

Pacific Journal of Mathematics

**A NOTE ON SECOND ORDER DIFFERENTIAL INEQUALITIES
AND FUNCTIONAL DIFFERENTIAL EQUATIONS**

HUGO TEUFEL

A NOTE ON SECOND ORDER DIFFERENTIAL
INEQUALITIES AND FUNCTIONAL
DIFFERENTIAL EQUATIONS

HUGO TEUFEL, JR.

Criteria are established for the nonexistence of eventually positive solutions of a second order differential inequality. The oscillation of all solutions of large classes of functional differential equations follows as corollaries.

1. Introduction. Study of the behavior of solutions of equations like

$$x'' + F(t, x, x') = 0,$$

where $xF \geq 0$, often entails study of the behavior of solutions of an inequality system like

$$(1) \quad x'' + H(t, x) \leq 0, \quad x \geq 0,$$

where $xF \geq xH \geq 0$ and H is selected for its tractability to analysis [6].

In this note it is shown that oscillation properties of large classes of equations

$$(2) \quad x'' + F(t, x(t), x(t - \tau(t))) = 0$$

can be established by use of inequalities like (1).

Thus, whenever feasible, inequalities like (1) should be primary objects of investigation.

2. Preliminaries. The inequality system discussed in this note is

$$(3) \quad x'' + a(t)N(x) \leq 0, \quad x \geq 0,$$

where $a(t)$ is nonnegative and continuous on $[0, \infty)$, and $N(x)$ is positive on $(0, \infty)$ and continuous and nondecreasing on $[0, \infty)$. Note that $N(0) > 0$ is permitted.

Three theorems are given on the nonexistence of eventually positive solutions of (3). Each theorem has a corollary concerning (2) where $F(t, u, v)$ is continuous on $[0, \infty) \times R_2$, and nondecreasing in u and v for $uv > 0$, $\tau(t)$ is continuous on $[0, \infty)$, and

$$(4) \quad \begin{aligned} F(t, u, u) &\geq a(t)N(u), \quad u \geq 0, \\ -F(t, u, u) &\geq a(t)N(-u), \quad u \leq 0. \end{aligned}$$

The term "solution" refers only to those solutions of equation (2) or inequality (3) which are defined and have a continuous second derivative on some interval $[T, \infty)$, $T \geq 0$. Inequality (3) does not restrict a solution at those t -values where it is negative.

P. K. Wong [7] has discussed an inequality system like

$$x'' - a(t)N(x) > 0, \quad x > 0.$$

3. The results. Theorem 1 is suggested by an oscillation criterion for a special case of the equation (1) due to F. V. Atkinson [1].

THEOREM 1. *Suppose*

$$(7) \quad \int_0^\infty ta(t)dt = \infty$$

and, if $\alpha > 0$,

$$(8) \quad \int_\alpha^\infty N^{-1}(u)du < \infty.$$

If $x(t)$ is a solution of (3) and $x(T_0) > 0$ at some T_0 in $[0, \infty)$, then $x(t)$ has a zero in (T_0, ∞) .

Proof. Suppose $x(t) > 0$ on $[T_0, \infty)$. Then $x(t)$ satisfies

$$(9) \quad x''(t) + a(t)N(x(t)) \leq 0.$$

If $x'(t)$ has a zero in $[T_0, \infty)$ then, by the conditions on a, N , (9) implies $x''(t) \leq 0$, $x''(t) \neq 0$, on $[T_0, \infty)$. Thus, it is readily seen that $x(t)$ must have a zero in $[T_0, \infty)$.

Therefore, suppose $x'(t) > 0$ on $[T_0, \infty)$. An integration of (9) over $[s, t]$, $T_0 \leq s < t$, gives, by neglect of positive $x'(t)$,

$$-x'(s) \leq -\int_s^t a(r)N(x(r))dr \leq -N(x(s))\int_s^t a(r)dr.$$

Division by $N(x(s))$ and an integration over $[T_0, t]$ gives

$$-\int_{x(T_0)}^{x(t)} N^{-1}(u)du \leq -\int_{T_0}^t (s - T_0)a(s)ds.$$

Clearly, if t is sufficiently large (7) and (8) are contradicted. This proves the theorem.

COROLLARY 1. *Given (2), suppose there exist functions a, N , which satisfy the conditions of the theorem and condition (4). Suppose $\tau = \sup_t \tau(t) < \infty$. Then, each solution of (2) has a zero in each interval $[T_0, \infty)$ (is oscillatory).*

Proof. If $x(t)$ is a solution of (2), but is not oscillatory, then $x(t) > 0$ on some interval $[T_0 - \tau, \infty)$. And, necessarily, $x'(t) > 0$ on $[T_0, \infty)$, whereby, $x(t - \tau(t)) \geq x(t - \tau)$ on $[T_0, \infty)$.

If $\tau \leq 0$, $x(t - \tau) \geq x(t)$ and $x(t)$ satisfies (9); thus, by Theorem 1, there is a contradiction.

If $\tau > 0$, then $x''(t) \leq 0$ on $[T_0, \infty)$ implies

$$(10) \quad x(t) - x(t - \tau) \leq x'(T_0)\tau, t \geq T_0 + \tau .$$

Hence, there is a β , $0 < \beta < 1$, such that $\beta x(t) \leq x(t - \tau)$ and $x(t)$ is a solution of (3) on $[T_0 + \tau, \infty)$ with $N(\beta x)$ in place of $N(x)$. The resulting contradiction proves the corollary.

Theorem 2 corresponds to the equation (1) where $F \equiv a(t)|x|^\gamma \operatorname{sgn} x$, $0 < \gamma < 1$, whereas Theorem 1 holds for this F with $\gamma > 1$. The proof is based on a proof given by J. W. Heidel [2].

Notice that if on $[t_0, \infty)$ $f(t) > 0$, $f'(t) > 0$ continuous and non-increasing, an integration shows that for each ν , $0 < \nu < 1$,

$$(11) \quad f(t) \geq \nu t f'(t), t \geq (1 - \nu)^{-1}t_0 .$$

THEOREM 2. *Suppose that in (3) $N(x)$ satisfies*

$$(12) \quad N(uv) \geq \eta(u)N(v) ,$$

for $u, v \geq 0$, u bounded, v large, and for some continuous function $\eta(u)$. Suppose, also,

$$(13) \quad \int^\infty N(t)a(t)dt = \infty ,$$

and the possibly improper integral,

$$(14) \quad \int_0^v \eta^{-1}(u)du ,$$

exists for each finite v . If $x(t)$ is a solution of (3) and $x(T_0) > 0$, then $x(t)$ must have a zero in (T_0, ∞) .

Proof. If $x(t) > 0$ on $[T_0, \infty)$, then $x'(t) > 0$ and $x(t) \geq 1/2(tx'(t))$ on $[T_1, \infty)$, $T_1 = 2T_0$. Therefore, $N(x(t)) \geq \eta(1/2(x'(t)))N(t)$ and (9) leads to

$$x''(t)\eta^{-1}(x'(t)) + N(t)a(t) \leq 0, \quad t \geq T_1 .$$

An integration produces

$$\int_{x'(T_1)}^{x'(t)} \eta^{-1}\left(\frac{1}{2}u\right)du + \int_T^t N(s)a(s)ds \leq 0 .$$

Since $x'(t)$ is nonincreasing this inequality with (13) contradicts the boundedness of (14). Hence, $x'(t)$ must have a zero in (T_0, ∞) and the theorem follows.

COROLLARY 2. *If there exist functions $a(t)$, $N(u)$, satisfying the conditions of the theorem and also satisfying (4), and if $\sup_t \tau(t) < \infty$, then all solutions of (2) oscillate.*

P. Waltman gave an example for an equation like (2) in which $t - \tau(t)$ increases more slowly than t , and the equation (2) has a non-oscillatory solution, while (2) with $\tau(t) \equiv 0$ has only oscillatory solutions. Corollary 3 slightly generalizes a sufficient condition for oscillation given in [5] for such cases. It holds for the linear equation

$$x''(t) + a(t)x(t - \tau(t)) = 0 .$$

THEOREM 3. *In (3) let $a(t)$ and $N(x)$ have the properties mentioned, and in addition let*

$$(15) \quad \int^{\infty} a(t)dt = \infty .$$

Then; if $x(t)$ is a solution of (3), $x(T_0) > 0$ implies $x(t)$ has a zero in (T_0, ∞) .

Proof. If the theorem is false $x(t) > x(T_0)$ on (T_0, ∞) and $N(x(t))$ in (9) is greater than $N(x(T_0))$. An integration produces a contradiction to $x'(t) > 0$ on $[T_0, \infty)$. This proves the theorem.

COROLLARY 3. *Given (3) suppose there exist functions $a(t)$, $N(u)$, satisfying (4). If $a(t)$ satisfies (15) and $t - \tau(t) \rightarrow \infty$ as $t \rightarrow \infty$, then all solutions of (2) are oscillatory.*

REMARK. The corollaries could have been given for equations more general than (2) where $F \equiv F(t, x(t - \tau_1(t)), \dots, x(t - \tau_n(t)))$ as in [3]. Also the $\tau(t)$ could have been expressed as explicitly dependent on $x(t)$ in certain ways. In [4] equations (2) which are nearly linear are discussed.

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Pacific Journal of Mathematics

Vol. 41, No. 2

December, 1972

Tom M. (Mike) Apostol, <i>Arithmetical properties of generalized Ramanujan sums</i>	281
David Lee Armacost and William Louis Armacost, <i>On p-thetic groups</i>	295
Janet E. Mills, <i>Regular semigroups which are extensions of groups</i>	303
Gregory Frank Bachelis, <i>Homomorphisms of Banach algebras with minimal ideals</i>	307
John Allen Beachy, <i>A generalization of injectivity</i>	313
David Geoffrey Cantor, <i>On arithmetic properties of the Taylor series of rational functions. II</i>	329
Václáv Chvátal and Frank Harary, <i>Generalized Ramsey theory for graphs. III. Small off-diagonal numbers</i>	335
Frank Rimi DeMeyer, <i>Irreducible characters and solvability of finite groups</i>	347
Robert P. Dickinson, <i>On right zero unions of commutative semigroups</i>	355
John Dustin Donald, <i>Non-openness and non-equidimensionality in algebraic quotients</i>	365
John D. Donaldson and Qazi Ibadur Rahman, <i>Inequalities for polynomials with a prescribed zero</i>	375
Robert E. Hall, <i>The translational hull of an N-semigroup</i>	379
John P. Holmes, <i>Differentiable power-associative groupoids</i>	391
Steven Kenyon Ingram, <i>Continuous dependence on parameters and boundary data for nonlinear two-point boundary value problems</i>	395
Robert Clarke James, <i>Super-reflexive spaces with bases</i>	409
Gary Douglas Jones, <i>The embedding of homeomorphisms of the plane in continuous flows</i>	421
Mary Joel Jordan, <i>Period H-semigroups and t-semisimple periodic H-semigroups</i>	437
Ronald Allen Knight, <i>Dynamical systems of characteristic 0</i>	447
Kwangil Koh, <i>On a representation of a strongly harmonic ring by sheaves</i>	459
Hui-Hsiung Kuo, <i>Stochastic integrals in abstract Wiener space</i>	469
Thomas Graham McLaughlin, <i>Supersimple sets and the problem of extending a retracing function</i>	485
William Nathan, <i>Open mappings on 2-manifolds</i>	495
M. J. O'Malley, <i>Isomorphic power series rings</i>	503
Sean B. O'Reilly, <i>Completely adequate neighborhood systems and metrization</i>	513
Qazi Ibadur Rahman, <i>On the zeros of a polynomial and its derivative</i>	525
Russell Daniel Rupp, Jr., <i>The Weierstrass excess function</i>	529
Hugo Teufel, <i>A note on second order differential inequalities and functional differential equations</i>	537
M. J. Wicks, <i>A general solution of binary homogeneous equations over free groups</i>	543