

# Pacific Journal of Mathematics

**ON PSEUDO-CONFORMAL MAPPINGS OF CIRCULAR  
DOMAINS**

STEFAN BERGMAN

## ON PSEUDO-CONFORMAL MAPPINGS OF CIRCULAR DOMAINS

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In the present paper we investigate the condition whether the bounded domain  $B$  of  $C^2$  is a pseudo-conformal image of a circular domain, say  $C$ . Under the assumption that this is the case and that the invariant  $J_B(z_1, z_2; \bar{z}_1, \bar{z}_2)$  is not a constant, we characterize the center of a circular domain. This characterization is invariant with respect to pseudoconformal transformations. Assuming that  $B$  is a pseudoconformal image of a circular domain  $C$  and that there is in  $B$  one and only one point, say  $(t_1, t_2)$  which satisfies the conditions mentioned above, we determine the representative  $R(B; t_1, t_2)$  of  $B$ . If  $B$  is a pseudo-conformal image of a circular domain  $C$  and  $(t_1, t_2)$  is the image in  $B$  of the center of  $C$ , then the representative  $R(B; t_1, t_2)$  is a circular domain. The pair of functions  $v^{10}, v^{01}$  mapping  $B$  onto  $R(B; t_1, t_2)$  can be written explicitly in terms of the kernel function of  $B$ .

A homeomorphism  $T$  of a domain, say  $B$ , of the  $z_1, z_2$ -space,  $z_k = x_k + iy_k, k = 1, 2$ , by a pair of holomorphic function

$$(1) \quad z_k^* = z_k^*(z_1, z_2), \quad (z_1, z_2) \in B,$$

of two complex variables is denoted a PCT (*pseudo-conformal transformation*).

A domain which admits the group

$$(2) \quad z_k^* = z_k e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi, z_k = x_k + iy_k,$$

of PCT's onto itself (automorphisms) is called a *circular domain*.

We assume that at every boundary point  $Q$  of  $C$  the Levi expression is negative (see (11), p. 11 and (16), p. 12, of [1]). (Hypothesis 1)

To decide whether a domain, say  $B$ , belongs to a given class of domains, for instance, whether  $B$  is a pseudo-conformal image of a circular domain  $C$ , is one of the interesting problems of the theory of PCT's. In the following we shall show that the theory of the kernel function permits us to answer this question in certain instances. In addition, if,  $B = T(C)$ , we shall determine the function pair mapping  $B$  onto the circular domain  $C$ .

REMARK. Concerning the application of the kernel function in the theory of conformal mapping of simply and multiply connected domains onto canonical domains and onto each other, see [3], Chapter VI, [5] and [6].

The first step in our approach (under the assumption that  $B = T(C)$ , i.e., that  $B$  is a pseudo-conformal image of a circular domain  $C$ ) is the determination of the image of the center  $O$  of  $C$  in  $B$ .

Using the considerations on p. 183 ff. of [3], we assume that the invariant (with respect to PCT's)

$$(3) \quad J_B = J \equiv \frac{K}{T_{11}T_{22} - |T_{12}|^2}, \quad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},$$

is known and is *not constant*. (Hypothesis 2)

In accordance with the considerations on p. 183 of [3]  $J_B$  is invariant with respect to PCT's. Consequently,

$$(4) \quad J_C(z, \bar{z}) = J_B(z^*, \bar{z}^*).$$

$J_C(z, \bar{z})$  is a real analytic function of  $z_1, z_2, \bar{z}_1, \bar{z}_2$  in  $C$ . From formula (33), p. 19, of [2] it follows that

$$(5) \quad J_C(z, \bar{z}) > 0, \quad z \in C.$$

Since we assumed that at every boundary point of  $C$  Levi's expression,  $L(\Phi)$ , is negative, it follows that for every boundary point  $Q$  of  $C$

$$(6) \quad \lim_{z \rightarrow Q} J_C(z, \bar{z}) = \frac{2}{9\pi^2}$$

holds (see (1), p. 12, of [2]).<sup>1</sup> Since  $J_C(z, \bar{z})$  is not constant in  $C$ , it must assume its maximum or minimum in  $C$ .

In Theorems 1 – 3 we shall discuss some properties of a connected set which includes the origin  $O$  where  $J_C(z, \bar{z})$  has a (local) maximum or a minimum. These properties are preserved in PCT's and enable us to determine the image  $T(O)$  of the center  $O$  in  $B = T(C)$ .

**THEOREM 1.** *If  $J(z_1, z_2)$  has a maximum, minimum or minimax at an isolated point  $P$  of  $B = T(C)$ , then*

$$(7) \quad P = T(O).$$

*Proof.* Let  $\tilde{C} = C \cap [y_2 = 0]$ . The function  $J(z_1, x_2)$  is defined in  $\tilde{C}$ . If  $J(z_1, x_2)$  assumes a value, say  $c_0$ , at a point  $z_1^0, x_2^0, (z_1^0, x_2^0) \neq O$ , then  $J(z_1, z_2) = c_0$  along a line

$$(8) \quad o^1(z_1^0, x_2^0) = [z_1 = z_1^0 e^{i\varphi}, z_2 = x_2^0 e^{i\varphi}, 0 \leq \varphi \leq 2\pi],$$

the orbit of  $(z_1^0, x_2^0)$ . If and only if  $z_1^0 = 0, x_2^0 = 0$  (i.e., if  $(z_1^0, x_2^0)$  is the origin  $O$ ),  $o^1$  degenerates to a point. In accordance with (4), if  $J_B(z^*, \bar{z}^*)$

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\* We assume here that we approach  $Q$  in the sense  $A^I$  in a way described in [2], p. 10.

has a minimum or a maximum at an isolated point  $P, P \in B$ , then (7) holds.

**THEOREM 2.** *It is impossible that  $J(z, \bar{z})$  has a maximum or a minimum along a (one-dimensional) connected set including  $O$ .*

(We assume here that the set does not include a segment of an orbit.)

*Proof.* Suppose that

$$(9) \quad p = \bigcup_{\alpha=0}^a P(\alpha), \quad a > 0, P(0) = (0, 0) = O$$

is a (one-dimensional) connected set consisting of points  $P(s), 0 \leq s \leq a$ , where  $J_c(z, \bar{z})$  assumes a minimum or a maximum. Then to every point  $P_\nu(\alpha), 0 < \alpha \leq a$ , corresponds the orbit  $o^1(P_\nu(\alpha))$ , along which  $J_c(z, \bar{z})$  assumes a constant value. Thus  $J_c(z, \bar{z})$  assumes the same value on

$$(10) \quad \bigcup_{\alpha=0}^a o^1(P_\nu(\alpha)), \quad 0 \leq \alpha \leq a.$$

Each  $o^1(P_\nu(\alpha)), \alpha$  constant,  $\alpha > 0$ , is one dimensional. Two different orbits  $o^1(P_\nu(\alpha_1))$  and  $o^1(P_\nu(\alpha_2)), \alpha_1 \neq \alpha_2$ , are disjoint and therefore (10) is a two-dimensional set.  $T[\bigcup_{\alpha=0}^a o^1(P_\nu(\alpha))]$  is also two dimensional since  $T$  is a homeomorphism.

In the following we shall consider two cases where  $J(z, \bar{z})$  equals a maximum or a minimum in a three-dimensional segment  $s^3$ .

**THEOREM 3a.** *Suppose that  $J_c(z, \bar{z}) = \text{maximum}$  (or *minimum*) in a (three-dimensional) set  $s^3, P \in s^3$ . We assume that  $s^3$  is connected in  $C, s^3 - P$  is a sum of two (or in general  $n, n < \infty$ ) disconnected sets. Then  $P$  is the center  $O$  of  $C$ .*

*If  $B = T(C)$ , i.e., if  $B$  is a pseudo-conformal image of  $C$ , then  $T(s^3)$  has the property indicated above and if  $T(s^3) - P^*$  is a sum of  $n, n > 1$ , disconnected parts, then  $P^* = T(O)$ .*

*Proof.* If two parts, say  $s_1^3$  and  $s_2^3, s_k^3 \in C$ , are connected at one point, say  $(z_1^0, z_2^0), |z_1^0|^2 + |z_2^0|^2 > 0$ , then they are connected along a line segment

$$(11) \quad o^1(z_1^0, z_2^0) = [z_1^0 e^{i\varphi}, z_2^0 e^{i\varphi}, 0 \leq \varphi \leq 2\pi].$$

If we delete one point, say  $Q = (z_1^0 e^{i\varphi_1}, z_2^0 e^{i\varphi_1})$ , from (11), then  $s_1^3$  and  $s_2^3$  will still be connected along

$$(12) \quad [z_1^0 e^{i\varphi}, z_2^0 e^{i\varphi}, 0 \leq \varphi \leq 2\pi] - (z_1^0 e^{i\varphi_1}, z_2^0 e^{i\varphi_1}).$$

Thus by deleting the point  $Q, s_1^3$  and  $s_2^3$  can become disconnected only

if  $Q = O = (0, 0)$ , in which case the orbit (11) degenerates to a point.

**THEOREM 3b.** *Suppose that  $J(z, \bar{z}) = \text{maximum}$  (or *minimum*),  $z \in C$ , is a connected set  $s^3, P \in s^3, s^3 = s_1^3 \cup s_2^3 \cup P$ , and that one can define sufficiently small neighborhoods  $N(s_1^3)$  and  $N(s_2^3)$  such that  $N(s_1^3) \cup N(s_2^3) \cup P$  is connected and  $N(s_1^3) \cup N(s_2^3)$  is not connected. Then  $P = O$ .*

*Proof.* Suppose that the point  $Q = (z_1^0, z_2^0) \neq O$  belongs to  $s^3$ . Then the orbit (11) must also belong to  $s^3$ . Let  $P_k \in N(s_k^3), k = 1, 2$ . According to the assumption of the the theorem, one can connect  $P_1$  and  $P_2$  by a line segment passing the point  $Q = (z_1^0, z_2^0)$ . Since

$$(13) \quad |z_1^0|^2 + |z_2^0|^2 > 0 ,$$

one can also connect  $P_1$  and  $P_2$  by a segment passing by the point  $Q^* = (z_1^0 e^{i\varphi_1}, z_2^0 e^{i\varphi_1}), 0 < \varphi_1 < 2\pi$ .  $N(s_1^3) - P$  and  $N(s_2^3) - P$  become disconnected only if the assumption (13) does not hold, i.e., if  $P = O = (0, 0)$ .

**REMARK.** It is interesting to give an example of a set  $s^3$  and to describe a construction of  $N(s^3)$  possessing the properties indicated in Theorem 3b. Suppose that  $s_1^3$  lies in  $x_2 > 0, s_2^3$  in  $x_2 < 0$ , and that they are connected by a one-dimensional set which lies in  $s^3$  and which includes  $O$  (it lies in  $x_2 = 0$ ). Let  $Q \neq O, Q \in s^3$ . To construct the desired neighborhood, we draw around every point  $Q \in s^3 - O$  a hypersphere  $H(Q, \rho)$  with the center at  $Q$  and of radius  $\rho > 0, \rho = \rho(Q) < d(Q)$ , where  $d(Q)$  denotes the distance between  $Q$  and  $x_2 = 0$ . Then  $N(s_1^3) = [\cup H(Q, 1/2d(Q)), Q \in s_1^3]$ . Obviously  $N(s^3)$  has no points lying in  $x_2 = 0$ . Naturally, instead  $x_2 = 0$  one can use another hypersurface possessing the necessary property.

**THEOREM 4.** *Suppose that  $J$  assumes its maximum (or minimum) on a two-dimensional connected set  $s^2, O \in s^2$ . We assume that  $s^2 - O$  is a sum of  $n$  disconnected segments,  $1 < n < \infty$ . Then  $O$  is the center of  $C$ .*

*Proof.* The proof proceeds as the proof of Theorem 3a. To every point  $P, P \neq O$ , of  $s^2 \cap (y_2 = 0)$  corresponds the orbit (8), i.e.,

$$s^2 = \bigcup_{\beta=0}^{\alpha} o^1(P_\nu(\alpha)) , \quad 0 \leq \alpha < a .$$

Suppose that  $P_1$  and  $P_2$  are two points of  $s^2$  which lie in different orbits  $o^1(P_k), k = 1, 2$ . If the line segment connecting  $P_1$  and  $P_2$  passes through  $O$ , the segments  $s_k^2, P_k \in s_k^2 - O, s_1^2 \cup s_2^2 = s^2 - O$  are disconnected. We note that if we delete a point  $Q, Q \neq O$ , then  $s_1^2$  and  $s_2^2$  can be

connected by a segment passing through another point of the orbit  $o^1(Q)$ . Only for  $Q = O$  this orbit shrinks to a point.

REMARK. It is not necessary to consider the case where the domain  $s$  (where  $J_B(z, \bar{z}) = \text{maximum (or minimum)}$ ) is four dimensional. If this holds, then

$$(14) \quad J_B(z, \bar{z}) = \text{const} , \quad (z_1, z_2) \in B ,$$

is in contradiction with Hypothesis 2.

Obviously there exist situations for which our procedures do not enable us to determine the location of  $T(O)$  in  $B$ . For example, suppose that  $s^2$  is a segment in  $C \cap [x_2 = 0, y_2 = 0]$ .

In some of these cases we can use in addition to  $J$  a second invariant (with respect to PCT's)  $J_2(z, \bar{z})$  which is linearly independent of  $J(z, \bar{z})$ . Concerning conditions for such domains  $B$ , see [4]. We assume that the intersection

$$(15) \quad [J(z, \bar{z}) = \text{const} = c_1] \cap [J_2(z, \bar{z}) = \text{const} = c_2]$$

either includes an isolated point or a closed Jordan curve.

THEOREM 5a. *Suppose that the set (15) in  $B$  consists of disconnected components and one of these components is an isolated point, say  $Q$ . Then  $Q = T(O)$ .*

*Proof.* Suppose that (15) in  $C$  is a point  $(z_1^0, z_2^0) \neq O$ . Since  $C$  admits the group (2) of PCT's onto itself, the orbit

$$(16) \quad o^1(z_1^0, z_2^0) = [z_k = z_k^0 e^{i\varphi}, 0 \leq \varphi \leq 2\pi, k = 1, 2]$$

must belong to (15). (16) is a closed Jordan curve and its image  $T(o^1(z_1^0, z_2^0))$  is also a closed Jordan curve. It degenerates to a point only if  $(z_1^0, z_2^0) = O$ .

THEOREM 5b. *Suppose that (15) includes a closed Jordan curve, say  $p^1$ . If we draw around every point  $R \in p^1$  a (invariant) hypersphere  $\bar{\sigma}^3(R, \rho)$  of radius  $\rho$ , then for  $\rho$  sufficiently small all hyperspheres  $\bar{\sigma}^3(R, \rho)$ ,  $R \in p^1$ , have no common point. If  $\rho$  increases, there exists a smallest  $\rho$ , say  $\rho = \rho_0$ , such that all  $\bar{\sigma}^3(R, \rho_0)$  have a common point, say  $Q$ . Then*

$$(17) \quad Q = T(O) .$$

*Proof.* Since the construction described in Theorem 5b is invariant with respect to PCT, we can consider it either in  $C$  or in  $B = T(C)$ . We shall consider it in  $C$ . By PCT (2)  $p^1$  goes onto itself. Therefore

the invariant distance, say  $\rho_0$ , of every point  $R \in p^1$  from  $O$  is the same. For  $\rho < \rho_0$  all hyperspheres  $\bar{\sigma}^3(R, \rho)$  have no common point. Suppose that there exists a point, say  $S$ ,  $S \neq O$ , such that the invariant distance between  $p^1$  and  $S$  is  $\rho_1 < \rho_0$ . Then  $p^1$  would be simultaneously an orbit around two different points,  $O$  and  $S$ . But the orbits around the two different points cannot coincide. If  $\rho = \rho_0$ , where  $\rho_0$  is the invariant distance of  $p^1$  from  $O$ , all (closed) hyperspheres  $\bar{\sigma}^3(R, \rho_0)$  will have the point  $O$  in common. Thus (17) holds.

Suppose that the domain  $B$  (in the  $z_1, z_2$ -space) is a pseudoconformal image of a circular domain  $C$ , i.e.,  $B = T(C)$ . The previous considerations in most cases enable us to determine in  $B$  the image  $t = T(O)$ ,  $t = (t_1, t_2)$  of the center  $O$  of  $C$ . In the following we shall indicate, using the above result, how we can determine the pair  $v^{10}(z, t)$ ,  $v^{01}(z, t)$ ,  $z = (z_1, z_2)$ ,  $t = (t_1, t_2)$  of analytic functions which transform  $B$  onto a circular domain.

We shall use, without proof, the following:

LEMMA. *A circular domain, say  $C$ , is transformed by a linear PCT again onto a circular domain.*

A mapping pair  $w_k(z_1, z_2)$  is said to be normalized at  $t = (t_1, t_2)$  if

$$(21) \quad w_k(t_1, t_2) = t_k, \quad \left. \frac{\partial w_k(z_1, z_2)}{\partial z_n} \right|_{z_p=t_p} = \delta_{kn},$$

$\delta_{kn} = 0$  for  $k \neq n$ ,  $\delta_{kn} = 1$  for  $k = n$ ,  $k = 1, 2$ ,  $p = 1, 2$ .

In (50), (51), pp. 188, 189 of [3] the pair  $V_t = (v^{10}(z, t), v^{01}(z, t))$  normalized at  $t$  is given (in terms of the kernel function  $K_B(z, t)$ ) which maps  $B$  onto the representative  $R(B, t)$  of  $B$ , see Theorem, p. 186 of [3].

THEOREM 6. *Suppose that  $B$  is a pseudo-conformal image of a circular domain  $C$ , i.e.,  $B = T(C)$ , where  $T$  is a PCT. Let  $t^* = (t_1^*, t_2^*)$  be the image in  $B$  of the center  $O$  of  $C$ . Then the representative  $R(B, t^*)$ ,  $t^* = T(O)$ , is a circular domain. Here  $R(B, t^*)$ ,  $t^* = T(O)$ , is the domain which we obtain from  $B$  using the PCT*

$$(22) \quad z_1 = v^{10}(z, t), \quad z_2 = v^{01}(z, t).$$

$(v^{10}, v^{01})$  is the pair of functions introduced in (50), (51), pp. 188-189 of [3].

*Proof.* According to our considerations  $R(B, t)$  and  $C$  are both representatives of  $B$  with respect to the same point  $t = T(O)$ . According to [3], p. 190, two representatives of  $B$  with respect to the same point can be transformed into each other by a linear PCT, say by

$$(23) \quad \begin{aligned} v^{*10} &= \alpha_{1\bar{1}}v^{10} + \alpha_{1\bar{2}}v^{01}, \\ v^{*01} &= \alpha_{2\bar{1}}v^{10} + \alpha_{2\bar{2}}v^{01}. \end{aligned} \quad \alpha_{\nu\bar{\mu}} \text{ constants,}$$

In accordance with Lemma 1,  $R(B, t)$  is also a circular domain since  $C$  is a circular domain and (23) is a linear PCT.

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George E. Andrews, <i>Two theorems of Gauss and allied identities proved arithmetically</i> .....	563
Stefan Bergman, <i>On pseudo-conformal mappings of circular domains</i> .....	579
Beverly L. Brechner, <i>On the non-monotony of dimension</i> .....	587
Richard Anthony Brualdi and John H. Mason, <i>Transversal matroids and Hall's theorem</i> .....	601
Philip Throop Church and James Timourian, <i>Differentiable maps with 0-dimensional critical set. I</i> .....	615
John H. E. Cohn, <i>Squares in some recurrent sequences</i> .....	631
Robert S. Cunningham, Edgar Andrews Rutter and Darrell R. Turnidge, <i>Rings of quotients of endomorphism rings of projective modules</i> .....	647
Eldon Dyer and S. Eilenberg, <i>An adjunction theorem for locally equiconnected spaces</i> .....	669
Michael W. Evans, <i>On commutative P. P. rings</i> .....	687
Ronald Lewis Graham, Hans Sylvain Witsenhausen and Hans Zassenhaus, <i>On tightest packings in the Minkowski plane</i> .....	699
Stanley P. Gudder, <i>Partial algebraic structures associated with orthomodular posets</i> .....	717
Karl Edwin Gustafson and Gunter Lumer, <i>Multiplicative perturbation of semigroup generators</i> .....	731
Kurt Kreith and Curtis Clyde Travis, Jr., <i>Oscillation criteria for selfadjoint elliptic equations</i> .....	743
Lawrence Louis Larmore, <i>Twisted cohomology theories and the single obstruction to lifting</i> .....	755
Jorge Martinez, <i>Tensor products of partially ordered groups</i> .....	771
Robert Alan Morris, <i>The inflation-restriction theorem for Amitsur cohomology</i> .....	791
Leo Sario and Cecilia Wang, <i>The class of <math>(p, q)</math>-biharmonic functions</i> .....	799
Manda Butchi Suryanarayana, <i>On multidimensional integral equations of Volterra type</i> .....	809
Kok Keong Tan, <i>Fixed point theorems for nonexpansive mappings</i> .....	829