PERIOD $H$-SEMIGROUPS AND $t$-SEMISIMPLE PERIODIC $H$-SEMIGROUPS

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An $H$-semigroup is a semigroup such that every right and every left congruence is a two-sided congruence on the semigroup. It is known that the set of idempotents of an $H$-semigroup form a subsemigroup. A semigroup is $t$-semisimple provided the intersection of all its maximal modular congruences is the identity relation. Let $S$ be a periodic $H$-semigroup such that the subsemigroup $E$ of idempotents of $S$ is commutative. In this paper it is shown that $S$ is a semilattice of disjoint one-idempotent $H$-semigroups, and that every subgroup of $S$ is a Hamiltonian group. Moreover, if $S$ is $t$-semisimple, then $S$ is an inverse semigroup such that the one-idempotent $H$-semigroups of the semilattice are the maximal subgroups of $S$, and a complete characterization is given.

If $\sigma$ is an equivalence relation on a semigroup $S$ and $a$ is equivalent to $b$, then we shall write $a\sigma b$. The $\sigma$-class containing $a$ will be denoted by $\sigma a$. An equivalence relation $\sigma$ on a semigroup $S$ is a right (left) congruence if $a, b \in S$ and $a\sigma b$ imply $(ac)\sigma (bc)$, $((ea)\sigma (eb))$. If an equivalence relation is both a right and a left congruence, we shall call it a two-sided congruence, or, more briefly, a congruence. We use the natural partial ordering on relations and say that $\sigma \leq \rho$ if and only if $a, b \in S$ and $a\sigma b$ imply $a\rho b$. Clearly the identity relation $I$ and the universal relation $V$ are congruences and $I \leq \sigma \leq V$, for each congruence $\sigma$ on $S$. A congruence $\sigma \neq V$ is called maximal if, for each congruence $\sigma'$ on $S$ such that $\sigma \leq \sigma' \leq V$, either $\sigma = \sigma'$ or $\sigma' = V$. A congruence $\sigma$ on $S$ is called modular if there is an element $e$ of $S$ such that $(ea)\sigma a$ and $(ae)\sigma a$ for all $a$ in $S$. The element $e$ is called an identity for $\sigma$. The intersection of all the maximal modular congruences on $S$ is called the $t$-radical of $S$ [4] and it will be denoted by $\tau$.

1. Preliminary definitions and results. In his initial paper on $H$-semigroups, Oehmke [3] obtained several useful results. For reference we summarize those results which are essential to this work. The set $E$ of idempotents of an $H$-semigroup $S$ forms a subsemigroup. For each $a \in E$, the subset $R_a$ of $E$ is the set of all $b \in E$ such that $ab = b$ and $ba = a$. Similarly, the set $L_a$ of $E$ is the set of all $b \in E$ such that $ba = b$ and $ab = a$. The collection of all $R_a(L_a)$ induces a decomposition of $E$ and the corresponding equivalence
relation is a right (left) congruence. The set of all $W_a$, where $W_a = L_aR_a$, $a \in E$, is a semilattice where the commutative multiplication operation (denoted by $\circ$) is defined as $W_a \circ W_b = W_{ab}$, and where the partial ordering relation is defined by $W_a \leq W_b$ if and only if $W_a \circ W_b = W_a$. If there is a minimal $W_a$ in the set, then it is unique. It follows that either $W_a = L_a$ or $W_a = R_a$ and, for all $a \in E$, either $W_a$ is trivial, that is, $W_a = \{a\}$, or $W_a$ is minimal. If $W_a$ is minimal and $W_a = R_a$, then $R_a = \{ac\}$, for all $c \in S$. If $W_a$ is minimal and $W_a = L_a$, then for any $c$ in $S$ we have $cL_a = \{ca\}$. If there is no minimal $W_a$, then each $W_a$ contains a single element. It then follows that $E$ is commutative. These results yield the following theorem.

**Theorem 1.** Let $W_a$ be minimal and $W_a = \{x_i : i \in I\}$. Then $S = \bigcup\{S_i : i \in I\}$ where the $S_i$ are disjoint $H$-subsemigroups of $S$. If $R_a = W_a$ then $S_iS_j = \{x_j\}$, for $i \neq j$, and $S_i$ is the set of all $b$ such that $R_ab = \{x_i\}$. If $L_a = W_a$ then $S_iS_j = \{x_i\}$, for $i \neq j$, and $S_i$ is the set of all $b$ such that $bL_a = \{x_j\}$. For any $i$, the set $E_i$ of idempotents of $S_i$ is a commutative subsemigroup [3].

By Theorem 1, we can reduce the study of $H$-semigroups to the study of those $H$-semigroups in which the idempotents form a commutative subsemigroup.

An element $b$ of a semigroup $S$ is an inverse of an element $a$ of $S$ provided $aba = a$ and $bab = b$. Then $e = ab$ is an idempotent of $S$ such that $ea = a$, and $f = ba$ is an idempotent of $S$ such that $af = a$. $S$ is an inverse semigroup provided every element of $S$ has a unique inverse. The inverse of an element $a$ of an inverse semigroup $S$ will be denoted by $a^{-1}$ so that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

A left (right) zero of a semigroup $S$ is an element $a$ of $S$ such that $as = a$ ($sa = a$), for each $s \in S$.

An element $a$ of a semigroup $S$ is regular provided $a \in aSa$. Then $a$ has at least one inverse in $S$, namely $bab$, where $aba = a$.

All of the definitions following Theorem 1 are taken from [1].

Let $T$ be the set of regular elements of an $H$-semigroup $S$. Let $a, b \in T$. Then there exist $s_1, s_2$ in $S$ such that $a = as_1a$, where $as_1, s_1a \in E$, and $b = bs_2b$, where $bs_2, s_2b \in E$. We assume that $E$ is a semilattice, that is, $E$ is a commutative idempotent semigroup with the induced ordering given by $e \leq f$ if and only if $ef = e$. Then

$$ab = a(s_1a)(bs_2)b = a(bs_2)(s_1a)b = ab(s_2s_1)ab.$$ 

Hence $ab \in T$ and $T$ is a subsemigroup of $S$. Since $s_1as_1$ is an inverse of $a$ in $S$, then $s_1as_1$ is in $T$ and $a \in aTa$. Hence $T$ is a regular
semigroup. It follows that $T$ is an inverse semigroup [1, p. 28]. Thus $T$ is an inverse subsemigroup of $S$. Let $c$ be a left zero of $S$. Then $c \in T$ and $c^{-1} = c$. Let $s \in S$. Then $scc = c$ and $scsc = sc$ imply $sc \in T$ and $c^{-1} = sc$. Hence $sc = c$. Since $s$ was arbitrary in $S$, then $c$ is a right zero of $S$. Analogously, if $c$ is a right zero of $S$, then $c$ is a left zero of $S$. Hence $S$ has at most one (left, right) zero.

If $S$ is an $H$-semigroup and $I$ is a right (left) ideal of $S$, then for $b \in S$, $bI \subseteq I(b \subseteq I)$ or $bI = \{c\}$, where $c$ is a left zero ($Ib = \{c\}$, where $c$ is a right zero) [3]. Using this, we get that a right (left) ideal of an $H$-semigroup $S$ such that $E$ is commutative is a two-sided ideal, and it follows that, for each $e$ in $E$, for each $a$ in $S$, $ea = a$ if and only if $ae = a$.

**Theorem 2.** Let $S$ be an $H$-semigroup such that the subsemigroup $E$ of idempotents of $S$ is a semilattice. Then the set $T$ of regular elements of $S$ is an inverse semigroup which is a semilattice of disjoint groups.

**Proof.** Let $a \in T$. Then there exists a unique element $a^{-1}$ in $T$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Since $aa^{-1}$, $a^{-1}a \in E$, we have $a(aa^{-1}) = a$ and $(a^{-1}a)a = a$. Hence

$$a^{-1}a = a^{-1}(aaa^{-1}) = (a^{-1}aa)a^{-1} = aa^{-1}.$$ 

It follows that $T$ is a union of disjoint groups [1, ex. 10, p. 34]. Let $G_e = \{b \in T: bb^{-1} = e\}$. Then $G_e$ is a maximal subgroup of $T$ and $T = \bigcup \{G_e: e \in E\}$, where $G_e \cap G_f = \emptyset$ for $e \neq f$. As in [2], we get that $T$ is a semilattice of disjoint groups.

2. For the remainder of this work, unless otherwise indicated, we assume not only that $S$ is an $H$-semigroup such that the subsemigroup $E$ of idempotents of $S$ is a semilattice, but also that $S$ is a periodic semigroup [1, p. 20]. Let $P_e = \{s \in S: s^n = e\}$ for some positive integer $n$. Let $T$ be the inverse subsemigroup of regular elements of $S$. Clearly $P_e \cap T = G_e \subseteq P_e$. Let $P_e - G_e = W_e$ and let $a \in W_e$, where $a^n = e$. Then

$$(ae)^n = (a^{n+1})^n = (a^n)^{n+1} = e \implies ae \in P_e,$$

and

$$ae(ae)^{n-1}ae = (ae)(ae)^n = ae^2 = ae \implies ae \in T.$$ 

Hence, $ae = aa^n = a^na = ea \in G_e$ and, for each $b$ in $G_e$, $ab = aeb \in G_e$ and $ba = bea \in G_e$, so that $G_e$ is an ideal in $P_e$. Let $T_e = \bigcup \{P_f: e \leq f\}$. 
LEMMA 3.1. \( ae \in G_e \iff a \in T_e \).

Proof. Let \( a \in T_e \). Then there exists \( f \geq e \) such that \( a \in P_f \) and \( af \in G_f \). Hence \( afe \in G_{ef} \), that is, \( ae \in G_e \). Conversely, if \( ae \in G_e \), then there exists \( b \in G_e \) such that \( aeb = ab = e \). Say \( a \in P_f \), where \( ax = f \). Then \( fb^n \in G_{ef} \) and

\[
fb^n = a^n b^n = a^{n-1} a b b^{n-1} = a^{n-1} e b^{n-1} = \ldots = ab \cdot ab = ab = e .
\]

Thus \( fb^n \in G_{ef} \cap G_e \). But this implies \( ef = e \) so that \( e \leq f \). Hence \( a \in T_e \).

LEMMA 3.2. For each \( e \) in \( E \), \( T_e \) is a subsemigroup of \( S \), and if \( a \in T_e \) and there exists \( b \in S \) such that \( ab \in T_e \), then \( b \in T_e \).

Proof. Let \( a, b \in T_e \), say \( a \in P_f \) and \( b \in P_h \), where \( e \leq f, h \). Then \( af \in G_f \) and \( bh \in G_h \) imply that \( afbh = abf h \in G_{fh} \) so that \( ab \in T_{fh} \). Now \( ef = e \) and \( eh = e \) imply that \( efh = e \) so that \( e \leq f, h \). Hence \( ab \in T_e \) and \( T_e \) is a subsemigroup of \( S \). Let \( S - T_e = T' \) and suppose \( e \) is not minimum so that \( T'_e \neq \emptyset \). Let \( a \in T_e \) and suppose there exists \( b \in S \) such that \( ab \in T_e \). Assume \( b \in T_e \). Then \( abe \in G_e \) and \( be \in G_e \) imply \( abe(be)^{-1} = ae \) is in \( G_e \) so that \( a \in T_e \), contradiction.

LEMMA 3.3. For each \( f \) in \( E \), \( T_f \) is an H-semigroup of \( S \), and if \( f \) is not minimum in \( E \), then \( T'_f \neq \emptyset \) and \( T'_f \) is an ideal of \( S \).

Proof. Let \( f \in E \). Let \( U_a = \{ b \in S : xb \in T_f \} \). Define \( \sigma \) on \( S \) by

\[
\alpha \sigma b \iff U_a = U_b .
\]

Clearly \( \sigma \) is a (right) congruence on \( S \). Let \( a, b \in T_f \). Then, using Lemma 3.2, we have

\[
x \in U_a \iff ax \in T_f \iff x \in T_f \iff bx \in T_f \iff x \in U_b .
\]

Thus \( U_a = U_b \) and \( \alpha \sigma b \). Further, if \( \alpha \sigma b \) and \( a \in T_f \), then, for each \( x \) in \( T_f \), \( x \in U_a = U_b \). In particular, \( a \in U_b \) so that \( ba \in T_f \) and, using Lemma 3.2, \( b \in T_f \). Thus \( T_f \) is an equivalence class of \( \sigma \). Since \( f \in U_f, U_f \neq \emptyset \). Let \( a \in S \).

\[
x \in U_a \iff ax \in T_f \iff fax \in T_f \iff x \in U_{fa} .
\]

Then \( U_a = U_{fa} \) and \( (fa) \sigma a \), for each \( a \) in \( S \). Let \( x \in U_{af} \). Then \( afx \in T_f \). Now \( (fx) \sigma x \) implies \( (afx) \sigma (ax) \), so that \( ax \in T_f \) and \( x \in U_a \).
Then $U_{af} \subseteq U_a$. Let $x \in U_a$. Then $ax \in T_f$ and $(fax)\sigma(ax)$. As before, $(fx)\sigma x$ implies $(afx)\sigma(ax)$. Hence, $(fax)\sigma(afx)$ implies $afx \in T_f$ so that $x \in U_{af}$. Then $U_a \subseteq U_{af}$ and $(af)\sigma a$, for each $a$ in $S$. Therefore $f$ is an identity for $\sigma$ and $\sigma$ is modular. Let $\rho$ be any congruence on $S$ such that $T_f$ is an equivalence class of $\rho$ and assume $\sigma < \rho$. Then there exist $a, b$ in $S$ such that $a\rho b$ and $a\rho b$, that is, there exists $x \in U_a$ such that $x \in U_b$, which implies that $ax \in T_f$ and $bx \in T_f$. But $a\rho b$ implies $(ax)\rho(bx)$ so that $bx \in T_f$, contradiction. Therefore, $\sigma = \rho$ and $\sigma$ is maximal with respect to having $T_f$ as a $\sigma$-class. Let $a \in T'_f$ and assume $x \in U_a$. Then $ax \in T_f$. Thus we have

$$
(ax)\sigma f \implies (a^2x)\sigma(af)\sigma a \implies (a^2x^2)\sigma(ax) \\
\implies (a^2x^2)\sigma f \implies (a^2x^2)\sigma(af)\sigma a \implies (a^2x^2)\sigma(ax) \\
\implies (a^2x^2)\sigma f \implies \cdots \\
\implies (a^2x^n)\sigma f, \text{ for each positive integer } n.
$$

Let $a^i = h$, where $h \in T_f$. Since $ax \in T'_f$, then $x \in T'_f$. Let $x^i = k$, where $k \in T_f$. Then we have

$$
(a^i x^i)\sigma f \implies (hk)\sigma f \implies hk \in T_f.
$$

But $h, k \in T_f$ implies $hk \in T_f$, contradiction. Hence, for each $a \in T'_f$, $U_a = \emptyset$. It follows that $T'_f$ is a $\sigma$-class and $T'_f$ is an ideal of $S$. Let $\rho$ be any right congruence on $T_f$. Define $\rho'$ on $S$ by

$$
a \rho' b \iff a, b \in T_f \text{ and } a \rho b \text{ or } a, b \in T'_f.
$$

Clearly $\rho'$ is a congruence on $S$ and the restriction of $\rho'$ to $T_f$ is $\rho$. Thus $\rho$ is a left congruence on $T_f$. By analogous proof, any left congruence on $T_f$ is a right congruence. Thus $T_f$ is an $H$-semigroup of $S$.

With the preceding lemmas, we are now in a position to prove the main results of this section.

**Theorem 3.** If $S$ is a periodic $H$-semigroup such that the subsemigroup $E$ of idempotents of $S$ is commutative, then $S$ is a semilattice of disjoint one-idempotent $H$-semigroups. Moreover, every subgroup of $S$ is a Hamiltonian group.

**Proof.** First we show that for each $e$ in $E$, $G_e$ is a Hamiltonian group. If $e = 0$, then $G_e$ is trivially Hamiltonian. Assume $e \neq 0$. Let $\sigma$ be a right congruence on $G_e$, let $H_e$ be the subgroup of $G_e$ induced by $\sigma$ and let $a, b \in T_e$. Write
By a straight-forward argument, \( \sigma^{(e)} \) is an equivalence relation on \( T_e \), so we need only show right compatibility. Accordingly, assume \( a \sigma^{(e)} b \) and \( c \in T_e \). Then \( (ea) \sigma (eb) \) and \( ec \in G_e \) imply \( (eac) \sigma (ebec) \) so that \( (a) \sigma^{(e)} (b) \sigma^{(e)} (bc) \). Clearly, \( \sigma^{(e)} \) restricted to \( G_e \) is \( \sigma \). Since \( T_e \) is an \( H \)-semigroup, then \( \sigma^{(e)} \) is a congruence on \( T_e \). Hence \( \sigma \) is a congruence on \( G_e \). Similarly, any left congruence on \( G_e \) is a congruence so that \( G_e \) is Hamiltonian.

We can now prove that, for each \( f \in E \), \( P_f \) is an \( H \)-semigroup. Let \( a, b \in P_f \). Since \( a, b \in T_f \), then \( ab \in T_f \). Assume \( ab \in P_f \). Then \( ab \in P_k \subseteq T_k \), where \( f < k \), for some \( k \in E \), so that \( a, b \in T_k' \). But then \( ab \in T_k' \), since \( T_k' \) is an ideal, contradiction. Therefore \( ab \in P_f \) and \( P_f \) is a semigroup of \( S \). Let \( \sigma \) be any right congruence on \( P_f \). Then \( \sigma \) induces a normal subgroup \( H_f \) of \( G_f \). Define \( \sigma' \) on \( T_f \) by

\[
a \sigma' b \iff a, b \in P_f \quad \text{and} \quad a \sigma b \text{ or } H_f a = H_f b.
\]

A straight-forward argument shows that \( \sigma' \) is a congruence on \( T_f \). Similarly, any left congruence on \( P_f \) is a congruence. Therefore \( P_f \) is an \( H \)-semigroup.

Suppose there exists \( a \in P_e \), \( b \in P_f \) such that \( ab \in P_{ef} \), say \( ab \in P_k \), for some \( k \in E \). Now \( a \in P_e \) implies \( ae \in G_e \), and \( b \in P_f \) implies \( bf \in G_f \) so that \( abef \in G_{ef} \) and \( ab \in T_{ef} \). Then \( ef < k \). If \( a \in T_k' \) or \( b \in T_k' \), then \( ab \in T_k' \), since \( T_k' \) is an ideal. Thus we must have \( a, b \in T_k' \). But then \( k \leq e, f \) so that \( k \leq ef \), contradiction. Thus \( ab \in P_{ef} \). Since, for each \( a \) in \( S \), \( \langle a \rangle \) has exactly one idempotent [1, p. 20], it follows that \( P_e \cap P_f = \emptyset \) for \( e \neq f \). This completes the proof of Theorem 3.

The obvious corollary follows from Theorem 1.

**Corollary 3.1.** If \( S \) is a periodic \( H \)-semigroup, then either the idempotents of \( S \) are commutative and \( S \) is a semilattice of disjoint one-idempotent \( H \)-semigroups; or the idempotents of \( S \) are not commutative and \( S = \bigcup \{ S_i; i \in I \} \), where the \( S_i \) are disjoint, the idempotents of each \( S_i \) are commutative and each \( S_i \) is a semilattice of disjoint one-idempotent \( H \)-semigroups. Moreover, every subgroup of \( S \) is a Hamiltonian group.

3. In this section we examine the \( t \)-semisimple periodic \( H \)-semigroups. However, our first result in this investigation is more general.

**Theorem 4.** If \( S \) is a \( t \)-semisimple \( H \)-semigroup, then the
idempotents of $S$ are commutative.

Proof. Let $S$ be a $t$-semisimple $H$-semigroup and assume that the idempotents of $S$ are not commutative. Then $S = \bigcup \{S_i; i \in I\}$, as in Theorem 1. Let $\sigma$ be a maximal modular congruence on $S$ with identity $x$. Say $x \in S_i$. Let $s \in S$, say $s \in S_j$, $i \neq j$. Since either $S_i S_j = \{x_j\}$, where $x_j$ is the zero of $S_j$, or $S_i S_j = \{x_i\}$, where $x_i$ is the zero of $S_i$, then $(xs)\sigma s \sigma (sx)$ implies $x_i \sigma x_j$ or $x_j \sigma x_i$. In either case, for every modular congruence $\sigma$ on $S$, $W_{a} = \{x_i; i \in I\}$ is contained in a $\sigma$-class. Since $S$ is $t$-semisimple then $W_{a}$ must be a singleton set. But then the idempotents of $S$ are commutative, contrary to the assumption.

In identifying the maximal modular congruences on a periodic $H$-semigroup where $E$ is a semilattice, we find the classification to be quite similar to that of inverse $H$-semigroups [2].

**Lemma 5.1.** If $\sigma$ is a maximal modular congruence on the periodic $H$-semigroup $S$, where the idempotents of $S$ are commutative, then either $\sigma$ is cancellative or $\sigma$ has exactly two equivalence classes, one of which is an ideal of non-identities for $\sigma$ and the other the semigroup of identities for $\sigma$.

Proof. Let $\sigma$ be a maximal modular congruence on the periodic $H$-semigroup $S$ where the idempotents of $S$ form a semilattice. Let $a$ be an identity for $\sigma$, say $a \in P_f$, where $a^* = f$. Then, for each $s$ in $S$,

$$(as)\sigma s \implies (a^s)\sigma (as)\sigma s \implies \cdots \implies (a^s)\sigma s \implies (fs)\sigma s,$$

and similarly $(sf)\sigma s$. Hence $f$ is an identity for $\sigma$.

Suppose $\sigma$ is cancellative. Let $e, f \in E$, where $e$ is an identity for $\sigma$. Then

$$(ef)\sigma f \implies (ef)\sigma (ff) \implies e\sigma f.$$ 

Hence $E \subseteq \sigma_e$, the $\sigma$-class containing $e$. Conversely, suppose $E \subseteq \sigma_e$ and assume $(ac)\sigma (bc)$ where $c \in P_f$. Since $e$ is an identity for $\sigma$ and, for each $f$ in $E$, $e \sigma f'$, then $(fs)\sigma \sigma (sf')$, for each $s$ in $S$, so that each idempotent is an identity for $\sigma$. Let $e^* = f$. Then $(ac)\sigma (bc)$ implies $(ae^*)\sigma (be^*)$ so that $(af)\sigma (bf)$, and, since $(af)\sigma a$ and $(bf)\sigma b$, then $a \sigma b$ and $\sigma$ is right cancellative. Similarly, $\sigma$ is left cancellative.

Suppose $\sigma$ is not cancellative and let $e \in E$ be an identity for $\sigma$. If $h$ is an identity for $\sigma$, where $h \in E$, then $h \sigma (eh) \sigma e$ and $h \in \sigma_e$. Since $\sigma$ is not cancellative, there exists $f \in E$ such that $f \in \sigma_e$, so that $f$ is not an identity for $\sigma$. Let $I = \{f \in E; f$ is not an identity
for \( \sigma \). Let \( J = \bigcup \{ P_f : f \in I \} \). It follows that \( I \) is an ideal in \( E \), \( J \) is an ideal in \( S \) and \( J' \) is a semigroup of \( S \). Oehmke [4] has shown that if \( \sigma \) is a maximal congruence on \( S \) and \( J \) is any ideal of \( S \), then either \( J \) is contained in a \( \sigma \)-class \( S_0 \) (which is also an ideal of \( S \)) or \( J \) contains an element of each \( \sigma \)-class. If \( x \in \sigma \cap J \) then \( x\sigma e \) and \( x \in P_f \) for some \( f \) in \( I \), where \( x = f \). But

\[
x\sigma e \rightarrow x^2 \sigma(xe) \quad \text{and} \quad (xe)\sigma e \rightarrow x^2 \sigma(xe) \\
\rightarrow x^2 \sigma e \rightarrow \cdots \rightarrow x^m \sigma e \rightarrow f \sigma e.
\]

Then \( f \not\in I \), contradiction. Hence \( \sigma_0 \cap J = \emptyset \) and \( J \subseteq S_0 \). Suppose there exists \( b \in S_0 \) such that \( b \in J \), say \( b \in P_h \), where \( h = e \). Let \( f \in I \subseteq S_0 \). Then \( b \sigma f \) implies \((bh)\sigma(fh)\) and \((bf)\sigma f\); and \( h \sigma e \) implies \((fh)\sigma(bf)\). But then \((b^m bh)\sigma(b^{m-1}bf)\) and \( h \sigma(hf) \). It follows that \( h \sigma f \) and \( f \not\in I \), contradiction. Thus \( J = S_0 \). Since \( J \) is an ideal and \( J' \) is a semigroup, the relation \( \sigma^* \), defined by \( a \sigma^* b = a, b \in J \) or \( a, b \in J' \), is a maximal modular congruence on \( S \) [2]. Clearly \( \sigma \subseteq \sigma^* \). Hence \( \sigma = \sigma^* \). Moreover, for each \( a \) in \( J' \), say \( a \in P_f \), and for each \( s \) in \( s \), \( a \sigma e \) implies \((as)\sigma \sigma(sa)\), so that \( J' \) is the semigroup of identities for \( \sigma \). And for each \( b \) in \( J \), say \( b \in P_f \), \( b \) cannot be an identity for \( \sigma \), since then \( f \) would be an identity for \( \sigma \).

Using Lemma 5.1, we can establish the following characterization.

**Theorem 5.** A periodic \( H \)-semigroup \( S \) is \( t \)-semisimple if and only if \( S \) is an inverse semigroup such that for each pair of groups \( G_e, G_f \) in the semilattice, with \( f \neq e \), the homomorphism \( \varphi_{f,e} \) on \( G_f \) into \( G_e \), defined by \( a \varphi_{f,e} = ae \), is a monomorphism; and, for each \( e \) in \( E \), for each \( a \neq e \) in \( G_e \), there exists a subsemigroup \( T_p \) of \( S \) such that \( a \in T_p \) and for each \( f \) in \( E \), \( T_p \cap G_f = H_f \), where \( H_f = G_f \) or \( H_f \) is a maximal subgroup of prime index \( p \) in \( G_f \).

**Proof.** Define \( \rho \) on \( S \) by \( x \rho y \) if and only if there exists \( e \) in \( E \) such that \( ex = ey \). Clearly, \( \rho \) is a congruence on \( S \). If \( \sigma \) is any maximal modular cancellative congruence on \( S \) and \( x, y \in S \) such that \( x \rho y \), then there exists \( e \) in \( E \) such that \( ex = ey \). Hence \((ex)\sigma(ey)\) and \( x\sigma e \). Thus \( \rho \leq \alpha \) where \( \alpha \) is the intersection of all the maximal modular cancellative congruences on \( S \). In view of Lemma 3.3, it is clear that the intersection \( \beta \) of all the maximal modular non-cancellative congruences of \( S \) separates \( S \) into its subsemigroups \( P_f \), where \( f \in E \). Let \( e < f \) and define \( \psi_{f,e} \) from \( P_f \) into \( P_e \) by \( a \psi_{f,e} = ea \). Clearly, \( \psi_{f,e} \) is a homomorphism from \( P_f \) into \( G_e \). Suppose \( S \) is \( t \)-semisimple, that is, \( \tau = \tau \). If \( \psi_{f,e} \) is not a monomorphism then there exist \( a \neq b \) in \( P_f \) with \( ea = eb \) so that \( a \sigma b \). This implies \( a \sigma b \). Since also \( a \sigma b \), then \( a \tau b \) and \( \tau \neq \tau \), contradiction. Thus if \( S \) is
t-semisimple, then every homomorphism $\psi_{f,e}$ is a monomorphism from $P_f$ into $G_e$. Suppose there exists $e$ in $E$ such that $G_e \subseteq P_e$. Then there exists $b \in W_e$ such that $eb = a \in G_e$, that is, $eb = ea$. Then, as before, $a\tau b$ and $\tau \neq \iota$, which is a contradiction. Hence, for each $e$ in $E$, $P_e = G_e$ and $S$ is an inverse semigroup. Considering the characterization of $t$-semisimple inverse $H$-semigroups in [2], the proof is complete.

The corollaries parallel those in [2].

**Corollary 5.1.** $S$ is a periodic $H$-semigroup all of whose maximal modular congruences are cancellative if and only if $S$ is a one-idempotent periodic $H$-semigroup.

**Corollary 5.2.** $S$ is a $t$-semisimple periodic $H$-semigroup all of whose nontrivial maximal modular congruences are not cancellative if and only if $S$ is a semilattice.

**Corollary 5.3.** If $S$ is a $t$-semisimple periodic $H$-semigroup, then $S$ is a semilattice of disjoint $t$-semisimple Hamiltonian groups.

**Corollary 5.4.** If $S$ is a $t$-semisimple periodic $H$-semigroup, then $S$ is commutative.

**Corollary 5.5.** If $S$ is a periodic $H$-semigroup with a minimum idempotent $e$, then $S$ is $t$-semisimple if and only if for each semigroup $P_f$ in the semilattice with $f \geq e$, the homomorphism $\psi_{f,e}$ on $P_f$ into $P_e$, defined by $a\psi_{f,e} = ae$, is a monomorphism and $P_e$ is $t$-semisimple.

**Corollary 5.6.** If $S$ is a $t$-semisimple periodic $H$-semigroup with no nontrivial modular congruences, then $S$ is either a cyclic group of prime order or the unique semilattice of two elements.

**Corollary 5.7.** If $S$ is a periodic $H$-semigroup with zero, then $S$ is $t$-semisimple if and only if $S$ is a semilattice.

**References**


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