PERIOD $H$-SEMIGROUPS AND $t$-SEMISIMPLE PERIODIC $H$-SEMIGROUPS

MARY JOEL JORDAN
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MARY JOEL JORDAN, S. C.

An $H$-semigroup is a semigroup such that every right and every left congruence is a two-sided congruence on the semigroup. It is known that the set of idempotents of an $H$-semigroup form a subsemigroup. A semigroup is $t$-semisimple provided the intersection of all its maximal modular congruences is the identity relation. Let $S$ be a periodic $H$-semigroup such that the subsemigroup $E$ of idempotents of $S$ is commutative. In this paper it is shown that $S$ is a semilattice of disjoint one-idempotent $H$-semigroups, and that every subgroup of $S$ is a Hamiltonian group. Moreover, if $S$ is $t$-semisimple, then $S$ is an inverse semigroup such that the one-idempotent $H$-semigroups of the semilattice are the maximal subgroups of $S$, and a complete characterization is given.

If $\sigma$ is an equivalence relation on a semigroup $S$ and $a$ is equivalent to $b$, then we shall write $a \sigma b$. The $\sigma$-class containing $a$ will be denoted by $\sigma a$. An equivalence relation $\sigma$ on a semigroup $S$ is a right (left) congruence if $a, b \in S$ and $a \sigma b$ imply $(ac) \sigma (bc)$ and $(ca) \sigma (cb)$. If an equivalence relation is both a right and a left congruence, we shall call it a two-sided congruence, or, more briefly, a congruence. We use the natural partial ordering on relations and say that $\sigma \leq \rho$ if and only if $a, b \in S$ and $a \sigma b$ imply $a \rho b$. Clearly the identity relation $\iota$ and the universal relation $\nu$ are congruences and $\iota \leq \sigma \leq \nu$, for each congruence $\sigma$ on $S$. A congruence $\sigma = \nu$ is called maximal if, for each congruence $\sigma'$ on $S$ such that $\sigma \leq \sigma' \leq \nu$, either $\sigma = \sigma'$ or $\sigma' = \nu$. A congruence $\sigma$ on $S$ is called modular if there is an element $e$ of $S$ such that $(ea) \sigma a$ and $(ae) \sigma a$ for all $a$ in $S$. The element $e$ is called an identity for $\sigma$. The intersection of all the maximal modular congruences on $S$ is called the $t$-radical of $S$ [4] and it will be denoted by $\tau$.

1. Preliminary definitions and results. In his initial paper on $H$-semigroups, Oehmke [3] obtained several useful results. For reference we summarize those results which are essential to this work. The set $E$ of idempotents of an $H$-semigroup $S$ forms a subsemigroup. For each $a \in E$, the subset $R_a$ of $E$ is the set of all $b \in E$ such that $ab = b$ and $ba = a$. Similarly, the set $L_a$ of $E$ is the set of all $b \in E$ such that $ba = b$ and $ab = a$. The collection of all $R_a(L_a)$ induces a decomposition of $E$ and the corresponding equivalence
relation is a right (left) congruence. The set of all $W_a$, where $W_a = L_aR_a$, $a \in E$, is a semilattice where the commutative multiplication operation (denoted by $\cdot$) is defined as $W_a \cdot W_b = W_{ab}$, and where the partial ordering relation is defined by $W_a \leq W_b$ if and only if $W_a \cdot W_b = W_a$. If there is a minimal $W_a$ in the set, then it is unique. It follows that either $W_a = L_a$ or $W_a = R_a$ and, for all $a \in E$, either $W_a$ is trivial, that is, $W_a = \{a\}$, or $W_a$ is minimal. If $W_a$ is minimal and $W_a = R_a$, then $R_a c = \{ac\}$, for all $c \in S$. If $W_a$ is minimal and $W_a = L_a$, then for any $c$ in $S$ we have $cL_a = \{ca\}$. If there is no minimal $W_a$, then each $W_a$ contains a single element. It then follows that $E$ is commutative. These results yield the following theorem.

**Theorem 1.** Let $W_a$ be minimal and $W_a = \{x_i; i \in I\}$. Then $S = \bigcup\{S_i; i \in I\}$ where the $S_i$ are disjoint $H$-subsemigroups of $S$. If $R_a = W_a$ then $S_iS_j = \{x_i\}$, for $i \neq j$, and $S_i$ is the set of all $b$ such that $R_b = \{x_i\}$. If $L_a = W_a$ then $S_iS_j = \{x_i\}$, for $i \neq j$, and $S_i$ is the set of all $b$ such that $bL_a = \{x_j\}$. For any $i$, the set $E_i$ of idempotents of $S_i$ is a commutative subsemigroup [3].

By Theorem 1, we can reduce the study of $H$-semigroups to the study of those $H$-semigroups in which the idempotents form a commutative subsemigroup.

An element $b$ of a semigroup $S$ is an inverse of an element $a$ of $S$ provided $aba = a$ and $bab = b$. Then $e = ab$ is an idempotent of $S$ such that $ea = a$, and $f = ba$ is an idempotent of $S$ such that $af = a$. $S$ is an inverse semigroup provided every element of $S$ has a unique inverse. The inverse of an element $a$ of an inverse semigroup $S$ will be denoted by $a^{-1}$ so that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

A left (right) zero of a semigroup $S$ is an element $a$ of $S$ such that $as = a$ ($sa = a$), for each $s \in S$.

An element $a$ of a semigroup $S$ is regular provided $a \in aSa$. Then $a$ has at least one inverse in $S$, namely $bab$, where $aba = a$.

All of the definitions following Theorem 1 are taken from [1].

Let $T$ be the set of regular elements of an $H$-semigroup $S$. Let $a, b \in T$. Then there exist $s_1, s_2$ in $S$ such that $a = as_1a$, where $as_1, s_2a \in E$, and $b = bs_2b$, where $bs_2, s_2b \in E$. We assume that $E$ is a semilattice, that is, $E$ is a commutative idempotent semigroup with the induced ordering given by $e \leq f$ if and only if $ef = e$. Then

$$ab = a(s_1a)(bs_2)b = a(bs_2)(s_1a)b = ab(s_1s_2)ab.$$ 

Hence $ab \in T$ and $T$ is a subsemigroup of $S$. Since $s_1as_1$ is an inverse of $a$ in $S$, then $s_1as_1$ is in $T$ and $a \in aTa$. Hence $T$ is a regular
semigroup. It follows that $T$ is an inverse semigroup \cite[p. 28]{1}. Thus $T$ is an inverse subsemigroup of $S$. Let $c$ be a left zero of $S$. Then $c \in T$ and $c^{-1} = c$. Let $s \in S$. Then $cscc = c$ and $scsc = sc$ imply $sc \in T$ and $c^{-1} = sc$. Hence $sc = c$. Since $s$ was arbitrary in $S$, then $c$ is a right zero of $S$. Analogously, if $c$ is a right zero of $S$, then $c$ is a left zero of $S$. Hence $S$ has at most one (left, right) zero.

If $S$ is an $H$-semigroup and $I$ is a right (left) ideal of $S$, then for $b \in S$, $bI \subseteq I(\mu \subseteq I)$ or $bI = \{e\}$, where $e$ is a left zero ($\mu = \{e\}$, where $e$ is a right zero) \cite{3}. Using this, we get that a right (left) ideal of an $H$-semigroup $S$ such that $E$ is commutative is a two-sided ideal, and it follows that, for each $e$ in $E$, for each $a$ in $S$, $ea = a$ if and only if $ae = a$.

**THEOREM 2.** Let $S$ be an $H$-semigroup such that the subsemigroup $E$ of idempotents of $S$ is a semilattice. Then the set $T$ of regular elements of $S$ is an inverse semigroup which is a semilattice of disjoint groups.

Proof. Let $a \in T$. Then there exists a unique element $a^{-1}$ in $T$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Since $aa^{-1}, a^{-1}a \in E$, we have $a(aa^{-1}) = a$ and $(a^{-1}a)a = a$. Hence

$$a^{-1}a = a^{-1}(aa^{-1}) = (a^{-1}a)a^{-1} = aa^{-1}. $$

It follows that $T$ is a union of disjoint groups \cite[ex. 10, p. 34]{1}. Let $G_e = \{b \in T: bb^{-1} = e\}$. Then $G_e$ is a maximal subgroup of $T$ and $T = \bigcup \{G_e: e \in E\}$, where $G_e \cap G_f = \emptyset$ for $e \neq f$. As in \cite{2}, we get that $T$ is a semilattice of disjoint groups.

2. For the remainder of this work, unless otherwise indicated, we assume not only that $S$ is an $H$-semigroup such that the subsemigroup $E$ of idempotents of $S$ is a semilattice, but also that $S$ is a periodic semigroup \cite[p. 20]{1}. Let $P_e = \{s \in S: s^n = e \text{ for some positive integer } n\}$. Let $T$ be the inverse subsemigroup of regular elements of $S$. Clearly $P_e \cap T = G_e \subseteq P_e$. Let $P_e - G_e = W_e$ and let $a \in W_e$, where $a^n = e$. Then

$$(ae)^n = (a^{n+1})^n = (a^n)^{n+1} = e \implies ae \in P_e,$$

and

$$ae(ae)^{n-1}ae = (ae)(ae)^n = ae^2 = ae \implies ae \in T.$$  

Hence, $ae = aa^n = a^n a = ea \in G_e$ and, for each $b$ in $G_e$, $ab = aeb \in G_e$ and $ba = bea \in G_e$, so that $G_e$ is an ideal in $P_e$. Let $T_e = \bigcup \{P_f: e \leq f\}$.  

LEMMA 3.1. ae ∈ G_e ⇐⇒ a ∈ T_e.

Proof. Let a ∈ T_e. Then there exists f ≥ e such that a ∈ P_f and af ∈ G_f. Hence af e ∈ G_{f_e}, that is, ae ∈ G_e. Conversely, if ae ∈ G_e, then there exists b ∈ G_e such that aeb = ab = e. Say a ∈ P_f, where a" = f. Then fb^n ∈ G_{f_e} and

\[ fb^n = a^n b^n = a^{n-1} a b^{n-1} = a^{n-1} e b^{n-1} = a^{n-1} b^{n-1} = \cdots = a b b = a b b = ab = e. \]

Thus fb^n ∈ G_{f_e} ∩ G_e. But this implies ef = e so that e ≤ f_e. Hence a ∈ T_e.

LEMMA 3.2. For each e in E, T_e is a subsemigroup of S, and if a ∈ T_e and there exists b ∈ S such that ab ∈ T_e, then b ∈ T_e.

Proof. Let a, b ∈ T_e, say a ∈ P_f and b ∈ P_h, where e ≤ f, h. Then af ∈ G_f and bh ∈ G_h imply that af bh = abf h ∈ G_{f h} so that ab ∈ T_{f h}. Now ef = e and eh = e imply that ef h = e so that e ≤ f h. Hence ab ∈ T_e and T_e is a subsemigroup of S. Let S − T_e = T'_e and suppose e is not minimum so that T'_e ≠ ∅. Let a ∈ T_e and suppose there exists b ∈ S such that ab ∈ T_e. Assume b ∈ T_e. Then abe ∈ G_e and be ∈ G_e imply abe(be)^{-1} = ae is in G_e so that a ∈ T_e, contradiction.

LEMMA 3.3. For each f in E, T_f is an H-semigroup of S, and if f is not minimum in E, then T'_f ≠ ∅ and T'_f is an ideal of S.

Proof. Let f ∈ E. Let U_a = \{b ∈ S : xb ∈ T_f\}. Define σ on S by

\[ aσb ⇐⇒ U_a = U_b. \]

Clearly σ is a (right) congruence on S. Let a, b ∈ T_f. Then, using Lemma 3.2, we have

\[ x ∈ U_a ⇐⇒ ax ∈ T_f ⇐⇒ x ∈ T_f ⇐⇒ bx ∈ T_f ⇐⇒ x ∈ U_b. \]

Thus U_a = U_b and aσb. Further, if aσb and a ∈ T_f, then, for each x in T_f, x ∈ U_a = U_b. In particular, a ∈ U_b so that ba ∈ T_f and, using Lemma 3.2, b ∈ T_f. Thus T_f is an equivalence class of σ. Since f ∈ U_f, U_f ≠ ∅. Let a ∈ S.

\[ x ∈ U_a ⇐⇒ ax ∈ T_f ⇐⇒ fax ∈ T_f ⇐⇒ x ∈ U_{ax}. \]

Then U_a = U_{af} and (fa)σa, for each a in S. Let x ∈ U_{af}. Then afx ∈ T_f. Now (fx)σx implies (afx)σ(ax), so that ax ∈ T_f and x ∈ U_a.
Then \( U_{af} \subseteq U_a \). Let \( x \in U_a \). Then \( ax \in T_f \) and \((fax)(ax)\). As before, \((fx)\sigma x \) implies \((afx)\sigma (ax)\). Hence, \((fax)\sigma (afx)\) implies \( afx \in T_f \) so that \( x \in U_{af} \). Then \( U_a \subseteq U_{af} \) and \((af)\sigma a\), for each \( a \) in \( S \). Therefore \( f \) is an identity for \( \sigma \) and \( \sigma \) is modular. Let \( \rho \) be any congruence on \( S \) such that \( T_f \) is an equivalence class of \( \rho \) and assume \( \sigma < \rho \). Then there exist \( a, b \) in \( S \) such that \( a\rho b \) and \( a\phi b \), that is, there exists \( x \in U_a \) such that \( x \in U_b \), which implies that \( ax \in T_f \) and \( bx \in T_f \). But \( a\rho b \) implies \((ax)\rho (bx)\) so that \( bx \in T_f \), contradiction. Therefore, \( \sigma = \rho \) and \( \sigma \) is maximal with respect to having \( T_f \) as a \( \sigma \)-class. Let \( a \in T'_f \) and assume \( x \in U_a \). Then \( ax \in T_f \). Thus we have

\[
(ax)\sigma f \rightarrow (a^2x)\sigma (af)\sigma a \rightarrow (a^2x)\sigma (ax) \\

\rightarrow (a^3x)\sigma f \rightarrow (a^3x)\sigma (af)\sigma a \rightarrow (a^3x)\sigma (ax) \\

\rightarrow (a^4x)\sigma f \rightarrow \cdots \\

\rightarrow (a^nx)\sigma f, \text{ for each positive integer } n.
\]

Let \( a^i = h \), where \( h \in T_f \). Since \( ax \in T'_f \), then \( x \in T'_f \). Let \( x^i = k \), where \( k \in T_f \). Then we have

\[
(a^ix^i)\sigma f \rightarrow (hk)\sigma f \rightarrow hk \in T_f.
\]

But \( h, k \in T_f \) implies \( hk \in T_f \), contradiction. Hence, for each \( a \in T'_f \), \( U_a = \emptyset \). It follows that \( T'_f \) is a \( \sigma \)-class and \( T'_f \) is an ideal of \( S \). Let \( \rho \) be any right congruence on \( T_f \). Define \( \rho' \) on \( S \) by

\[
\rho' (a, b) = a, b \in T_f \text{ and } a\rho b \text{ or } a, b \in T'_f.
\]

Clearly \( \rho' \) is a congruence on \( S \) and the restriction of \( \rho' \) to \( T_f \) is \( \rho \). Thus \( \rho \) is a left congruence on \( T_f \). By analogous proof, any left congruence on \( T_f \) is a right congruence. Thus \( T_f \) is an \( H \)-semigroup of \( S \).

With the preceding lemmas, we are now in a position to prove the main results of this section.

**Theorem 3.** If \( S \) is a periodic \( H \)-semigroup such that the subsemigroup \( E \) of idempotents of \( S \) is commutative, then \( S \) is a semilattice of disjoint one-idempotent \( H \)-semigroups. Moreover, every subgroup of \( S \) is a Hamiltonian group.

**Proof.** First we show that for each \( e \) in \( E \), \( G_e \) is a Hamiltonian group. If \( e = 0 \), then \( G_e \) is trivially Hamiltonian. Assume \( e \neq 0 \). Let \( \sigma \) be a right congruence on \( G_e \), let \( H_e \) be the subgroup of \( G_e \) induced by \( \sigma \) and let \( a, b \in T_f \). Write
By a straight-forward argument, $\sigma^{(e)}$ is an equivalence relation on $T_e$, so we need only show right compatibility. Accordingly, assume $a\sigma^{(e)}b$ and $c \in T_e$. Then $(ea)\sigma(eb)$ and $ec \in G_e$ imply $(eae)e\sigma(ceb)$ so that $(eae)e\sigma(ceb)$ and $(ae)e\sigma^{(e)}(bc)$. Clearly, $\sigma^{(e)}$ restricted to $G_e$ is $\sigma$. Since $T_e$ is an $H$-semigroup, then $\sigma^{(e)}$ is a congruence on $T_e$. Hence $\sigma$ is a congruence on $G_e$. Similarly, any left congruence on $G_e$ is a congruence so that $G_e$ is Hamiltonian.

We can now prove that, for each $f$ in $E$, $P_f$ is an $H$-semigroup. Let $a, b \in P_f$. Since $a, b \in T_f$, then $ab \in T_f$. Assume $ab \in P_f$. Then $ab \in P_e \subseteq T_e$, where $f < k$, for some $k \in E$, so that $a, b \in T'_e$. But then $ab \in T'_e$, since $T'_e$ is an ideal, contradiction. Therefore $ab \in P_f$ and $P_f$ is a semigroup of $S$. Let $\sigma$ be any right congruence on $P_f$. Then $\sigma$ induces a normal subgroup $H_f$ of $G_f$. Define $\sigma'$ on $T_f$ by

$$a\sigma'b \iff a, b \in P_f$$

and $a\sigma b$ or $H_f a = H_f b$.

A straight-forward argument shows that $\sigma'$ is a congruence on $T_f$. Similarly, any left congruence on $P_f$ is a congruence. Therefore $P_f$ is an $H$-semigroup.

Suppose there exists $a \in P_e$, $b \in P_f$, such that $ab \in P_{ef}$, say $ab \in P_k$, for some $k \in E$. Now $a \in P_e$ implies $ae \in G_e$, and $b \in P_f$ implies $bf \in G_f$ so that $abef \in G_{ef}$ and $ab \in T'_f$. Then $ef < k$. If $a \in T'_e$ or $b \in T''_e$, then $ab \in T''_e$, since $T''_e$ is an ideal. Thus we must have $a, b \in T_k$. But then $k \leq e, f$ so that $k \leq ef$, contradiction. Thus $ab \in P_{ef}$. Since, for each $a$ in $S$, $\langle a \rangle$ has exactly one idempotent [1, p. 20], it follows that $P_e \cap P_f = \emptyset$ for $e \neq f$. This completes the proof of Theorem 3.

The obvious corollary follows from Theorem 1.

**COROLLARY 3.1.** If $S$ is a periodic $H$-semigroup, then either the idempotents of $S$ are commutative and $S$ is a semilattice of disjoint one-idempotent $H$-semigroups; or the idempotents of $S$ are not commutative and $S = \bigcup \{S_i : i \in I\}$, where the $S_i$ are disjoint, the idempotents of each $S_i$ are commutative and each $S_i$ is a semilattice of disjoint one-idempotent $H$-semigroups. Moreover, every subgroup of $S$ is a Hamiltonian group.

3. In this section we examine the $t$-semisimple periodic $H$-semigroups. However, our first result in this investigation is more general.

**THEOREM 4.** If $S$ is a $t$-semisimple $H$-semigroup, then the
idempotents of \( S \) are commutative.

**Proof.** Let \( S \) be a \( t \)-semisimple \( H \)-semigroup and assume that the idempotents of \( S \) are not commutative. Then \( S = \bigcup \{ S_i; i \in I \} \), as in Theorem 1. Let \( \sigma \) be a maximal modular congruence on \( S \) with identity \( x \). Say \( x \in S_i \). Let \( s \in S \), say \( s \in S_j \), \( i \neq j \). Since either \( S_i S_j = \{ x_j \} \), where \( x_j \) is the zero of \( S_j \), or \( S_i S_j = \{ x_i \} \), where \( x_i \) is the zero of \( S_i \), then \((xs)\sigma\sigma(sx)\) implies \( x_i \sigma \sigma x_j \) or \( x_j \sigma \sigma x_i \). In either case, for every modular congruence \( \sigma \) on \( S \), \( W_a = \{ x_i; i \in I \} \) is contained is a \( \sigma \)-class. Since \( S \) is \( t \)-semisimple then \( W_a \) must be a singleton set. But then the idempotents of \( S \) are commutative, contrary to the assumption.

In identifying the maximal modular congruences on a periodic \( H \)-semigroup where \( E \) is a semilattice, we find the classification to be quite similar to that of inverse \( H \)-semigroups [2].

**Lemma 5.1.** If \( \sigma \) is a maximal modular congruence on the periodic \( H \)-semigroup \( S \), where the idempotents of \( S \) form a semilattice, then either \( \sigma \) is cancellative or \( \sigma \) has exactly two equivalence classes, one of which is an ideal of non-identities for \( \sigma \) and the other the semigroup of identities for \( \sigma \).

**Proof.** Let \( \sigma \) be a maximal modular congruence on the periodic \( H \)-semigroup \( S \) where the idempotents of \( S \) form a semilattice. Let \( a \) be an identity for \( \sigma \), say \( a \in P_f \), where \( a^* = f \). Then, for each \( s \) in \( S \),

\[
(as)\sigma as \rightarrow (s^*a)\sigma(as)\sigma as \rightarrow \cdots \rightarrow (a^n s)\sigma as \rightarrow (f s)\sigma as ,
\]

and similarly \((sf)s\sigma s\). Hence \( f \) is an identity for \( \sigma \).

Suppose \( \sigma \) is cancellative. Let \( e, f \in E \), where \( e \) is an identity for \( \sigma \). Then

\[
(ef)\sigma f \rightarrow (ef)\sigma ff \rightarrow e \sigma f .
\]

Hence \( E \subseteq \sigma_e \), the \( \sigma \)-class containing \( e \). Conversely, suppose \( E \subseteq \sigma_e \) and assume \((ac)\sigma bc)\) where \( c \in P_f \). Since \( e \) is an identity for \( \sigma \) and, for each \( f \) in \( E \), \( e \sigma f \), then \((fs)\sigma\sigma(fs)\), for each \( s \) in \( S \), so that each idempotent is an identity for \( \sigma \). Let \( c^m = f \). Then \((ac)\sigma bc)\) implies \((ac^m)\sigma bc)\) so that \((af)\sigma(bf)\), and, since \((af)\sigma a)\) and \((bf)\sigma b)\), then \( a \sigma b)\) and \( \sigma \) is right cancellative. Similarly, \( \sigma \) is left cancellative.

Suppose \( \sigma \) is not cancellative and let \( e \in E \) be an identity for \( \sigma \). If \( h \) is an identity for \( \sigma \), where \( h \in E \), then \( h \sigma eh)\sigma e)\) and \( h \in \sigma_e \). Since \( \sigma \) is not cancellative, there exists \( f \in E \) such that \( f \in \sigma_e \), so that \( f \) is not an identity for \( \sigma \). Let \( I = \{ f \in E; f \) is not an identity
for \(\sigma\). Let \(J = \bigcup \{P_f : f \in I\}\). It follows that \(I\) is an ideal in \(E\), \(J\) is an ideal in \(S\) and \(J'\) is a semigroup of \(S\). Oehmke [4] has shown that if \(\sigma\) is a maximal congruence on \(S\) and \(J\) is any ideal of \(S\), then either \(J\) is contained in a \(\sigma\)-class \(S_o\) (which is also an ideal of \(S\)) or \(J\) contains an element of each \(\sigma\)-class. If \(x \in \sigma_* \cap J\) then \(x\sigma e\) and \(x \in P_f\) for some \(f \in I\), where \(x^m = f\). But

\[
\begin{align*}
x\sigma e & \rightarrow x^2\sigma(xe) \text{ and } (xe)\sigma e \rightarrow x^2\sigma(xe) \rightarrow \cdots \rightarrow x^*\sigma e \rightarrow f\sigma e.
\end{align*}
\]

Then \(f \in I\), contradiction. Hence \(\mu \cap J = \emptyset\) and \(J \subseteq S_o\). Suppose there exists \(b \in S\) such that \(b \in J\), say \(b \in P_h\), where \(h\sigma e\). Let \(f \in I \subseteq S_o\). Then \(baf\) implies \((bh)\sigma(fh)\) and \((bf)\alpha f\); and \(h\sigma e\) implies \((fh)\sigma(f)\) so that \((bh)\sigma(bf)\). But then \((b^m-bh)\sigma(b^m-bf)\) and \(h\sigma(h,f)\). It follows that \(h\sigma f\) and \(f \in I\), contradiction. Thus \(J = S_o\). Since \(J\) is an ideal and \(J'\) is a semigroup, the relation \(\sigma^*\), defined by \(a\sigma^* b \iff a, b \in J\) or \(a, b \in J'\), is a maximal modular congruence on \(S\) [2]. Clearly \(\sigma \leq \sigma^*\). Hence \(\sigma = \sigma^*\). Moreover, for each \(a \in J'\), say \(a \in P_e\), and for each \(s \in S, a\sigma e\) implies \((as)\sigma s\sigma(sa)\), so that \(J'\) is the semigroup of identities for \(\sigma\). And for each \(b \in J\), say \(b \in P_f\), \(b\) cannot be an identity for \(\sigma\), since then \(f\) would be an identity for \(\sigma\).

Using Lemma 5.1, we can establish the following characterization.

**Theorem 5.** A periodic \(H\)-semigroup \(S\) is \(t\)-semisimple if and only if \(S\) is an inverse semigroup such that for each pair of groups \(G_e, G_f\) in the semilattice, with \(f \geq e\), the homomorphism \(\psi_{f,e}\) on \(G_f\) into \(G_e\), defined by \(a\psi_{f,e} = ae\), is a monomorphism; and, for each \(e\) in \(E\), for each \(a \neq e\) in \(G_e\), there exists a subsemigroup \(T_p\) of \(S\) such that \(a \in T_p\) for each \(f \in E, T_p \cap G_f = H_f\), where \(H_f = G_f\) or \(H_f\) is a maximal subgroup of prime index \(p\) in \(G_f\).

**Proof.** Define \(\rho\) on \(S\) by \(x\rho y\) if and only if there exists \(e \in E\) such that \(ex = ey\). Clearly, \(\rho\) is a congruence on \(S\). If \(\sigma\) is any maximal modular cancellative congruence on \(S\) and \(x, y \in S\) such that \(x\rho y\), then there exists \(e \in E\) such that \(ex = ey\). Hence \((ex)\sigma(ey)\) and \(x\sigma y\). Thus \(\rho \leq \alpha\) where \(\alpha\) is the intersection of all the maximal modular cancellative congruences on \(S\). In view of Lemma 3.3, it is clear that the intersection \(\beta\) of all the maximal modular non-cancellative congruences of \(S\) separates \(S\) into its subsemigroups \(P_f\), where \(f \in E\). Let \(e < f\) and define \(\psi_{f,e}\) from \(P_f\) into \(P_e\) by \(a\psi_{f,e} = eo\). Clearly, \(\psi_{f,e}\) is a homomorphism from \(P_f\) into \(G_e\). Suppose \(S\) is \(t\)-semisimple, that is, \(\tau = \iota\). If \(\psi_{f,e}\) is not a monomorphism then there exist \(a \neq b \in P_f\) with \(ea = eb\) so that \(a\rho b\). This implies \(a\sigma b\). Since also \(a\beta b\), then \(a\tau b\) and \(\tau \neq \iota\), contradiction. Thus if \(S\) is
$t$-semisimple, then every homomorphism $\psi_{f,e}$ is a monomorphism from $P_f$ into $G_e$. Suppose there exists $e$ in $E$ such that $G_e \subseteq P_e$. Then there exists $b \in W_e$ such that $eb = a \in G_e$, that is, $eb = ea$. Then, as before, $arb$ and $\tau \neq \iota$, which is a contradiction. Hence, for each $e$ in $E$, $P_e = G_e$ and $S$ is an inverse semigroup. Considering the characterization of $t$-semisimple inverse $H$-semigroups in [2], the proof is complete.

The corollaries parallel those in [2].

**COROLLARY 5.1.** $S$ is a periodic $H$-semigroup all of whose maximal modular congruences are cancellative if and only if $S$ is a one-idempotent periodic $H$-semigroup.

**COROLLARY 5.2.** $S$ is a $t$-semisimple periodic $H$-semigroup all of whose nontrivial maximal modular congruences are not cancellative if and only if $S$ is a semilattice.

**COROLLARY 5.3.** If $S$ is a $t$-semisimple periodic $H$-semigroup, then $S$ is a semilattice of disjoint $t$-semisimple Hamiltonian groups.

**COROLLARY 5.4.** If $S$ is a $t$-semisimple periodic $H$-semigroup, then $S$ is commutative.

**COROLLARY 5.5.** If $S$ is a periodic $H$-semigroup with a minimum idempotent $e$, then $S$ is $t$-semisimple if and only if for each semigroup $P_f$ in the semilattice with $f \geq e$, the homomorphism $\psi_{f,e}$ on $P_f$ into $P_e$, defined by $a\psi_{f,e} = ae$, is a monomorphism and $P_e$ is $t$-semisimple.

**COROLLARY 5.6.** If $S$ is a $t$-semisimple periodic $H$-semigroup with no nontrivial modular congruences, then $S$ is either a cyclic group of prime order or the unique semilattice of two elements.

**COROLLARY 5.7.** If $S$ is a periodic $H$-semigroup with zero, then $S$ is $t$-semisimple if and only if $S$ is a semilattice.

**REFERENCES**

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