

# Pacific Journal of Mathematics

**ON THE ZEROS OF A POLYNOMIAL AND ITS DERIVATIVE**

QAZI IBADUR RAHMAN

## ON THE ZEROS OF A POLYNOMIAL AND ITS DERIVATIVE

Q. I. RAHMAN

Let all the zeros of a polynomial  $p(z)$  of degree  $n$  lie in  $|z| \leq 1$ . Given a complex number  $a$  what is the radius of the smallest disk centred at  $a$  containing at least one zero of the polynomial  $((z-a)p(z))'$ ? According to Theorem 1 the answer is  $(|a| + 1)/(n+1)$  if  $|a| > (n+2)/n$ . Theorem 2 which states that if both the zeros of the quadratic polynomial  $p(z)$  lie in  $|z| \leq 1$  and  $|a| \leq 2$  then  $((z-a)p(z))'$  has at least one zero in

$$|z-a| \leq \{3|a| + (12-3|a|^2)^{1/2}\}/6$$

completely settles the case  $n = 2$ .

For  $|a| \leq 1$  the question is equivalent to a problem in [1, (see problem 4.5)] which reads as follows: Is it true that if all the zeros  $z_1, z_2, \dots, z_n$  of the polynomial  $p(z) = c \prod_{\nu=1}^n (z-z_\nu)$  lie in the disk  $|z| \leq 1$  then  $p'(z)$  has at least one zero in each of the disks  $|z-z_\nu| \leq 1, \nu = 1, 2, \dots, n$ ? It has been shown by Rubinstein [2] that if all the zeros of the polynomial  $p(z)$  lie in  $|z| \leq 1$  and  $p(1) = 0$  then at least one zero of  $p'(z)$  lies in the disk  $|z-1| \leq 1$ . On the other hand, the example  $z^n - 1$  shows that a disk of radius less than 1 may not contain a zero of  $p'(z)$ . Thus when  $|a| = 1$  the answer to our question is 1.

If  $a$  is arbitrary the problem is trivial for  $n = 1$  and the answer to the question is  $(|a| + 1)/2 = (|a| + 1)/(n+1)$ .

For polynomials of arbitrary degree  $n$  we prove

**THEOREM 1.** *If all the zeros of a polynomial  $p(z)$  of degree  $n$  lie in the closed unit disk then  $((z-a)p(z))'$  has one and only one zero in  $|z-a| \leq (|a| + 1)/(n+1)$  provided  $|a| > (n+2)/n$ . The remaining  $n-1$  zeros of  $((z-a)p(z))'$  lie in  $|z| \leq 1$ . The example  $p(z) = (z+e^{i\alpha})^n$  where  $\alpha = \arg a$  shows that the result is best possible.*

The disk  $|z-a| \leq (|a| + 1)/(n+1)$  may contain more than one zero of  $((z-a)p(z))'$  if  $|a| = (n+2)/n$ . That it contains at least one follows from the fact that the zeros of  $((z-a)p(z))'$  are continuous functions of  $a$ .

The next theorem gives a solution of the problem when

$$|a| \leq (n+2)/n \quad \text{and} \quad n = 2.$$

**THEOREM 2.** *If both the zeros of the quadratic polynomial  $p(z)$  lie in  $|z| \leq 1$  and  $|a| \leq 2$  then  $((z-a)p(z))'$  has at least one zero in*

$$|z-a| \leq \{3|a| + (12-3|a|^2)^{1/2}\}/6.$$

*The example*

$$p(z) = z^2 - 2 \left[ \frac{3-a(12-3a^2)^{1/2}}{3a-(12-3a^2)^{1/2}} \right] z + 1, \quad 0 \leq a \leq 2$$

*shows that the result is best possible.*

For the proof of Theorem 2 we shall need the following lemma [3, p. 36].

**LEMMA.** *If both the zeros of the polynomial*

$$A(z) = a_0 + \binom{2}{1} a_1 z + a_2 z^2$$

*lie in  $|z| \geq r$  and those of*

$$B(z) = b_0 + \binom{2}{1} b_1 z + b_2 z^2$$

*lie in  $|z| > s$  then both the zeros of the polynomial*

$$C(z) = a_0 b_0 + \binom{2}{1} a_1 b_1 z + a_2 b_2 z^2$$

*lie in  $|z| > rs$ .*

*Proof of Theorem 1.* Let

$$p(z) = c \prod_{\nu=1}^n (z-z_\nu)$$

where by hypothesis  $|z_\nu| \leq 1$ ,  $\nu = 1, 2, \dots, n$ . For a given  $z_0$  with  $|z_0| > 1$  the transformation  $1/(z_0-z)$  maps the closed unit disk onto some disk  $D(z_0)$  in the finite plane. Thus all the numbers  $1/(z_0-z_1)$ ,  $1/(z_0-z_2)$ ,  $\dots$ ,  $1/(z_0-z_n)$  belong to  $D(z_0)$  and hence so does their arithmetic mean  $\mu(z_0)$ . But there exists a unique point  $\phi(z_0)$  in the disk  $|z| \leq 1$  such that  $\mu(z_0) = 1/(z_0-\phi(z_0))$ . Consequently

$$p'(z_0)/p(z) = n/(z_0-\phi(z_0)).$$

Since  $z_0$  in an arbitrary point outside the unit disk we get the representation

$$p'(z)/p(z) = n/(z - \phi(z))$$

where  $\phi(z) = z - n \{p(z)/p'(z)\}$  is holomorphic and of absolute value at most 1 in  $|z| > 1$ .

If  $|a| > 1$  then

$$p'(z)/p(z) = n\psi(z)/\{(z-a)\psi(z)-1\}$$

where  $\psi(z) = 1/(\phi(z)-a)$  is holomorphic in  $|z| > 1$  and

$$(1) \quad 1/(|a|+1) \leq |\psi(z)| \leq 1/(|a|-1).$$

Since

$$\{(z-a)p'(z)+p(z)\}/p(z) = \{(n+1)(z-a)\psi(z)-1\}/\{(z-a)\psi(z)-1\}$$

the zeros of  $((z-a)p(z))'$  are the same as the zeros of  $(n+1)(z-a)\psi(z)-1$ . Now if  $|a| > (n+2)/n$  and  $(|a|+1)/(n+1) < |z-a| < |a|-1$  then from (1)

$$|(n+1)(z-a)\psi(z)| > 1.$$

Hence by Rouché's theorem  $(n+1)(z-a)\psi(z)-1$ ,  $(n+1)(z-a)\psi(z)$  have the same number of zeros in  $|z-a| \leq (|a|+1)/(n+1)$ , namely 1. Given  $\xi \in \{z: |z| \leq 1\} \cup \{z: |z-a| \leq (|a|+1)/(n+1)\}$  we can draw a contour  $C$  such that  $\{z: |z-a| \leq (|a|+1)/(n+1)\}$  and the point  $\xi$  lie in  $C_i$  (the bounded domain determined by  $C$ ) whereas  $\{z: |z| \leq 1\}$  lies in  $C_e$  (the unbounded domain determined by  $C$ ). According to the above reasoning  $((z-a)p(z))'$  has one and only one zero in  $C_i$ . Since we know that the zero lies in  $|z-a| \leq (|a|+1)/(n+1)$  the point  $\xi$  cannot be a zero of  $((z-a)p(z))'$ . Hence the remaining  $n-1$  zeros of  $((z-a)p(z))'$  lie in  $|z| \leq 1$ .

REMARK. Theorem 1 may be refined by observing that  $(n+1)(z-a)\psi(z)-1 \equiv (n+1)(z-a)(\phi(z)-a)^{-1}-1$  can vanish only if  $z-na/(n+1) = \phi(z)/(n+1)$ . Hence in fact  $((z-a)p(z))'$  has one and only one zero in  $D = \{z: |z-na/(n+1)| \leq 1/(n+1)\}$ . By considering  $p(z) = (z-z_0)^n$  with an appropriate  $z_0$  in the closed unit disk we see that any given point of  $D$  can be a zero of  $((z-a)p(z))'$ .

*Proof of Theorem 2.* Without loss of generality we may suppose  $0 \leq a \leq 2$ . Let

$$p(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2$$

and put

$$f(z) = ((z-a)p(z))' = (\alpha_0 - a\alpha_1) + 2(\alpha_1 - a\alpha_2)z + 3\alpha_2 z^2, \\ s = \{3a + (12 - 3a^2)^{1/2}\}/6.$$

We wish to prove that  $f(z)$  must vanish is  $|z-a| \leq s$ . If not, both the zeros of

$$B(z) = f(z+a) = \alpha_0 + a\alpha_1 + a^2\alpha_2 + \binom{2}{1}(\alpha_1 + 2a\alpha_2)z + 3\alpha_2z^2$$

lie in  $|z| > s$ . Since both the zeros of

$$A(z) = 1 + \binom{2}{1}(1/2)z + (1/3)z^2$$

lie on  $|z| = \sqrt{3}$  the lemma implies that both the zeros of the polynomial

$$C(z) = \alpha_0 + a\alpha_1 + a^2\alpha_2 + (\alpha_1 + 2a\alpha_2)z + \alpha_2z^2 \equiv p(z+a)$$

lie in  $|z| > \sqrt{3}s$ , i. e., the polynomial  $p(z)$  does not vanish in  $|z-a| \leq \sqrt{3}s$ . We can therefore find a positive number  $\varepsilon$  such that the disk  $|z - (a-2s)| \leq s - \varepsilon$  contains both the zeros of  $p(z)$ . Now it can be easily deduced from Theorem 1 that  $((z-a)p(z))'$  has one and only one zero in  $|z-a| \leq s - \varepsilon/3$ . This completes the proof of Theorem 2.

#### REFERENCES

1. W. K. Hayman, *Research Problems in Function Theory*, The Athlone press of the University of London, London 1967.
2. Z. Rubinstein, *On a problem of Ilyeff*, Pacific J. Math., **26** (1968), 159-161.
3. G. Szegő, *Bemerkungen zu einem Satz von J.H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Z., **13** (1922), 28-55.

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