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PERIODIC H-SEMIGROUPS AND t-SEMISIMPLE PERIODIC H-SEMIGROUPS

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An H-semigroup is a semigroup such that every right and every left congruence is a two-sided congruence on the semigroup. It is known that the set of idempotents of an H-semigroup form a subsemigroup. A semigroup is t-semisimple provided the intersection of all its maximal modular congruences is the identity relation. Let S be a periodic H-semigroup such that the subsemigroup E of idempotents of S is commutative. In this paper it is shown that S is a semilattice of disjoint one-idempotent H-semigroups, and that every subgroup of S is a Hamiltonian group. Moreover, if S is t-semisimple, then S is an inverse semigroup such that the one-idempotent H-semigroups of the semilattice are the maximal subgroups of S, and a complete characterization is given.

If σ is an equivalence relation on a semigroup S and a is equivalent to b, then we shall write $a\sigma b$. The σ-class containing a will be denoted by σ_a. An equivalence relation σ on a semigroup S is a right (left) congruence if $a, b \in S$ and $a\sigma b$ imply $(ac)\sigma(bc)((ca)\sigma(cb))$. If an equivalence relation is both a right and a left congruence, we shall call it a two-sided congruence, or, more briefly, a congruence. We use the natural partial ordering on relations and say that $\sigma \leqq \rho$ if and only if $a, b \in S$ and $a\sigma b$ imply $a\rho b$. Clearly the identity relation ι and the universal relation ν are congruences and $\iota \leqq \sigma \leqq \nu$, for each congruence σ on S. A congruence $\sigma \neq \nu$ is called maximal if, for each congruence σ' on S such that $\sigma \leqq \sigma' \leqq \nu$, either $\sigma = \sigma'$ or $\sigma' = \nu$. A congruence σ on S is called modular if there is an element e of S such that $(ea)\sigma a$ and $(ae)\sigma a$ for all a in S. The element e is called an identity for σ. The intersection of all the maximal modular congruences on S is called the t-radical of S [4] and it will be denoted by τ.

1. **Preliminary definitions and results.** In his initial paper on H-semigroups, Oehmke [3] obtained several useful results. For reference we summarize those results which are essential to this work. The set E of idempotents of an H-semigroup S forms a subsemigroup. For each $a \in E$, the subset R_a of E is the set of all $b \in E$ such that $ab = b$ and $ba = a$. Similarly, the set L_a of E is the set of all $b \in E$ such that $ba = b$ and $ab = a$. The collection of all $R_a(L_a)$ induces a decomposition of E and the corresponding equivalence

relation is a right (left) congruence. The set of all W_a, where $W_a = L_aR_a$, $a \in E$, is a semilattice where the commutative multiplication operation (denoted by $\circ$) is defined as $W_a \circ W_b = W_{ab}$, and where the partial ordering relation is defined by $W_a \leqq W_b$ if and only if $W_a \circ W_b = W_a$. If there is a minimal W_a in the set, then it is unique. It follows that either $W_a = L_a$ or $W_a = R_a$ and, for all $a \in E$, either W_a is trivial, that is, $W_a = \{a\}$, or W_a is minimal. If W_a is minimal and $W_a = R_a$, then $R_ac = \{ac\}$, for all $c \in S$. If W_a is minimal and $W_a = L_a$, then for any c in S we have $cL_a = \{ca\}$. If there is no mimimal W_a, then each W_a contains a single element. It then follows that E is commutative. These results yield the following theorem.

THEOREM 1. *Let W_a be minimal and $W_a = \{x_i\colon i \in I\}$. Then $S = \bigcup\{S_i\colon i \in I\}$ where the S_i are disjoint H-subsemigroups of S. If $R_a = W_a$ then $S_iS_j = \{x_j\}$, for $i \neq j$, and S_i is the set of all b such that $R_ab = \{x_i\}$. If $L_a = W_a$ then $S_iS_j = \{x_i\}$, for $i \neq j$, and S_i is the set of all b such that $bL_a = \{x_j\}$. For any i, the set E_i of idempotents of S_i is a commutative subsemigroup* [3].

By Theorem 1, we can reduce the study of H-semigroups to the study of those H-semigroups in which the idempotents form a commutative subsemigroup.

An element b of a semigroup S is an inverse of an element a of S provided $aba = a$ and $bab = b$. Then $e = ab$ is an idempotent of S such that $ea = a$, and $f = ba$ is an idempotent of S such that $af = a$. S is an inverse semigroup provided every element of S has a unique inverse. The inverse of an element a of an inverse semigroup S will be denoted by a^{-1} so that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

A left (right) zero of a semigroup S is an element a of S such that $as = a$ ($sa = a$), for each $s \in S$.

An element a of a semigroup S is regular provided $a \in aSa$. Then a has at least one inverse in S, namely bab, where $aba = a$.

All of the definitions following Theorem 1 are taken from [1].

Let T be the set of regular elements of an H-semigroup S. Let $a, b \in T$. Then there exist s_1, s_2 in S such that $a = as_1a$, where as_1, $s_1a \in E$, and $b = bs_2b$, where bs_2, $s_2b \in E$. We assume that E is a semilattice, that is, E is a commutative idempotent semigroup with the induced ordering given by $e \leqq f$ if and only if $ef = e$. Then

$$ab = a(s_1a)(bs_2)b = a(bs_2)(s_1a)b = ab(s_2s_1)ab\ .$$

Hence $ab \in T$ and T is a subsemigroup of S. Since s_1as_1 is an inverse of a in S, then s_1as_1 is in T and $a \in aTa$. Hence T is a regular semigroup. It follows that T is an inverse semigroup [1, p. 28]. Thus T is an inverse subsemigroup of S. Let c be a left zero of S. Then $c \in T$ and $c^{-1} = c$. Let $s \in S$. Then $cscc = c$ and $sccsc = sc$ imply $sc \in T$ and $c^{-1} = sc$. Hence $sc = c$. Since s was arbitrary in S, then c is a right zero of S. Analogously, if c is a right zero of S, then c is a left zero of S. Hence S has at most one (left, right) zero.

If S is an H-semigroup and I is a right (left) ideal of S, then for $b \in S$, $bI \subseteqq I (Ib \subseteqq I)$ or $bI = \{c\}$, where c is a left zero ($Ib = \{c\}$, where c is a right zero) [3]. Using this, we get that a right (left) ideal of an H-semigroup S such that E is commutative is a two-sided ideal, and it follows that, for each e in E, for each a in S, $ea = a$ if and only if $ae = a$.

THEOREM 2. *Let S be an H-semigroup such that the subsemigroup E of idempotents of S is a semilattice. Then the set T of regular elements of S is an inverse semigroup which is a semilattice of disjoint groups.*

Proof. Let $a \in T$. Then there exists a unique element a^{-1} in T such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Since aa^{-1}, $a^{-1}a \in E$, we have $a(aa^{-1}) = a$ and $(a^{-1}a)a = a$. Hence

$$a^{-1}a = a^{-1}(aaa^{-1}) = (a^{-1}aa)a^{-1} = aa^{-1} .$$

It follows that T is a union of disjoint groups [1, ex. 10, p. 34]. Let $G_e = \{b \in T: bb^{-1} = e\}$. Then G_e is a maximal subgroup of T and $T = \bigcup \{G_e: e \in E\}$, where $G_e \cap G_f = \varnothing$ for $e \neq f$. As in [2], we get that T is a semilattice of disjoint groups.

2. For the remainder of this work, unless otherwise indicated, we assume not only that S is an H-semigroup such that the subsemigroup E of idempotents of S is a semilattice, but also that S is a periodic semigroup [1, p. 20]. Let $P_e = \{s \in S: s^n = e$ for some positive integer $n\}$. Let T be the inverse subsemigroup of regular elements of S. Clearly $P_e \cap T = G_e \subseteqq P_e$. Let $P_e - G_e = W_e$ and let $a \in W_e$, where $a^n = e$. Then

$$(ae)^n = (a^{n+1})^n = (a^n)^{n+1} = e \Longrightarrow ae \in P_e ,$$

and

$$ae(ae)^{n-1}ae = (ae)(ae)^n = ae^2 = ae \Longrightarrow ae \in T .$$

Hence, $ae = aa^n = a^na = ea \in G_e$ and, for each b in G_e, $ab = aeb \in G_e$ and $ba = bea \in G_e$, so that G_e, is an ideal in P_e. Let $T_e = \bigcup\{P_f: e \leqq f\}$.

LEMMA 3.1. $ae \in G_e \Leftrightarrow a \in T_e$.

Proof. Let $a \in T_e$. Then there exists $f \geqq e$ such that $a \in P_f$ and $af \in G_f$. Hence $afe \in G_{fe}$, that is, $ae \in G_e$. Conversely, if $ae \in G_e$, then there exists $b \in G_e$ such that $aeb = ab = e$. Say $a \in P_f$, where $a^n = f$. Then $fb^n \in G_{ef}$ and

$$\begin{aligned} fb^n &= a^n b^n = a^{n-1}abb^{n-1} = a^{n-1}eb^{n-1} \\ &= a^{n-1}b^{n-1} = \cdots = aabb = aeb = ab = e \,. \end{aligned}$$

Thus $fb^n \in G_{ef} \cap G_e$. But this implies $ef = e$ so that $e \leqq f$. Hence $a \in T_e$.

LEMMA 3.2. *For each e in E, T_e is a subsemigroup of S, and if $a \notin T_e$ and there exists $b \in S$ such that $ab \in T_e$, then $b \notin T_e$.*

Proof. Let $a, b \in T_e$, say $a \in P_f$ and $b \in P_h$, where $e \leqq f, h$. Then $af \in G_f$ and $bh \in G_h$ imply that $afbh = abfh \in G_{fh}$ so that $ab \in T_{fh}$. Now $ef = e$ and $eh = e$ imply that $efh = e$ so that $e \leqq fh$. Hence $ab \in T_e$ and T_e is a subsemigroup of S. Let $S - T_e = T'_e$ and suppose e is not minimum so that $T'_e \neq \varnothing$. Let $a \notin T_e$ and suppose there exists $b \in S$ such that $ab \in T_e$. Assume $b \in T_e$. Then $abe \in G_e$ and $be \in G_e$ imply $abe(be)^{-1} = ae$ is in G_e so that $a \in T_e$, contradiction.

LEMMA 3.3. *For each f in E, T_f is an H-semigroup of S, and if f is not minimum in E, then $T'_f \neq \varnothing$ and T'_f is an ideal of S.*

Proof. Let $f \in E$. Let $U_x = \{b \in S: xb \in T_f\}$. Define σ on S by

$$a\sigma b \Longleftrightarrow U_a = U_b \,.$$

Clearly σ is a (right) congruence on S. Let $a, b \in T_f$. Then, using Lemma 3.2, we have

$$x \in U_a \Longleftrightarrow ax \in T_f \Longleftrightarrow x \in T_f \Longleftrightarrow bx \in T_f \Longleftrightarrow x \in U_b \,.$$

Thus $U_a = U_b$ and $a\sigma b$. Further, if $a\sigma b$ and $a \in T_f$, then, for each x in T_f, $x \in U_a = U_b$. In particular, $a \in U_b$ so that $ba \in T_f$ and, using Lemma 3.2, $b \in T_f$. Thus T_f is an equivalence class of σ. Since $f \in U_f$, $U_f \neq \varnothing$. Let $a \in S$.

$$x \in U_a \Longleftrightarrow ax \in T_f \Longleftrightarrow fax \in T_f \Longleftrightarrow x \in U_{fa} \,.$$

Then $U_a = U_{fa}$ and $(fa)\sigma a$, for each a in S. Let $x \in U_{af}$. Then $afx \in T_f$. Now $(fx)\sigma x$ implies $(afx)\sigma(ax)$, so that $ax \in T_f$ and $x \in U_a$.

Then $U_{af} \subseteqq U_a$. Let $x \in U_a$. Then $ax \in T_f$ and $(fax)\sigma(ax)$. As before, $(fx)\sigma x$ implies $(afx)\sigma(ax)$. Hence, $(fax)\sigma(afx)$ implies $afx \in T_f$ so that $x \in U_{af}$. Then $U_a \subseteqq U_{af}$ and $(af)\sigma a$, for each a in S. Therefore f is an identity for σ and σ is modular. Let ρ be any congruence on S such that T_f is an equivalence class of ρ and assume $\sigma < \rho$. Then there exist a, b in S such that $a\rho b$ and $a\not\sigma b$, that is, there exists $x \in U_a$ such that $x \notin U_b$, which implies that $ax \in T_f$ and $bx \notin T_f$. But $a\rho b$ implies $(ax)\rho(bx)$ so that $bx \in T_f$, contradiction. Therefore, $\sigma = \rho$ and σ is maximal with respect to having T_f as a σ-class. Let $a \in T'_f$ and assume $x \in U_a$. Then $ax \in T_f$. Thus we have

$$\begin{aligned}(ax)\sigma f &\Longrightarrow (a^2x)\sigma(af)\sigma a \Longrightarrow (a^2x^2)\sigma(ax)\\ &\Longrightarrow (a^2x^2)\sigma f \Longrightarrow (a^3x^2)\sigma(af)\sigma a \Longrightarrow (a^3x^3)\sigma(ax)\\ &\Longrightarrow (a^3x^3)\sigma f \Longrightarrow \cdots\\ &\Longrightarrow (a^nx^n)\sigma f\,, \quad \text{for each positive integer } n\,.\end{aligned}$$

Let $a^i = h$, where $h \notin T_f$. Since $ax \in T'_f$, then $x \in T'_f$. Let $x^j = k$, where $k \notin T_f$. Then we have

$$(a^{ij}x^{ij})\sigma f \Longrightarrow (hk)\sigma f \Longrightarrow hk \in T_f\,.$$

But $h, k \notin T_f$ implies $hk \notin T_f$, contradiction. Hence, for each $a \in T'_f$, $U_a = \varnothing$. It follows that T'_f is a σ-class and T'_f is an ideal of S. Let ρ be any right congruence on T_f. Define ρ' on S by

$$a\rho' b \Longleftrightarrow a, b \in T_f \quad \text{and} \quad a\rho b \text{ or } a, b \in T'_f\,.$$

Clearly ρ' is a congruence on S and the restriction of ρ' to T_f is ρ. Thus ρ is a left congruence on T_f. By analogous proof, any left congruence on T_f is a right congruence. Thus T_f is an H-semigroup of S.

With the preceding lemmas, we are now in a position to prove the main results of this section.

THEOREM 3. *If S is a periodic H-semigroup such that the subsemigroup E of idempotents of S is commutative, then S is a semilattice of disjoint one-idempotent H-semigroups. Moreover, every subgroup of S is a Hamiltonian group.*

Proof. First we show that for each e in E, G_e is a Hamiltonian group. If $e = 0$, then G_e is trivially Hamiltonian. Assume $e \neq 0$. Let σ be a right congruence on G_e, let H_e be the subgroup of G_e induced by σ and let $a, b \in T_e$. Write

$$a\sigma^{(e)}b \Longleftrightarrow (ea)\sigma(eb) \ .$$

By a straight-forward argument, $\sigma^{(e)}$ is an equivalence relation on T_e, so we need only show right compatibility. Accordingly, assume $a\sigma^{(e)}b$ and $c \in T_e$. Then $(ea)\sigma(eb)$ and $ec \in G_e$ imply $(eaec)\sigma(ebec)$ so that $(eac)\sigma(ebc)$ and $(ac)\sigma^{(e)}(bc)$. Clearly, $\sigma^{(e)}$ restricted to G_e is σ. Since T_e is an H-semigroup, then $\sigma^{(e)}$ is a congruence on T_e. Hence σ is a congruence on G_e. Similarly, any left congruence on G_e is a congruence so that G_e is Hamiltonian.

We can now prove that, for each f in E, P_f is an H-semigroup. Let $a, b \in P_f$. Since $a, b \in T_f$, then $ab \in T_f$. Assume $ab \notin P_f$. Then $ab \in P_k \subsetneqq T_k$, where $f < k$, for some $k \in E$, so that $a, b \in T'_k$. But then $ab \in T'_k$, since T'_k is an ideal, contradiction. Therefore $ab \in P_f$ and P_f is a semigroup of S. Let σ be any right congruence on P_f. Then σ induces a normal subgroup H_f of G_f. Define σ' on T_f by

$$a\sigma' b \Longleftrightarrow a, b \in P_f \quad \text{and} \quad a\sigma b \text{ or } H_f a = H_f b \ .$$

A straight-forward argument shows that σ' is a congruence on T_f. Similarly, any left congruence on P_f is a congruence. Therefore P_f is an H-semigroup.

Suppose there exists $a \in P_e$, $b \in P_f$ such that $ab \notin P_{ef}$, say $ab \in P_k$, for some $k \in E$. Now $a \in P_e$ implies $ae \in G_e$, and $b \in P_f$ implies $bf \in G_f$ so that $abef \in G_{ef}$ and $ab \in T_{ef}$. Then $ef < k$. If $a \in T'_k$ or $b \in T'_k$, then $ab \in T'_k$, since T'_k is an ideal. Thus we must have $a, b \in T_k$. But then $k \leqq e, f$ so that $k \leqq ef$, contradiction. Thus $ab \in P_{ef}$. Since, for each a in S, $\langle a \rangle$ has exactly one idempotent [1, p. 20], it follows that $P_e \cap P_f = \varnothing$ for $e \neq f$. This completes the proof of Theorem 3.

The obvious corollary follows from Theorem 1.

COROLLARY 3.1. *If S is a periodic H-semigroup, then either the idempotents of S are commutative and S is a semilattice of disjoint one-idempotent H-semigroups; or the idempotents of S are not commutative and $S = \bigcup\{S_i : i \in I\}$, where the S_i are disjoint, the idempotents of each S_i are commutative and each S_i is a semilattice of disjoint one-idempotent H-semigroups. Moreover, every subgroup of S is a Hamiltonian group.*

3. In this section we examine the t-semisimple periodic H-semigroups. However, our first result in this investigation is more general.

THEOREM 4. *If S is a t-semisimple H-semigroup, then the* *idempotents of S are commutative.*

Proof. Let S be a t-semisimple H-semigroup and assume that the idempotents of S are not commutative. Then $S = \bigcup\{S_i: i \in I\}$, as in Theorem 1. Let σ be a maximal modular congruence on S with identity x. Say $x \in S_i$. Let $s \in S$, say $s \in S_j$, $i \neq j$. Since either $S_iS_j = \{x_j\}$, where x_j is the zero of S_j, or $S_iS_j = \{x_i\}$, where x_i is the zero of S_i, then $(xs)\sigma s\sigma(sx)$ implies $x_i\sigma s\sigma x_j$ or $x_j\sigma s\sigma x_i$. In either case, for every modular congruence σ on S, $W_a = \{x_i: i \in I\}$ is contained is a σ-class. Since S is t-semisimple then W_a must be a singleton set. But then the idempotents of S are commutative, contrary to the assumption.

In identifying the maximal modular congruences on a periodic H-semigroup where E is a semilattice, we find the classification to be quite similar to that of inverse H-semigroups [2].

LEMMA 5.1. *If σ is a maximal modular congruence on the periodic H-semigroup S, where the idempotents of S are commutative, then either σ is cancellative or σ has exactly two equivalence classes, one of which is an ideal of non-identities for σ and the other the semigroup of identities for σ.*

Proof. Let σ be a maximal modular congruence on the periodic H-semigroup S where the idempotents of S form a semilattice. Let a be an identity for σ, say $a \in P_f$, where $a^n = f$. Then, for each s in S,

$$(as)\sigma s \Longrightarrow (a^2s)\sigma(as)\sigma s \Longrightarrow \cdots \Longrightarrow (a^ns)\sigma s \Longrightarrow (fs)\sigma s\ ,$$

and similarly $(sf)\sigma s$. Hence f is an identity for σ.

Suppose σ is cancellative. Let $e, f \in E$, where e is an identity for σ. Then

$$(ef)\sigma f \Longrightarrow (ef)\sigma(ff) \Longrightarrow e\sigma f\ .$$

Hence $E \subseteqq \sigma_e$, the σ-class containing e. Conversely, suppose $E \subseteqq \sigma_e$ and assume $(ac)\sigma(bc)$ where $c \in P_f$. Since e is an identity for σ and, for each f in E, $e\sigma f$, then $(fs)\sigma s\sigma(sf)$, for each s in S, so that each idempotent is an identity for σ. Let $c^m = f$. Then $(ac)\sigma(bc)$ implies $(ac^m)\sigma(bc^m)$ so that $(af)\sigma(bf)$, and, since $(af)\sigma a$ and $(bf)\sigma b$, then $a\sigma b$ and σ is right cancellative. Similarly, σ is left cancellative.

Suppose σ is not cancellative and let $e \in E$ be an identity for σ. If h is an identity for σ, where $h \in E$, then $h\sigma(eh)\sigma e$ and $h \in \sigma_e$. Since σ is not cancellative, there exists $f \in E$ such that $f \notin \sigma_e$, so that f is not an identity for σ. Let $I = \{f \in E: f$ is not an identity for $\sigma\}$. Let $J = \bigcup\{P_f : f \in I\}$. It follows that I is an ideal in E, J is an ideal in S and J' is a semigroup of S. Oehmke [4] has shown that if σ is a maximal congruence on S and J is any ideal of S, then either J is contained in a σ-class S_0 (which is also an ideal of S) or J contains an element of each σ-class. If $x \in \sigma_e \cap J$ then $x\sigma e$ and $x \in P_f$ for some f in I, where $x^m = f$. But

$$\begin{aligned} x\sigma e &\Longrightarrow x^2\sigma(xe) \text{ and } (xe)\sigma e \Longrightarrow x^2\sigma e \Longrightarrow x^3\sigma(xe) \\ &\Longrightarrow x^3\sigma e \Longrightarrow \cdots \Longrightarrow x^m\sigma e \Longrightarrow f\sigma e\,. \end{aligned}$$

Then $f \notin I$, contradiction. Hence $\sigma_e \cap J = \varnothing$ and $J \subseteqq S_0$. Suppose there exists $b \in S_0$ such that $b \notin J$, say $b \in P_h$, where $h\sigma e$. Let $f \in I \subseteqq S_0$. Then $b\sigma f$ implies $(bh)\sigma(fh)$ and $(bf)\sigma f$; and $h\sigma e$ implies $(fh)\sigma f$ so that $(bh)\sigma(bf)$. But then $(b^{n-1}bh)\sigma(b^{n-1}bf)$ and $h\sigma(hf)$. It follows that $h\sigma f$ and $f \notin I$, contradiction. Thus $J = S_0$. Since J is an ideal and J' is a semigroup, the relation σ^*, defined by $a\sigma^* b \Leftrightarrow a, b \in J$ or $a, b \in J'$, is a maximal modular congruence on S [2]. Clearly $\sigma \leqq \sigma^*$. Hence $\sigma = \sigma^*$. Moreover, for each a in J', say $a \in P_e$, and for each s in S, $a\sigma e$ implies $(as)\sigma s\sigma(sa)$, so that J' is the semigroup of identities for σ. And for each b in J, say $b \in P_f$, b cannot be an identity for σ, since then f would be an identity for σ.

Using Lemma 5.1, we can establish the following characterization.

THEOREM 5. *A periodic H-semigroup S is t-semisimple if and only if S is an inverse semigroup such that for each pair of groups G_e, G_f in the semilattice, with $f \geqq e$, the homomorphism $\varphi_{f,e}$ on G_f into G_e, defined by $a\varphi_{f,e} = ae$, is a monomorphism; and, for each e in E, for each $a \neq e$ in G_e, there exists a subsemigroup T_p of S such that $a \notin T_p$ and for each f in E, $T_p \cap G_f = H_f$, where $H_f = G_f$ or H_f is a maximal subgroup of prime index p in G_f.*

Proof. Define ρ on S by $x\rho y$ if and only if there exists e in E such that $ex = ey$. Clearly, ρ is a congruence on S. If σ is any maximal modular cancellative congruence on S and $x, y \in S$ such that $x\rho y$, then there exists e in E such that $ex = ey$. Hence $(ex)\sigma(ey)$ and $x\sigma y$. Thus $\rho \leqq \alpha$ where α is the intersection of all the maximal modular cancellative congruences on S. In view of Lemma 3.3, it is clear that the intersection β of all the maximal modular noncancellative congruences of S separates S into its subsemigroups P_f, where $f \in E$. Let $e < f$ and define $\psi_{f,e}$ from P_f into P_e by $a\psi_{f,e} = ea$. Clearly, $\psi_{f,e}$ is a homomorphism from P_f into G_e. Suppose S is t-semisimple, that is, $\tau = \iota$. If $\psi_{f,e}$ is not a monomorphism then there exist $a \neq b$ in P_f with $ea = eb$ so that $a\rho b$. This implies $a\sigma b$. Since also $a\beta b$, then $a\tau b$ and $\tau \neq \iota$, contradiction. Thus if S is

t-semisimple, then every homomorphism $\psi_{f,e}$ is a monomorphism from P_f into G_e. Suppose there exists e in E such that $G_e \subset P_e$. Then there exists $b \in W_e$ such that $eb = a \in G_e$, that is, $eb = ea$. Then, as before, $a\tau b$ and $\tau \neq \iota$, which is a contradiction. Hence, for each e in E, $P_e = G_e$ and S is an inverse semigroup. Considering the characterization of t-semisimple inverse H-semigroups in [2], the proof is complete.

The corollaries parallel those in [2].

COROLLARY 5.1. *S is a periodic H-semigroup all of whose maximal modular congruences are cancellative if and only if S is a one-idempotent periodic H-semigroup.*

COROLLARY 5.2. *S is a t-semisimple periodic H-semigroup all of whose nontrivial maximal modular congruences are not cancellative if and only if S is a semilattice.*

COROLLARY 5.3. *If S is a t-semisimple periodic H-semigroup, then S is a semilattice of disjoint t-semisimple Hamiltonian groups.*

COROLLARY 5.4. *If S is a t-semisimple periodic H-semigroup, then S is commutative.*

COROLLARY 5.5. *If S is a periodic H-semigroup with a minimum idempotent e, then S is t-semisimple if and only if for each semigroup P_f in the semilattice with $f \geqq e$, the homomorphism $\psi_{f,e}$ on P_f into P_e, defined by $a\psi_{f,e} = ae$, is a monomorphism and P_e is t-semisimple.*

COROLLARY 5.6. *If S is a t-semisimple periodic H-semigroup with no nontrivial modular congruences, then S is either a cyclic group of prime order or the unique semilattice of two elements.*

COROLLARY 5.7. *If S is a periodic H-semigroup with zero, then S is t-semisimple if and only if S is a semilattice.*

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