ON A REPRESENTATION OF A STRONGLY HARMONIC RING
BY SHEAVES

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A ring $R$ is strongly harmonic provided that if $M_1$, $M_2$ are a pair of distinct maximal modular ideals of $R$, then there exist ideals $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \nsubseteq M_1$, $\mathfrak{B} \nsubseteq M_2$ and $\mathfrak{A} \mathfrak{B} = 0$. Let $\mathcal{M}(R)$ be the maximal modular ideal space of $R$. If $M \in \mathcal{M}(R)$, let $O(M) = \{ r \in R \mid \text{for some } y \in M, rxy = 0 \text{ for every } x \in R \}$. Define $\mathcal{B}(R) = \bigcup \{ R/O(M) \mid M \in \mathcal{M}(R) \}$.

If $R$ is a strongly harmonic ring with $1$, then $R$ is isomorphic to the ring of global sections of the sheaf of local rings $\mathcal{B}(R)$ over $\mathcal{M}(R)$. Let $\Gamma(\mathcal{M}(R), \mathcal{B}(R))$ be the ring of global sections of $\mathcal{B}(R)$ over $\mathcal{M}(R)$. For every unitary (right) $R$-module $A$, let $A_M = \{ a \in A \mid aRx = 0 \text{ for some } x \in M \}$ and let $\widetilde{A} = \bigcup \{ A/M \mid M \in \mathcal{M}(R) \}$. Define $\delta(M) = a + A_M$ and $\rho(M) = r + O(M)$ for every $a \in A$, $r \in R$ and $m \in \mathcal{M}(R)$.

Then the mapping $\xi_A: a \mapsto \widetilde{a}$ is a semi-linear isomorphism of $A$ onto $\Gamma(\mathcal{M}(R), \mathcal{B}(R))$—module $\Gamma(\mathcal{M}(R), \widetilde{A})$ in the sense that $\xi_A(aR) = \widetilde{a}r$ for every $a \in A$ and $r \in R$.

1. If $R$ is a ring with $1$, $R$ is called harmonic (or regular) if the maximal modular ideal space, say $\mathcal{M}(R)$, with the hull-kernel topology, is a Hausdorff space (refer [5]). A ring $R$ is strongly harmonic provided that for any pair of distinct maximal modular ideals $M_1$, $M_2$ there exist ideals $\mathfrak{A}$, $\mathfrak{B}$ in $R$ such that $\mathfrak{A} \nsubseteq M_1$, $\mathfrak{B} \nsubseteq M_2$ and $\mathfrak{A} \mathfrak{B} = 0$. For any nonempty subset $S$ of a ring $R$ define $(S) = \{ r \in R \mid sr = 0 \text{ for every } s \in S \}$ and if $a \in R$ let $aR$, be the principal right ideal generated by $a$. If $M$ is a prime ideal of a ring $R$ let $O(M) = \{ r \in R \mid (rR)^{-1} \nsubseteq M \}$. An ideal $\mathfrak{A}$ of a ring $R$ is called $M$-primary for some maximal modular ideal $M$ of $R$ provided that $M/\mathfrak{A}$ is the unique maximal modular ideal of $R/\mathfrak{A}$ and if $\mathfrak{A}'$ is an ideal of $R$ such that $\mathfrak{A}' \subseteq \mathfrak{A}$ and $\mathfrak{A}' \neq \mathfrak{A}$ then $R/\mathfrak{A}'$ is no longer a local ring (here by a local ring we mean a ring with the unique maximal modular ideal). The principal results in this paper are as follows: Let $R$ be a ring such that if $R/S$ is a local ring for some ideal $S$ of $R$ then $R/S$ has a unit. Then $R$ is strongly harmonic if and only if $O(M)$ is $M$-primary for every maximal modular ideal $M$ of $R$. If $R$ is a strongly harmonic ring with $1$ then $R$ is isomorphic to $\Gamma(\mathcal{M}(R), \mathcal{B}(R))$ the ring of global sections of the sheaf of local rings $\mathcal{B}(R) = \bigcup \{ R/O(M) \mid M \in \mathcal{M}(R) \}$ over $\mathcal{M}(R)$ and if $A$ is a unitary right $R$-module then the mapping $\xi_A: a \mapsto \widetilde{a}$ is a semi-linear isomorphism of $A$ onto $\Gamma(\mathcal{M}(R), \mathcal{B}(R))$—
module $\Gamma(\mathcal{M}(R), \hat{A})$ in the sense that $\xi_A$ is a group isomorphism satisfying $\xi_A(\alpha r) = \hat{a} \hat{r}$ for $\alpha \in A$, $r \in R$ where $\hat{a}(M) = a + A_M$, $\hat{r}(M) = r + O(M)$ for $M \in \mathcal{M}(R)$ and $\hat{A} = \bigcup \{A/A_M | M \in \mathcal{M}(R)\}$, the disjoint union of the family of right $R$-modules $A/A_M$ indexed by $\mathcal{M}(R)$, and $A_M = \{a \in A | (aR)_1 \not\subseteq M\}$. If $R$ is a ring with 1 such that it contains no nonzero nilpotent elements then $R$ is biregular (see [2: p. 104] for definition) if and only if every prime ideal of $R$ is a maximal ideal. Our results here generalize S. Teleman's result that in case $1 \in R$, a strongly semi-simple harmonic ring or a von Neumann algebra can be represented as a ring of global sections of the sheaf of local algebras over its maximal modular ideal space (refer [5], [6] and [7]). The author wishes to express his gratitude to Professors K. H. Hofmann and S. Teleman for their many invaluable suggestions for the preparation of this paper.

2. Let $R$ be a ring and $A$ be a right $R$-module. For each prime ideal $M$ of $R$, define $A_M = \{a \in A | (aR)_1 \not\subseteq M\}$ where $aR_i$ is the submodule of $A$ which is generated by the element $a$ and $(aR)_1 = \{r \in R | aRr = 0\}$. 

**Proposition 2.1.** $A_M$ is a submodule of $A$.

**Proof.** Let $a, b \in A_M$. Then $(a - b)R_1 \subseteq aR_1 + bR_1$ and $((a - b)R)_1 \supseteq (aR_1 + bR_1)_1 = (aR_1)_1 \cap (bR_1)_1 \supseteq (aR_1)_1 \cap (bR_1)_1$. Hence if $a - b \not\in A_M$ then $(aR_1)_1 \cap (bR_1)_1 \not\subseteq M$ and either $(aR_1)_1 \not\subseteq M$ or $(bR_1)_1 \not\subseteq M$ since $M$ is a prime ideal of $R$. Hence either $a \not\in A_M$ or $b \not\in A_M$. This is impossible. Thus $a - b \in A_M$. Now if $r \in R$ and $a \in A_M$ then $arR_1 \subseteq aR_1$ and $(arR_1)_1 \supseteq (aR_1)_1$. Since $(aR_1)_1 \not\subseteq M$, $(arR_1)_1 \not\subseteq M$ and $ar \in A_M$.

**Corollary 2.2.** If $A$ is $R$, whose module multiplication is given by the ring multiplication, then $A_M$ is an ideal of $R$ which is contained in $M$ for any prime ideal $M$ of $R$.

**Proof.** If $O(M)$ is already a right ideal of $R$ by 2.2. Let $r \in R$ and $a \in O(M)$. Then $(raR)_1 \supseteq (aR)_1$. Since $(aR)_1 \not\subseteq M$, $(raR)_1 \not\subseteq M$ and $ra \in O(M)$.

**Proposition 2.3.** If $A$ is a right $R$-module for some ring $R$ then $AO(M) \subseteq A_M$ for any prime ideal $M$ of $R$.

**Proof.** Since $A_M$ is a submodule of $A$, it suffices to show that if $a \in A$ and $x \in O(M)$ then $ax \in A_M$. But this is immediate since $(axR)_1 \supseteq (xR)_1$ and $(xR)_1 \not\subseteq M$. 


**Theorem 2.4.** Let \( R \) be a ring such that if \( \mathcal{P} \) is a proper ideal of \( R \) then there is a maximal modular ideal \( M \) in \( R \) such that \( \mathcal{P} \subseteq M \). Let \( A \) be a right \( R \)-module such that if \( aR = 0 \) for some \( a \in A \) then \( a = 0 \). Then \( \bigcap \{A_M | M \text{ is a maximal modular ideal of } R \} \) is zero.

**Proof.** Let \( a \in \bigcap \{A_M | M \text{ is a maximal modular ideal of } R \} \) such that \( a \neq 0 \). Then \((aR)^\perp \neq R\), for if \((aR)^\perp = R\) then \( aR = 0 \) and \( a = 0 \). Since \((aR)^\perp \neq R\), \((aR)^\perp \) is a proper ideal of \( R \). Hence there is a maximal modular ideal \( M \) in \( R \) such that \((aR)^\perp \subseteq M\). This means that \( a \in A_M \) and \( a \notin \bigcap \{A_M | M \text{ is a maximal modular ideal of } R \} \). This is a contradiction.

**Corollary 2.5.** If \( R \) is a ring with 1 and \( A \) is a unitary right \( R \)-module, then \( \bigcap \{A0(M) | M \text{ is a maximal ideal of } R \} \) is zero.

**Proof.** By 2.4, \( \bigcap \{A_M | M \text{ is a maximal ideal of } R \} = 0 \). Since \( A0(M) \subseteq A_M \) for any prime ideal of \( R \) by 2.3, the conclusion now follows.

**Definition 2.6.** We say that a ring \( R \) is **strong harmonic** provided that for any pair of distinct maximal modular ideals \( M_1, M_2 \) there exist ideals \( \mathcal{A}, \mathcal{B} \) in \( R \) such that \( \mathcal{A} \nsubseteq M_1, \mathcal{B} \nsubseteq M_2 \) and \( \mathcal{A} \mathcal{B} = 0 \).

**Proposition 2.7.** If \( R \) is strongly harmonic, then \( \mathcal{M}(R) \) is Hausdorff.

**Proof.** If \( M_1, M_2 \) are distinct maximal modular ideals of \( R \), then, by definition, there exist ideals \( \mathcal{A} \) and \( \mathcal{B} \) such that \( \mathcal{A} \nsubseteq M_1, \mathcal{B} \nsubseteq M_2 \) and \( \mathcal{A} \mathcal{B} = 0 \). Therefore, two open sets \( \{M \in \mathcal{M}(R) | \mathcal{A} \nsubseteq M \} \) and \( \{M \in \mathcal{M}(R) | \mathcal{B} \nsubseteq M \} \) are disjoint.

**Example 2.8.** Let \( R \) be a strongly semi-simple ring, that is a ring in which the intersection of maximal modular ideals is zero. If the maximal modular ideal space, \( \mathcal{M}(R) \) with the hull-kernel topology, is a Hausdorff space, then \( R \) is strongly harmonic.

**Example 2.9.** If \( R \) is a ring with 1 such that it is strongly harmonic then it is harmonic. However, if \( 1 \in R \) then a strongly harmonic ring may not be harmonic. For example, let \( R \) be the algebra of sequences \((a_n)_{n \geq 0}\) of \( 2 \times 2 \)-matrices over the field of complex numbers \( C \), such that \( a_n \to \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \) for \( n \to \infty \) for some \( \lambda \in C \). Then...
the intersection of the maximal modular ideals of \( R \) is zero and \( \mathcal{M}(R) \) is Hausdorff. Hence \( R \) is strongly harmonic; however, it is not harmonic.

**Example 2.10.** Let \( R \) be a von Neumann algebra. Then for any distinct pair of maximal ideals \( M_i, M_j \) there exist central idempotents \( e_i, e_j \) in \( R \) such that \( e_i \notin M_j, e_j \notin M_i \) and such that \( e_i \cdot e_j = 0 \). Hence \( R \) is strongly harmonic.

**Example 2.11.** Let \( Q \) be the field of rational numbers and let \( p_1, p_2, \ldots, p_l \) be a finite number of distinct prime numbers. Let \( R = \{ m/n \in Q \mid n \) is not divisible by any \( p_i, 1 \leq i \leq l \} \). Then \( \mathcal{M}(R) \) consist of \( l \) points and it is a Hausdorff space. However, since \( R \) is an integral domain, \( R \) is not strongly harmonic if \( l > 1 \).

**Definition 2.12.** Let \( R \) be a ring and \( M \) be a maximal modular ideal of \( R \). An ideal \( \mathfrak{P} \) in \( R \) is said to be \( M \)-primary, for some maximal modular ideal \( M \) of \( R \), provided that \( \mathfrak{P} \subseteq M, R/\mathfrak{P} \) is a ring with a unique maximal modular ideal \( M/\mathfrak{P} \), and if \( P \) is an ideal of \( R \) such that \( P \subseteq \mathfrak{P} \) and \( P \neq \mathfrak{P} \), then \( R/P \) is not a local ring. Here, by a *local ring* we mean a ring with a unique maximal modular ideal.

**Proposition 2.13.** Let \( R \) be a ring and \( M \) be a maximal modular ideal of \( R \). If an \( M \)-primary ideal, say \( \mathfrak{P} \), exists, then it is unique.

*Proof.* Let \( \mathfrak{P} \) be a \( M \)-primary ideal of \( R \). If either \( \mathfrak{P} \subseteq \mathfrak{P} \) or \( \mathfrak{P} \subseteq \mathfrak{P} \) then, by definition, \( \mathfrak{P} = \mathfrak{P} \). So assume \( \mathfrak{P} \cap \mathfrak{P} \) is properly contained in \( \mathfrak{P} \) or \( \mathfrak{P} \). Then the ideal \( \mathfrak{P} \mathfrak{P} \) is properly contained in \( \mathfrak{P} \) and \( R/\mathfrak{P} \mathfrak{P} \) is not a local ring. Hence there is a maximal modular ideal \( N \) in \( R \) such that \( N \neq M \) and \( \mathfrak{P} \mathfrak{P} \subseteq N \). Since \( N \) is a prime ideal, this means that either \( \mathfrak{P} \subseteq N \) or \( \mathfrak{P} \subseteq N \). In either case, this means that \( \mathfrak{P} \) or \( \mathfrak{P} \) is not \( M \)-primary. This is a contradiction.

**Proposition 2.14.** Let \( R \) be a ring such that if \( R/\mathfrak{P} \) is a local ring for some ideal \( \mathfrak{P} \) in \( R \), then \( R/\mathfrak{P} \) has a unit. If \( R/\mathfrak{O}(M) \) is a local ring for some maximal modular ideal \( M \) in \( R \), then \( \mathfrak{O}(M) \) is \( M \)-primary.

*Proof.* Observe that \( \mathfrak{O}(M) \subseteq M \). Hence \( M/\mathfrak{O}(M) \) is the unique maximal modular ideal of the local ring \( R/\mathfrak{O}(M) \). Let \( \mathfrak{P} \) be an ideal of \( R \) such that \( \mathfrak{P} \subseteq \mathfrak{O}(M), \mathfrak{P} \neq \mathfrak{O}(M) \) and \( R/\mathfrak{P} \) is a local ring. Let \( t \in \mathfrak{O}(M) \) such that \( t \in \mathfrak{P} \). Then \( (tR) \mathfrak{P} \subseteq M \). If \( \mathfrak{P} \mathfrak{P} \neq \mathfrak{P} \), then...
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$R$ then there is a maximal modular ideal $N$ in $R$ such that \( \mathcal{P} + (tR)^{\perp} \subseteq N \), since $R/\mathcal{P}$ has a unit. Since $(tR)^{\perp} \nsubseteq M$, this means that $M \neq N$. This is impossible. Hence $R = \mathcal{P} + (tR)^{\perp}$. Let $e + \mathcal{P}$ be the identity of $R/\mathcal{P}$ for some $e \in R$. Then $e = p + s$ for some $p \in \mathcal{P}$ and $s \in (tR)^{\perp}$. Hence $te = tp$ and $t - te = t - tp \in \mathcal{P}$. This means that $t \in \mathcal{P}$ and this is a contradiction. Thus $O(M)$ must be $M$-primary.

**Theorem 2.15.** Let $R$ be a ring such that if $R/\mathcal{P}$ is a local ring for some ideal $\mathcal{P}$, then it has a unit. Then $R$ is strongly harmonic if, and only if, $O(M)$ is $M$-primary for every maximal modular ideal $M$ in $R$.

\textit{Proof.} Assume $R$ is strongly harmonic. By 2.14, it suffices to show that $R/O(M)$ is a local ring for each maximal modular ideal $M$ of $R$. If $R/O(M)$ is not a local ring for some maximal modular ideal $M$, then there is a maximal modular ideal $N$ in $R$ such that $N \neq M$ and $O(M) \subseteq N$. Since $R$ is strongly harmonic, there exist ideals $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \not\subseteq N$, $\mathcal{B} \not\subseteq M$ and $\mathcal{A} \mathcal{B} = 0$. This means that $\mathcal{A} \subseteq O(M)$. Since $O(M) \subseteq N$, $\mathcal{A} \subseteq N$. This is a contradiction. Conversely, assume $O(M)$ is $M$-primary for each maximal modular ideal $M$ of $R$. Let $M_1, M_2$ be two distinct maximal modular ideals of $R$. Then $O(M_1) \nsubseteq M_2$ and $O(M_2) \nsubseteq M_1$. Hence there exist $a \in O(M_1)$ such that $a \notin M_2$ and $b \in O(M_2)$ such that $b \notin M_1$. Then (b), the ideal generated by $b$, is not contained in $M$. Let $\mathcal{A} = (b)$ and let $\mathcal{B} = (bR)^{\perp}$. Then $\mathcal{A} \nsubseteq M_1$, $\mathcal{B} \nsubseteq M_2$ and $\mathcal{A} \mathcal{B} = 0$.

**Remark 2.16.** If $R$ is a strongly semi-simple ring with 1 such that $\mathcal{M}(R)$, the maximal modular ideal space of $R$, is a Hausdorff space, then by [5: Theorem 6.5] and [5: Theorem 6.15], the $M$-primary ideal exists for each maximal modular ideal $M$ in $R$. In this case, the $M$-primary ideal $p(M)$ is given by the set $\{x \in R \mid \text{supp}(RxR) \cap \{M\} = \phi\}$, where $\text{supp}(RxR) = \{M \in \mathcal{M}(R) \mid RxR \nsubseteq M\}$ by [5: Theorem 6.14].

3. If $\mathcal{A}$ is an ideal of a ring $R$, let

\[
\text{supp}(\mathcal{A}) = \{M \in \mathcal{M}(R) \mid \mathcal{A} \nsubseteq M\}, \quad h(\mathcal{A}) = \mathcal{M}(R) \setminus \text{supp}(\mathcal{A}), \quad k(\mathcal{A}) = \bigcap \{M \in \mathcal{M}(R) \mid M \in F\}.
\]

**Theorem 3.1.** Let $R$ be a ring and let

\[
\mathcal{B}(R) = \bigcup \{R/O(M) \mid M \in \mathcal{M}(R)\},
\]

the disjoint union of a family of rings $\{R/O(M) \mid M \in \mathcal{M}(R)\}$. For
each \( r \in R \) define \( \hat{r} \) to be the function from \( \mathcal{A}(R) \) into \( \mathcal{B}(R) \) such that \( \hat{r}(M) = r + O(M) \) for each \( M \in \mathcal{A}(R) \). Let \( \tau = \{ \hat{r}(U) \mid r \in R \) and \( U \) is an open set in \( \mathcal{A}(R) \} \). Let \( \rho \) be a family of sets consisting of arbitrary unions of the members of \( \tau \). Then \( (\mathcal{B}(R), \rho) \) is a topological space and each point \( \hat{r}(M) \) of \( \mathcal{B}(R) \), \( r \in R \) and \( M \in \mathcal{A}(R) \), is contained in an open set which is homeomorphic to an open set of \( \mathcal{A}(R) \) under the canonical projection: \( \hat{r}(M) \to M \), that is, \( \mathcal{B}(R) \) is a sheaf of rings over \( \mathcal{A}(R) \).

**Proof.** In \( \eta \in \hat{r}_1(U) \cap \hat{r}_2(V) \) for some \( r_1, r_2 \in R \) and some open sets \( U, V \) in \( \mathcal{A}(R) \) then there is \( M \in U \cap V \) such that \( r_1 - r_2 \in O(M) \). Hence \( (r_1 - r_2)R_i \not\subseteq M \). Let \( W = U \cap V \cap \supp((r_1 - r_2)R_i) \). Then \( M \in W \) and \( \eta \in \hat{r}_1(W) \subseteq \hat{r}_1(U) \cap \hat{r}_2(V) \). Since \( W \) is an open set of \( \mathcal{A}(R) \), \( \hat{r}_1(W) \in \tau \) and hence \( (\mathcal{B}(R), \rho) \) is a topological space. In view of [1: 2.2 p. 151], it suffices to show that if \( \hat{r}(M) = 0 \) for some \( r \in R \) and \( M \in \mathcal{A}(R) \) then there exists an open set \( U \) of \( M \) such that \( \hat{r}(U) = 0 \). But this is immediate since if \( \hat{r}(M) = 0 \) then \( r \in O(M) \) and \( (rR_i) \not\subseteq M \). Therefore, if we let \( U = \supp((rR_i) \not\subseteq M) \) then \( \hat{r}(U) = 0 \) since \( r \in \bigcap \{ O(M) \mid M \in U \} \).

**Theorem 3.2.** Let \( R \) be a strongly harmonic ring. If \( F \) is a compact subset of \( \mathcal{A}(R) \) and \( M_0 \in F \) for some \( M_0 \in \mathcal{A}(R) \) then there exist ideals \( \mathcal{A} \) and \( \mathcal{B} \) such that \( \mathcal{A} \mathcal{B} = 0 \), \( M_0 \in \supp(\mathcal{A}) \) and \( F \subseteq \supp(\mathcal{B}) \).

**Proof.** Since \( R \) is strongly harmonic, for any \( M \in F \) there exist ideals \( \mathcal{A}', \mathcal{B}' \) in \( R \) such that \( M_0 \in \supp(\mathcal{A}') \), \( M \in \supp(\mathcal{B}') \) and \( \mathcal{A}' \mathcal{B}' = 0 \). Since \( F \) is compact, there exist a finite number of ideals, say \( \mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}, \mathcal{B}_1, \ldots, \mathcal{B}_n \) such that

\[
M_0 \in \bigcap_{i=1}^n \supp(\mathcal{A}_i) = \supp(\mathcal{A}_1 \mathcal{A}_2 \cdots \mathcal{A}_n)
\]

and \( F \subseteq \bigcup_{i=1}^n \supp(\mathcal{B}_i) = \supp(\sum_{i=1}^n \mathcal{B}_i) \) such that \( \mathcal{A}_i \mathcal{B}_i = 0 \) for all \( i = 1, 2, \ldots, n \), and \( \mathcal{A} \mathcal{B}_i \subseteq 0 \).

**Theorem 3.3.** Let \( R \) be a strongly harmonic ring. If \( F \) is a compact subset of \( \mathcal{A}(R) \) then \( F = h(\bigcap \{ O(M) \mid M \in F \}) \).

**Proof.** Since \( \bigcap_{M \in F} O(M) \subseteq k(F) \), \( F \subseteq h(\bigcap_{M \in F} O(M)) \). Suppose there is \( M_0 \in h(\bigcap_{M \in F} O(M)) \) such that \( M_0 \in F \). Then by 3.2 there exist ideals \( \mathcal{A}, \mathcal{B} \) in \( R \) such that \( M_0 \in \supp(\mathcal{A}) \), \( F \subseteq \supp(\mathcal{B}) \) and \( \mathcal{A} \mathcal{B} = 0 \). Hence if \( M \in F \) then \( \mathcal{B} \not\subseteq M \) and \( \mathcal{A} \subseteq O(M) \). Thus \( A \subseteq \bigcap_{M \in F} O(M) \). Since \( M_0 \in h(\bigcap_{M \in F} O(M)) \), this means that \( \mathcal{A} \subseteq M_0 \) and this is a contradiction.
THEOREM 3.4. Let $R$ be a strongly harmonic ring with $1$ and let $\mathcal{R}(R)$ be the sheaf of local rings over $\mathcal{M}(R)$, which is described in 3.1. If $F_0$ is a compact subset of $\mathcal{M}(R)$ and $\sigma$ is a section from $F_0$ into $\mathcal{R}(R)$, then there is $r \in R$ such that $\hat{r} |_{F_0} = \sigma$.

Proof. If $M_0 \in F_0$ then there exists an open set $U$ in $\mathcal{M}(R)$ which contains $M_0$ and $r \in R$ such that if $M \in U \cap F_0$ then $\sigma(M) = \hat{r}(M)$. Let $U_0 = \mathcal{M}(R) \setminus F_0$. Since $\mathcal{M}(R)$ is Hausdorff by 2.7, $F_0$ is a closed set. Hence $U_0$ is an open subset of $\mathcal{M}(R)$. There exist a finite number of points $M_1, M_2, \ldots, M_n$ in $F_0$, open sets $U_1, U_2, \ldots, U_n$ such that $M_i \in U_i$, $i = 1, 2, \ldots, n$, and $r_1, r_2, \ldots, r_n$ in $R$ such that $\sigma(M) = \hat{r}_i(M)$ for every $M \in U_i \cap F_0$ for every $i = 1, 2, \ldots, n$. Furthermore, $F_0 \subseteq \bigcup_{i=1}^{n} U_i$ and $\mathcal{M}(R) = \bigcup_{i=1}^{n} U_i$. Let $F_i = \mathcal{M}(R) \setminus U_i$ and let $I_i = \bigcap_{M \in F_i} O(M)$ for each $i = 0, 1, 2, \ldots, n$. Since $F_i$ is a closed subset of a compact space, it is compact. Hence $F_i = h(I_i)$ for each $i = 0, 1, 2, \ldots, n$ by 3.3. Since $\phi = \bigcap_{i=0}^{n} F_i = \bigcap_{i=0}^{n} h(I_i) = h(\bigcap_{i=0}^{n} I_i)$, $R = \bigcup_{i=0}^{n} I_i$ and $1 = \sum_{i=0}^{n} e_i$ for some $e_i \in I_i$, $i = 0, 1, 2, \ldots, n$. If $M \in F_i \cap F_0$, then $\hat{r}_i(M)\hat{e}_i(M) = O(M) = \sigma(M)\hat{e}_i(M)$. If $M \in U_i \cap F_0$, then $\hat{r}_i(M)\hat{e}_i(M) = \sigma(M)\hat{e}_i(M)$. Hence, for every $M \in F_0$, $\hat{r}_i(M)\hat{e}_i(M) = \sigma(M)\hat{e}_i(M)$. Thus if we let $r = e_0 + \sum_{i=1}^{n} r_i e_i$, then for every $M \in F_0$:

$$\hat{r}(M) = \hat{e}_0(M) + \sum_{i=1}^{n} \hat{r}_i(M)\hat{e}_i(M) = \sigma(M)\hat{e}_0(M) + \sum_{i=1}^{n} \sigma(M)\hat{e}_i(M) = \sigma(M)\left(\sum_{i=0}^{n} \hat{e}_i(M)\right) = \sigma(M).$$

COROLLARY 3.5. If $R$ is a strongly harmonic ring with $1$ then $R \cong \Gamma(\mathcal{M}(R), \mathcal{R}(R))$.

Proof. By 2.5, $r \mapsto \hat{r}$ is a monomorphism from $R$ into $\Gamma(\mathcal{M}(R), \mathcal{R}(R))$. Since $\mathcal{M}(R)$ is a compact space, by 3.4 if $\sigma \in \Gamma(\mathcal{M}(R), \mathcal{R}(R))$ then there is $r \in R$ such that $\sigma = \hat{r}$. Thus $r \mapsto \hat{r}$ is an isomorphism of $R$ onto $\Gamma(\mathcal{M}(R), \mathcal{R}(R))$.

DEFINITION 3.6. We say that a sheaf $\mathcal{R}$ over the space $X$ is soft provided that if $F$ is a compact subset of $X$ and $\sigma \in \Gamma(F, \mathcal{R})$ then there is $\hat{\sigma} \in \Gamma(X, \mathcal{R})$ such that $\hat{\sigma} |_{F} = \sigma$.

THEOREM 3.7.1 Let $R$ be a strongly harmonic ring with $1$. Then the sheaf $\mathcal{R}(R)$ of local rings which is constructed in 3.1 is soft. Conversely, if $\mathcal{R}$ is a soft sheaf of local rings over a Hausdorff compact space $\mathcal{M}$, then $\Gamma(\mathcal{M}, \mathcal{R})$ is a strongly harmonic ring.

1 The author is indebted to Professor S. Teleman for this theorem.
Proof. By 3.4, $R(R)$ is soft if $R$ is a strongly harmonic ring with 1. Suppose now that $R$ is a soft sheaf of local rings over a Hausdorff compact space $\mathcal{M}$. Let $R = \Gamma(\mathcal{M}, R)$. By Theorem 11 of [6: p. 712], $\mathcal{M}$ is homeomorphic to $\mathcal{M}(R)$. Hence we may take $R = \Gamma(\mathcal{M}(R), R)$. Since $\mathcal{M}$ is Hausdorff, if $M_i, M_j \in \mathcal{M}(R)$ such that $M_i \neq M_j$ then there exist open sets $U_i, i = 1, 2$, in $\mathcal{M}(R)$ such that $M_i \subseteq U_i$, $M_j \subseteq U_j$ and $U_i \cap U_j = \phi$. If $\sigma \in R$, define 
\[ |\sigma| = \{M \in \mathcal{M}(R) | \sigma(M) \neq 0\}. \]

Let $A_i = \{\sigma \in R | |\sigma| \subseteq U_i\}$, $i = 1, 2$. Clearly, $A_1, A_2$ are ideals of $R$ and $A_1A_2 = 0 = A_2A_1$ since $U_1 \cap U_2 = \phi$. There exists compact sets $K_1, K_2$ such that $M_i \subseteq K_i$ and $K_i \subseteq U_i$, $i = 1, 2$. Let $F_i = \mathcal{M}(R) \setminus U_i$. Since $R$ is soft there exist $\sigma_i$ in $\Gamma(\mathcal{M}(R), R)$ such that $\sigma_i(K_i) = 1$ and $\sigma_i(F_i) = 0$, $i = 1, 2$. Hence $A_i \nsubseteq M_i$ for $i = 1, 2$. Thus $R$ is strongly harmonic.

REMARK 3.8. Let $R$ be a ring and $A$ be a right $R$-module. We will associate with $A$ a sheaf if $R(R)$-modules over $\mathcal{M}(R)$ (refer [4] for definition). For $M \in \mathcal{M}(R)$, denote $\tilde{A} = \bigcup \{A/A_M | M \in \mathcal{M}(R)\}$, the disjoint union of a family of $R$-modules $A/A_M$ indexed by $\mathcal{M}(R)$. Let $\pi: \tilde{A} \to \mathcal{M}(R)$ be given by $\pi^{-1}(M) = A/A_M$. For $a \in A$ and $M \in \mathcal{M}(R)$, let $t_a(M)$ be the image of $a$, under the natural homomorphism of $A$ onto $A/A_M$. Topologize $\tilde{A}$ by taking all sets $t_a(U)$, with $a \in A$, $U$ is an open set in $\mathcal{M}(R)$, as a basis for the open sets. Then $\tilde{A}$ becomes a sheaf of $R(R)$-modules over $\mathcal{M}(R)$. The justification of this statement and proof of this result require only slight modifications of 3.1.

THEOREM 3.9. Let $R$ be a strongly harmonic ring with 1 and let $A$ be a unitary right $R$-module. Then the mapping $\tilde{\xi}_a: a \mapsto t_a$ is a semi-linear isomorphism of $A$ onto the $\Gamma(\mathcal{M}(R), R(R))$-module $\Gamma(\mathcal{M}(R), \tilde{A})$ in the sense that $\tilde{\xi}_a$ is a group isomorphism satisfying $\tilde{\xi}_a(ar) = t_a \cdot \tilde{a}$ for $a \in A$, $r \in R$ where $t_a(M) = a + A_M$ for all $m \in \mathcal{M}(R)$.

Proof. We omit the proof because it is only a variant of the proof of 3.4. However, it is worth noting that the full strength of 2.4 is needed here to prove that $\tilde{\xi}_a$ is an injection.

4. A ring is called biregular if every principal ideal of the ring is generated by a central idempotent. In [2], Dauns and Hofmann proved that if $R$ is a ring with 1 then $R$ is biregular if and only if $R$ is isomorphic to the ring of all global sections of a sheaf of simple rings over a Boolean space. By applying this theorem, we
will show that if $R$ is a ring with 1 such that it contains no nonzero nilpotent elements then $R$ is biregular if, and only if, every prime ideal of $R$ is a maximal ideal of $R$.

**Proposition 4.1.** If $R$ is a biregular ring then every prime ideal $M$ of $R$ is a maximal ideal of $R$.

**Proof.** If $R$ is biregular then so is the ring $R/M$ for any ideal $M$ of $R$. Hence if $M$ is a prime ideal then $R/M$ is a prime biregular ring. Therefore, $R/M$ contains no proper principal ideal for if $R/M$ contains a proper principal ideal, then $R/M$ would have two nonzero ideals whose product is zero. Thus $R/M$ is a simple ring and $M$ is a maximal ideal of $R$.

**Proposition 4.2.** Let $R$ be a ring and $M$ be a prime ideal of $R$. Define $O_M = \{x \in R \mid xy = 0$ for some $y \in M\}$. If $R$ contains no nonzero nilpotent elements then $O_M = O(M)$.

**Proof.** Clearly $O(M) \subseteq O_M$. If $x, y$ are elements of $R$ such that $xy = 0$ then $yx$ is zero since $yxy = 0$ and $R$ contains no nonzero nilpotent elements. Furthermore, if $r \in R$, $xry = 0$ since $xryxry = 0$. Thus $O(M) = O_M$.

**Proposition 4.3.** Let $R$ be a ring without nilpotent elements. If every prime ideal of $R$ is maximal, then $M = O(M)$ for every prime ideal $M$ of $R$.

**Proof.** If every prime ideal of $R$ is maximal, then every prime ideal is a maximal prime ideal. Hence by [3: 2.4], $M = O_M$ for each prime ideal $M$ of $R$. Thus by 4.2 $M = O(M)$.

**Proposition 4.4.** If $R$ is a ring with 1 such that $R$ contains no nonzero nilpotent elements and if every prime ideal of $R$ is maximal, then $\mathcal{M}(R)$ is a Boolean space.

**Proof.** This is a direct consequence of [3: 2.5].

**Theorem 4.5.** Let $R$ be a ring with 1 such that it contains no nonzero nilpotent elements. Then $R$ is biregular if, every prime ideal of $R$ is maximal.

**Proof.** If $R$ is biregular then by 4.1, every prime ideal is maximal. Conversely, suppose that every prime ideal of $R$ is maximal. Since $R$ is a ring without nilpotent elements, the intersection of
prime ideals of $R$ is zero. Since $\mathcal{M}(R)$ is a Hausdorff space by 4.4, if $M_1, M_2$ are two distinct elements in $\mathcal{M}(R)$, then there exist ideals $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \not\subseteq M_1$, $\mathfrak{B} \not\subseteq M_2$ and $\mathfrak{A} \mathfrak{B} = 0$. Hence $O(M)$ is $M$-primary for every $M \in \mathcal{M}(R)$ by 2.13 and thus $R \cong \Gamma(\mathcal{M}(R), \mathcal{B}(R))$ by 3.5. Since $\mathcal{M}(R)$ is a Boolean space by 4.4 and $M = O(M)$ by 4.3, $R$ is a biregular ring by [2: 2.19, p. 108].

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