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HUGO TEUFEL

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HUGO TEUFEL, JR.

Criteria are established for the nonexistence of eventually positive solutions of a second order differential inequality. The oscillation of all solutions of large classes of functional differential equations follows as corollaries.

1. **Introduction.** Study of the behavior of solutions of equations like

$$x'' + F(t, x, x') = 0 ,$$

where $xF \geqq 0$, often entails study of the behavior of solutions of an inequality system like

$$x'' + H(t, x) \leqq 0 , \quad x \geqq 0 , \tag{1}$$

where $xF \geqq xH \geqq 0$ and H is selected for its tractability to analysis [6].

In this note it is shown that oscillation properties of large classes of equations

$$x'' + F(t, x(t), x(t - \tau(t))) = 0 \tag{2}$$

can be established by use of inequalities like (1).

Thus, whenever feasible, inequalities like (1) should be primary objects of investigation.

2. **Preliminaries.** The inequality system discussed in this note is

$$x'' + a(t)N(x) \leqq 0 , \quad x \geqq 0 , \tag{3}$$

where $a(t)$ is nonnegative and continuous on $[0, \infty)$, and $N(x)$ is positive on $(0, \infty)$ and continuous and nondecreasing on $[0, \infty)$. Note that $N(0) > 0$ is permitted.

Three theorems are given on the nonexistence of eventually positive solutions of (3). Each theorem has a corollary concerning (2) where $F(t, u, v)$ is continuous on $[0, \infty) \times R_2$, and nondecreasing in u and v for $uv > 0$, $\tau(t)$ is continuous on $[0, \infty)$, and

$$\begin{aligned} F(t, u, u) &\geqq a(t)N(u) , \quad u \geqq 0 , \\ -F(t, u, u) &\geqq a(t)N(-u) , \quad u \leqq 0 . \end{aligned} \tag{4}$$

The term "solution" refers only to those solutions of equation (2) or inequality (3) which are defined and have a continuous second derivavative on some interval $[T, \infty)$, $T \geqq 0$. Inequality (3) does not restrict a solution at those t-values where it is negative.

P. K. Wong [7] has discussed an inequality system like

$$x'' - a(t)N(x) > 0 \,, \quad x > 0 \,.$$

3. **The results.** Theorem 1 is suggested by an oscillation criterion for a special case of the equation (1) due to F. V. Atkinson [1].

THEOREM 1. *Suppose*

$$\int^{\infty} ta(t)dt = \infty \tag{7}$$

and, if $\alpha > 0$,

$$\int_{\alpha}^{\infty} N^{-1}(u)du < \infty \,. \tag{8}$$

If $x(t)$ *is a solution of* (3) *and* $x(T_0) > 0$ *at some* T_0 *in* $[0, \infty)$, *then* $x(t)$ *has a zero in* (T_0, ∞).

Proof. Suppose $x(t) > 0$ on $[T_0, \infty)$. Then $x(t)$ satisfies

$$x''(t) + a(t)N(x(t)) \leqq 0 \,. \tag{9}$$

If $x'(t)$ has a zero in $[T_0, \infty)$ then, by the conditions on a, N, (9) implies $x''(t) \leqq 0$, $x''(t) \not\equiv 0$, on $[T_0, \infty)$. Thus, it is readily seen that $x(t)$ must have a zero in $[T_0, \infty)$.

Therefore, suppose $x'(t) > 0$ on $[T_0, \infty)$. An integration of (9) over $[s, t]$, $T_0 \leqq s < t$, gives, by neglect of positive $x'(t)$,

$$-x'(s) \leqq -\int_s^t a(r)N(x(r))dr \leqq -N(x(s))\int_s^t a(r)dr \,.$$

Division by $N(x(s))$ and an integration over $[T_0, t]$ gives

$$-\int_{x(T_0)}^{x(t)} N^{-1}(u)du \leqq -\int_{T_0}^{t} (s - T_0)a(s)ds \,.$$

Clearly, if t is sufficiently large (7) and (8) are contradicted. This proves the theorem.

COROLLARY 1. *Given* (2), *suppose there exist functions* a, N, *which satisfy the conditions of the theorem and condition* (4). *Suppose* $\tau = \sup_t \tau(t) < \infty$. *Then, each solution of* (2) *has a zero in each interval* $[T_0, \infty)$ (*is oscillatory*).

Proof. If $x(t)$ is a solution of (2), but is not oscillatory, then $x(t) > 0$ on some interval $[T_0 - \tau, \infty)$. And, necessarily, $x'(t) > 0$ on $[T_0, \infty)$, whereby, $x(t - \tau(t)) \geqq x(t - \tau)$ on $[T_0, \infty)$.

If $\tau \leqq 0$, $x(t - \tau) \geqq x(t)$ and $x(t)$ satisfies (9); thus, by Theorem 1, there is a contradiction.

If $\tau > 0$, then $x''(t) \leqq 0$ on $[T_0, \infty)$ implies

$$x(t) - x(t - \tau) \leqq x'(T_0)\tau,\ t \geqq T_0 + \tau\ . \tag{10}$$

Hence, there is a β, $0 < \beta < 1$, such that $\beta x(t) \leqq x(t - \tau)$ and $x(t)$ is a solution of (3) on $[T_0 + \tau, \infty)$ with $N(\beta x)$ in place of $N(x)$. The resulting contradiction proves the corollary.

Theorem 2 corresponds to the equation (1) where $F \equiv a(t)|x|^\gamma \operatorname{sgn} x$, $0 < \gamma < 1$, whereas Theorem 1 holds for this F with $\gamma > 1$. The proof is based on a proof given by J. W. Heidel [2].

Notice that if on $[t_0, \infty)$ $f(t) > 0$, $f'(t) > 0$ continuous and non-increasing, an integration shows that for each ν, $0 < \nu < 1$,

$$f(t) \geqq \nu t f'(t),\ t \geqq (1 - \nu)^{-1} t_0\ . \tag{11}$$

THEOREM 2. *Suppose that in* (3) $N(x)$ *satisfies*

$$N(uv) \geqq \eta(u) N(v)\ , \tag{12}$$

for $u, v \geqq 0$, u *bounded,* v *large, and for some continuous function* $\eta(u)$. *Suppose, also,*

$$\int^\infty N(t)a(t)dt = \infty\ , \tag{13}$$

and the possibly improper integral,

$$\int_0^v \eta^{-1}(u)du\ , \tag{14}$$

exists for each finite v. *If* $x(t)$ *is a solution of* (3) *and* $x(T_0) > 0$, *then* $x(t)$ *must have a zero in* (T_0, ∞).

Proof. If $x(t) > 0$ on $[T_0, \infty)$, then $x'(t) > 0$ and $x(t) \geqq 1/2(tx'(t))$ on $[T_1, \infty)$, $T_1 = 2T_0$. Therefore, $N(x(t)) \geqq \eta(1/2(x'(t)))N(t)$ and (9) leads to

$$x''(t)\eta^{-1}(x'(t)) + N(t)a(t) \leqq 0\ , \quad t \geqq T_1\ .$$

An integration produces

$$\int_{x'(T_1)}^{x'(t)} \eta^{-1}\Big(\frac{1}{2}u\Big)du + \int_T^t N(s)a(s)ds \leqq 0\ .$$

Since $x'(t)$ is nonincreasing this inequality with (13) contradicts the boundedness of (14). Hence, $x'(t)$ must have a zero in (T_0, ∞) and the theorem follows.

COROLLARY 2. *If there exist functions* $a(t)$, $N(u)$, *satisfying the conditions of the theorem and also satisfying* (4), *and if* $\sup_t \tau(t) < \infty$, *then all solutions of* (2) *oscillate.*

P. Waltman gave an example for an equation like (2) in which $t - \tau(t)$ increases more slowly than t, and the equation (2) has a non-oscillatory solution, while (2) with $\tau(t) \equiv 0$ has only oscillatory solutions. Corollary 3 slightly generalizes a sufficient condition for oscillation given in [5] for such cases. It holds for the linear equation

$$x''(t) + a(t)x(t - \tau(t)) = 0 .$$

THEOREM 3. *In* (3) *let* $a(t)$ *and* $N(x)$ *have the properties mentioned, and in addition let*

$$\int^{\infty} a(t)dt = \infty . \tag{15}$$

Then; if $x(t)$ *is a solution of* (3), $x(T_0) > 0$ *implies* $x(t)$ *has a zero in* (T_0, ∞).

Proof. If the theorem is false $x(t) > x(T_0)$ on (T_0, ∞) and $N(x(t))$ in (9) is greater than $N(x(T_0))$. An integration produces a contradiction to $x'(t) > 0$ on $[T_0, \infty)$. This proves the theorem.

COROLLARY 3. *Given* (3) *suppose there exist funtions* $a(t)$, $N(u)$, *satisfying* (4). *If* $a(t)$ *satisfies* (15) *and* $t - \tau(t) \to \infty$ *as* $t \to \infty$, *then all solutions of* (2) *are oscillatory.*

REMARK. The corollaries could have been given for equations more general than (2) where $F \equiv F(t, x(t - \tau_1(t)), \cdots, x(t-\tau_n(t)))$ as in [3]. Also the $\tau(t)$ could have been expressed as explicitly dependent on $x(t)$ in certain ways. In [4] equations (2) which are nearly linear are discussed.

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WICHITA STATE UNIVERSITY

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