ON THE ZEROS OF A POLYNOMIAL AND ITS DERIVATIVE

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Let all the zeros of a polynomial \( p(z) \) of degree \( n \) lie in \( |z| \leq 1 \). Given a complex number \( a \) what is the radius of the smallest disk centred at \( a \) containing at least one zero of the polynomial \( ((z-a)p(z))' \)? According to Theorem 1 the answer is \( (|a| + 1)/(n+1) \) if \( |a| > (n+2)/n \). Theorem 2 which states that if both the zeros of the quadratic polynomial \( p(z) \) lie in \( |z| \leq 1 \) and \( |a| \leq 2 \) then \( ((z-a)p(z))' \) has at least one zero in

\[
|z-a| \leq \frac{3}{2} |a| + \frac{12-3|a|^2}{6}.
\]

completely settles the case \( n = 2 \).

For \( |a| \leq 1 \) the question is equivalent to a problem in [1, (see problem 4.5)] which reads as follows: Is it true that if all the zeros \( z_1, z_2, \ldots, z_n \) of the polynomial \( p(z) = c \prod_{i=1}^{n} (z-z_i) \) lie in the disk \( |z| \leq 1 \) then \( p'(z) \) has at least one zero in each of the disks \( |z-z_\nu| \leq 1, \nu = 1, 2, \ldots, n \)? It has been shown by Rubinstein [2] that if all the zeros of the polynomial \( p(z) \) lie in \( |z| \leq 1 \) and \( p(1) = 0 \) then at least one zero of \( p'(z) \) lies in the disk \( |z-1| \leq 1 \). On the other hand, the example \( z^n - 1 \) shows that a disk of radius less than 1 may not contain a zero of \( p'(z) \). Thus when \( |a| = 1 \) the answer to our question is 1.

If \( a \) is arbitrary the problem is trivial for \( n = 1 \) and the answer to the question is \((|a| + 1)/2 = (|a| + 1)/(n+1)\).

For polynomials of arbitrary degree \( n \) we prove

**Theorem 1.** If all the zeros of a polynomial \( p(z) \) of degree \( n \) lie in the closed unit disk then \( ((z-a)p(z))' \) has one and only one zero in \( |z-a| \leq (|a| + 1)/(n+1) \) provided \( a > (n+2)/n \). The remaining \( n-1 \) zeros of \( ((z-a)p(z))' \) lie in \( |z| \leq 1 \). The example \( p(z) = (z+e^{i\alpha})^n \) where \( \alpha = \arg a \) shows that the result is best possible.

The disk \( |z-a| \leq (|a| + 1)/(n+1) \) may contain more than one zero of \( ((z-a)p(z))' \) if \( |a| = (n+2)/n \). That it contains at least one follows from the fact that the zeros of \( ((z-a)p(z))' \) are continuous functions of \( a \).

The next theorem gives a solution of the problem when
| a | ≤ (n+2)/n and n = 2.

**THEOREM 2.** If both the zeros of the quadratic polynomial $p(z)$ lie in $|z| ≤ 1$ and $|a| ≤ 2$ then $((z-a)p(z))'$ has at least one zero in $|z-a| ≤ \{3|a| + (12-3|a|^2)/2\}/6$.

The example

$p(z) = z^2 - 2 [(3-a(12-3a^2)^{1/2})/(3a-(12-3a^2)^{1/2})] z + 1, 0 ≤ a ≤ 2$

shows that the result is best possible.

For the proof of Theorem 2 we shall need the following lemma [3, p. 36].

**LEMMA.** If both the zeros of the polynomial

$$A(z) = a_0 + \binom{2}{1} a_1 z + a_2 z^2$$

lie in $|z| ≥ r$ and those of

$$B(z) = b_0 + \binom{2}{1} b_1 z + b_2 z^2$$

lie in $|z| > s$ then both the zeros of the polynomial

$$C(z) = a_0 b_0 + \binom{2}{1} a_1 b_1 z + a_2 b_2 z^2$$

lie in $|z| > rs$.

**Proof of Theorem 1.** Let

$$p(z) = c \prod_{\nu=1}^{n} (z-z_{\nu})$$

where by hypothesis $|z_{\nu}| ≤ 1, \nu = 1, 2, \cdots, n$. For a given $z_0$ with $|z_0| > 1$ the transformation $1/(z_0-z)$ maps the closed unit disk onto some disk $D(z_0)$ in the finite plane. Thus all the numbers $1/(z_0-z_1), 1/(z_0-z_2), \cdots, 1/(z_0-z_n)$ belong to $D(z_0)$ and hence so does their arithmetic mean $\mu(z_0)$. But there exists a unique point $\phi(z_0)$ in the disk $|z| ≤ 1$ such that $\mu(z_0) = 1/(z_0-\phi(z_0))$. Consequently

$$p'(z_0)/p(z) = n/(z_0-\phi(z_0)).$$

Since $z_0$ in an arbitrary point outside the unit disk we get the representation
\[ p'(z)/p(z) = n/(z-\phi(z)) \]

where \( \phi(z) = z - n\{p(z)/p'(z)\} \) is holomorphic and of absolute value at most 1 in \(|z| > 1\).

If \(|a| > 1\) then

\[ p'(z)/p(z) = n\psi(z)/\{(z-a)\psi(z)-1\} \]

where \( \psi(z) = 1/(\phi(z)-a) \) is holomorphic in \(|z| > 1\) and

(1) \[ 1/(|a|+1) \leq |\psi(z)| \leq 1/(|a|-1) . \]

Since

\[ \{(z-a)p'(z)+p(z)\}/p(z) = (n+1)(z-a)\psi(z)-1)}/\{(z-a)\psi(z)-1\} \]

the zeros of \((z-a)p(z)'\) are the same as the zeros of \((n+1)(z-a)\psi(z)-1\).

Now if \(|a| > (n+2)/n\) and \((|a|+1)/(n+1) < |z-a| < |a|-1\) then from (1)

\[ |(n+1)(z-a)\psi(z)| > 1 . \]

Hence by Rouché’s theorem \((n+1)(z-a)\psi(z)-1, (n+1)(z-a)\psi(z)\) have the same number of zeros in \(|z-a| \leq (|a|+1)/(n+1)\), namely 1.

Given \( \xi \in \{z: |z| \leq 1\} \cup \{z: |z-a| \leq (|a|+1)/(n+1)\} \) we can draw a contour \( C \) such that \( \{z: |z-a| \leq (|a|+1)/(n+1)\} \) and the point \( \xi \) lie in \( C_i \) (the bounded domain determined by \( C \)) whereas \( \{z: |z| \leq 1\} \) lies in \( C_\epsilon \) (the unbounded domain determined by \( C \)). According to the above reasoning \((z-a)p(z)\)' has one and only one zero in \( C_i \). Since we know that the zero lies in \( |z-a| \leq (|a|+1)/(n+1) \) the point \( \xi \) cannot be a zero of \((z-a)p(z)'\). Hence the remaining \( n-1 \) zeros of \((z-a)p(z)'\) lie in \(|z| \leq 1\).

Remark. Theorem 1 may be refined by observing that \((n+1)(z-a)\psi(z)-1 = (n+1)(z-a)(\phi(z)-a)^{-1} - 1\) can vanish only if \(z - na/(n+1) = \phi(z)/(n+1)\). Hence in fact \((z-a)p(z)'\) has one and only one zero in \( D = \{z: |z - na/(n+1)| \leq 1/(n+1)\} \). By considering \( p(z) = (z-z_i)^n \) with an appropriate \( z_i \) in the closed unit disk we see that any given point of \( D \) can be a zero of \((z-a)p(z)'\).

Proof of Theorem 2. Without loss of generality we may suppose \( 0 \leq a \leq 2 \). Let

\[ p(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \]

and put

\[ f(z) = ((z-a)p(z))' = (\alpha_0-\alpha_1) + 2(\alpha_1-\alpha_2)z + 3\alpha_2 z^2 , \]

\[ s = \{3a + (12-3a^2)z^2\}/6 . \]
We wish to prove that $f(z)$ must vanish is $|z-a| \leq s$. If not, both the zeros of

$$B(z) = f(z+a) = a_0 + a_1z + a_2z^2 + \frac{2}{1}(\alpha_1 + 2\alpha_2)z + 3\alpha_2z^2$$

lie in $|z| > s$. Since both the zeros of

$$A(z) = 1 + \frac{2}{1}(1/2)z + (1/3)z^2$$

lie on $|z| = \sqrt{3}$ the lemma implies that both the zeros of the polynomial

$$C(z) = \alpha_0 + a_1z + a_2z^2 + (\alpha_1 + 2\alpha_2)z + \alpha_2z^2 = p(z+a)$$

lie in $|z| > \sqrt{3}s$, i.e., the polynomial $p(z)$ does not vanish in $|z-a| \leq \sqrt{3}s$. We can therefore find a positive number $\varepsilon$ such that the disk $|z - (a-2s)| \leq s - \varepsilon$ contains both the zeros of $p(z)$. Now it can be easily deduced from Theorem 1 that $((z-a)p(z))'$ has one and only one zero in $|z-a| \leq s - \varepsilon/3$. This completes the proof of Theorem 2.

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