$  be the ray along  $L$  from  $x_0$ . The subspace  $P_1 - L_1$  is homeomorphic to a plane  $P_2$ . Suppose  $g$  is a homeomorphism of  $P_1 - L_1$  onto  $P_2$  sending  $C - \{x_0\}$  onto the  $y$ -axis of plane  $P_2$ . If  $x \in \text{Int } D$ , let  $H_1$  be the closed half plane defined by the  $y$ -axis of plane  $P_2$  containing  $g(x)$ . It follows that  $g^{-1}(H_1) = D - \{x_0\}$ . Now define  $F: P_2 \rightarrow P_2$  by

$$\begin{aligned} F(x, y) &= (x, (gf g^{-1}(0, y))_y) && \text{if } (x, y) \in P_2 - H_1 \\ &= gf g^{-1}(x, y) && \text{if } (x, y) \in H_1. \end{aligned}$$

$F$  is orientation preserving self-homeomorphism of plane  $P_2$  with no fixed points. If  $z \in P_2 - H_1$ , then clearly  $z$  is in the same fundamental region as the  $y$ -axis of  $P_2$ . Let  $x_1, x_2 \in H_1$  and let  $K$  be any compact set in  $P_2$ . Since  $g^{-1}(K)$  is a compact set in  $P_1$  and  $x_0 \notin g^{-1}(K)$ , it follows that there is an arc  $A$  joining  $g^{-1}(x_1)$  and  $g^{-1}(x_2)$  such that  $f^{\pm n}(A) \cap g^{-1}(K) = \emptyset$  if  $n > N$  for some  $N$ . But that implies  $F^{\pm n}(g(A)) \cap K = \emptyset$  if  $n > N$ . Thus  $P_2$  has exactly one fundamental region under  $F$ , and consequently can be embedded in a flow  $\Pi$  which leaves the  $y$ -axis of  $P_2$  invariant by Theorem 7. It follows that  $\Pi(H_1, r) = H$  for every real number  $r$ . Thus,  $f$  is restricted to  $D - \{x_0\}$  can be embedded in a flow  $\sigma_1$ . Now define  $\sigma: D \times R \rightarrow D$  by



$$\begin{aligned}\sigma(d, r) &= \sigma_1(d, r) && \text{if } d \in D - \{x_0\}, r \in R, \\ \sigma(x_0, r) &= x_0 && \text{if } r \in R.\end{aligned}$$

$\sigma$  is a continuous flow if it is continuous at  $x_0$ . Let  $U$  be an open neighborhood of  $x_0$  in  $D$ . Then  $D - U$  is compact in  $P_1$ . Thus  $g(D - U)$  is compact in  $P_2$ . There exists a compact set  $K \subset P_2$  such that if  $x \in P_2 - K$  and if  $|t - t_0| < 1$  then  $\Pi(x, t) \in P_2 - g(D - U)$ . From that it follows that if  $x \in [g^{-1}(P_2) - g^{-1}(K)] \cap D$  and if  $|r - r_0| < 1$ , then  $\sigma(x, r) \in U$ . Thus,  $\sigma$  is continuous at  $(x, r_0)$ .

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# Pacific Journal of Mathematics

Vol. 41, No. 2

December, 1972

Tom M. (Mike) Apostol, <i>Arithmetical properties of generalized Ramanujan sums</i> .....	281
David Lee Armacost and William Louis Armacost, <i>On <math>p</math>-thetic groups</i> .....	295
Janet E. Mills, <i>Regular semigroups which are extensions of groups</i> .....	303
Gregory Frank Bachelis, <i>Homomorphisms of Banach algebras with minimal ideals</i> .....	307
John Allen Beachy, <i>A generalization of injectivity</i> .....	313
David Geoffrey Cantor, <i>On arithmetic properties of the Taylor series of rational functions. II</i> .....	329
Václáv Chvátal and Frank Harary, <i>Generalized Ramsey theory for graphs. III. Small off-diagonal numbers</i> .....	335
Frank Rimi DeMeyer, <i>Irreducible characters and solvability of finite groups</i> ....	347
Robert P. Dickinson, <i>On right zero unions of commutative semigroups</i> .....	355
John Dustin Donald, <i>Non-openness and non-equidimensionality in algebraic quotients</i> .....	365
John D. Donaldson and Qazi Ibadur Rahman, <i>Inequalities for polynomials with a prescribed zero</i> .....	375
Robert E. Hall, <i>The translational hull of an <math>N</math>-semigroup</i> .....	379
John P. Holmes, <i>Differentiable power-associative groupoids</i> .....	391
Steven Kenyon Ingram, <i>Continuous dependence on parameters and boundary data for nonlinear two-point boundary value problems</i> .....	395
Robert Clarke James, <i>Super-reflexive spaces with bases</i> .....	409
Gary Douglas Jones, <i>The embedding of homeomorphisms of the plane in continuous flows</i> .....	421
Mary Joel Jordan, <i>Period <math>H</math>-semigroups and <math>t</math>-semisimple periodic <math>H</math>-semigroups</i> .....	437
Ronald Allen Knight, <i>Dynamical systems of characteristic 0</i> .....	447
Kwangil Koh, <i>On a representation of a strongly harmonic ring by sheaves</i> .....	459
Hui-Hsiung Kuo, <i>Stochastic integrals in abstract Wiener space</i> .....	469
Thomas Graham McLaughlin, <i>Supersimple sets and the problem of extending a retracing function</i> .....	485
William Nathan, <i>Open mappings on 2-manifolds</i> .....	495
M. J. O'Malley, <i>Isomorphic power series rings</i> .....	503
Sean B. O'Reilly, <i>Completely adequate neighborhood systems and metrization</i> .....	513
Qazi Ibadur Rahman, <i>On the zeros of a polynomial and its derivative</i> .....	525
Russell Daniel Rupp, Jr., <i>The Weierstrass excess function</i> .....	529
Hugo Teufel, <i>A note on second order differential inequalities and functional differential equations</i> .....	537
M. J. Wicks, <i>A general solution of binary homogeneous equations over free groups</i> .....	543