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In this paper an example is constructed of a compact Hausdorff space X with covering and large inductive dimensions 0, but containing subsets  $Y_n$ , with covering dimension n, and large inductive dimension at least n, each  $n, 1 \leq n \leq n$  $\infty$ . This result is an interesting contrast with the recent result of D. W. Henderson that there exists an infinite dimensional compact metric space with no *n*-dimensional compact subsets for  $1 \leq n < \infty$ . As a corollary of our results, we show that for each  $n, 1 \leq n \leq \infty$ , there exists a (necessarily non-metric) continuum  $M_n$  of covering dimension n, which contains subsets of all positive covering dimensions. Covering dimension is treated in  $\S2$ , while large inductive dimension is treated in §4. In §3, compactifications of  $Y_n$  are discussed and it is shown that the covering dimension of  $\beta Y_n = 0$  for  $1 \leq n < \infty$ . It is known (see Gillman and Jerison) that for a normal space N, cov dim  $N = cov \dim \beta N$ .

1. Introduction and construction of examples. For separable metric spaces,  $\operatorname{cov} \dim = \operatorname{ind} \dim = \operatorname{Ind} \dim$ , and each of these is *monotone*; that is,  $A \subseteq B$  implies  $\dim A \leq \dim B$ . However, it is well-known that although ind dim is always monotone [7, appendix],  $\operatorname{cov} \dim$  and Ind dim need not be monotone. In the appendix of [7], it is shown that if S denotes the Tychonoff Plank, and T denotes a particular (non-normal) subset of S, then ind dim  $S = \operatorname{cov} \dim S =$ Ind dim S = 0, while  $\operatorname{cov} \dim T > 0$  and Ind dim T > 0. However,  $\operatorname{cov} \dim T$  and Ind dim T are not determined in [7]. In this paper, we will prove, among other things, that the set T ( $Y_1$  in our notation) satisfies  $\operatorname{cov} \dim T = \operatorname{Ind} \dim T = 1$ .

We establish the following notation.

Let  $\Omega_0$  denote the first ordinal whose cardinal is infinite.

Let  $\Omega_n$ , for each positive integer n, denote the first ordinal greater than  $\Omega_{n-1}$ , whose cardinality is greater than the cardinality of  $\Omega_{n-1}$ . (Note that  $\Omega_n$  is a limit ordinal. See page 100 of [5].)

Let  $X_i$  be the set of ordinals from 0 to  $\Omega_i$ , inclusive, with the order topology;  $X_i = [0, \Omega_i]$ .

Let  $X = \prod_{i=0}^{\infty} X_i$ . Let  $p = (\Omega_0, \Omega_1, \dots, \Omega_i, \dots) \in X$ . Let  $Y_{\infty} = Y = X - \{p\}$ . Let  $Z_n = \prod_{i=0}^n X_i$ . Let  $p_n = (\Omega_0, \Omega_1, \dots, \Omega_n) \in Z_n$ . Let  $Y_n = Z_n - \{p_n\}$ .

The examples of this paper are X and the  $Y_n$ 's just constructed. We note that they are simple generalizations of the Tychonoff Plank. (See p. 132 of [9] or p. 154 of [7].) It is known that for any space A, Ind dim A = 0 iff cov dim A = 0, and that for a compact Hausdorff space A, ind dim A = 0 iff Ind dim A = 0 iff cov dim A = 0. (See [7, appendix] and [10].) Thus a totally disconnected, compact, Hausdorff space has dimension 0 in all three senses. The spaces  $Z_n, 1 \leq n < \infty$ , and X, constructed above, are totally disconnected, compact, Hausdorff, and hence, 0-dimensional in all three senses.

All spaces are Hausdorff.

We use the following definitions.

The small inductive dimension of a space A, ind A, is -1, iff  $A = \emptyset$ ; ind  $A \leq n$  iff for every point  $p \in A$  and open set U containing p, there is an open set V such that  $p \in V \subseteq U$  and ind  $(bdry V) \leq n-1$ . We say ind A = n iff ind  $A \leq n$  but ind  $A \leq n-1$ .

The large inductive dimension of a space A, Ind A, is -1 iff  $A = \emptyset$ ; Ind  $A \leq n$  iff for every closed set  $F \subseteq A$  and open set U containing F, there is an open set V such that  $F \subseteq V \subseteq U$  and Ind (bdry V)  $\leq n - 1$ . We say Ind A = n iff Ind  $A \leq n$  but Ind  $A \leq n - 1$ .

The covering dimension of a space A, cov dim A, or dim A, is -1 iff A is empty; dim  $A \leq n$  iff every finite open cover of A has a finite open refinement such that at most (n + 1) elements have a nonempty intersection. We say dim A = n iff dim  $A \leq n$  but dim  $A \leq n - 1$ .

See [7] and [10] for a further discussion of dimension theory. An *n*-dimensional cover of a space is a cover in which some

(n + 1) elements meet, but no (n + 2) elements meet. An *initial segment* of ordinals in  $X_n$  (or  $X_n - \{\Omega_n\}$ ) is the set of all ordinals less than some  $\alpha \in X_n$  (or  $X_n - \{\Omega_n\}$ ).

An end segment of ordinals is the complement of an initial segment.

A cofinal set S in  $X_n$  means if  $\alpha \in [0, \Omega_n)$  then there exists  $\beta$  such that  $\alpha < \beta < \Omega_n$  and  $\beta \in S$ .

2. The non-monotony of covering dimension. We have seen that dim X = 0. In this section we prove that dim  $Y_n = n$  for  $1 \leq n \leq \infty$ , and that X contains a copy of each  $Y_n$ . It is clear that  $Y_n$  is not normal for  $n \geq 1$ , since it contains, as a closed subset, the nonnormal subset T of the Tychonoff Plank S discussed in §1.

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We note that Dowker has given an example of a normal Hausdorff space N with a normal subset M such that dim N = Ind N = 0, but dim M = Ind M = 1. See [3] or pages 102-3 of [8].

LEMMA 2.1. Let  $\mathscr{U}: U_1, U_2, \dots, U_k$  be a finite open cover of  $Y_n$ . Let  $B_i = \{(x_0, x_1, \dots, x_i, \dots, x_n) \in Y_n | x_i \in [0, \Omega_i), x_j = \Omega_j \text{ for } j \neq i\}$ . Then some element of  $\mathscr{U}$  must contain a cofinal subset of  $B_i$ .

*Proof.* Suppose not. Since  $B_i$  is well-ordered by its *i*th coordinate, then for each  $j, 1 \leq j \leq k$ , there exists an  $\alpha_{i_j} \in [0, \Omega_i]$  such that  $(\Omega_0, \Omega_1, \dots, \alpha_{i_j}, \dots, \Omega_n)$  is the first element of  $B_i$  such that no element following it is in  $U_j$ . Let  $\alpha = \max{\{\alpha_{i_j}\}_{j=1}^k}$ . Then

$$p = (\Omega_0, \Omega_1, \cdots, \beta_i, \cdots, \Omega_n)$$

is not in any element of  $\mathcal{U}$ , if  $\beta_i$  is the *i*th coordinate of *p*, with  $\beta_i > \alpha$ . This is a contradiction.

THEOREM 2.1.  $Y_1$  is one dimensional.

**Proof.** Let  $\mathscr{U} = \{U_1, U_2\}$  be a cover of  $Y_1$  defined as follows:  $U_1 = Y_1 - B_0$ , where  $B_0 = \{(x, \Omega_1) | x \in [0, \Omega_0)\}$ ,  $U_2 = Y_1 - B_1$ , where  $B_1 = \{(\Omega_0, x) | x \in [0, \Omega_1)\}$ . We will show that any finite refinement  $\mathscr{V}$  of  $\mathscr{U}$  must contain some pair of elements which intersect. We first note that some element of  $\mathscr{V}$  must contain a cofinal subset of  $B_0$ , by Lemma 2.1. We assume that  $V_1$  is such an element.

We show that  $V_1$  must have a limit point in  $B_1$ . For each  $(\alpha, \Omega_1) \in V_1 \cap B_0$ , there exists  $\beta_\alpha$  such that  $(\alpha, \beta) \in V_1$  for  $\beta \geq \beta_\alpha$ , since  $V_1$  is an open set about  $(\alpha, \Omega_1)$  and the set  $\{(\alpha, x) \mid \alpha \text{ fixed}, x \in [0, \Omega_1]\}$  is homeomorphic to  $X_1$ . Then card  $\{\beta_\alpha \mid (\alpha, \Omega_1) \in V_1\} < \text{card } \Omega_1$ , and therefore  $\gamma = \sup \{\beta_\alpha \mid (\alpha, \Omega_1) \in V_1\} < \Omega_1$ . Then  $(\Omega_0, \gamma)$  is a limit point of  $V_1$ , but not in  $U_2$  and therefore not in  $V_1$ . Thus any element of  $\mathscr{V}$  containing  $(\Omega_0, \gamma)$  must meet  $V_1$ . It follows that  $Y_1$  is at least one dimensional.

We must now show that  $Y_1$  is at most one dimensional. Let  $\mathscr{U}$  be an arbitrary finite open cover of  $Y_1$ . We wish to find a finite refinement which is at most 1-dimensional—i.e. no three elements intersect. Recall that  $Y_1 = [0, \Omega_0] \times [0, \Omega_1] - \{(\Omega_0, \Omega_1)\}$ .

Now  $B_0 = [0, \Omega_0) \times \{\Omega_1\}$  is a closed subset of  $Y_1$  and is homeomorphic to the set of positive integers. Since  $\mathscr{U}$  is a finite open cover of  $Y_1$ , we note, incidentally, that at least one element of  $\mathscr{U}$  contains a cofinal subset of  $B_0$ . Let  $U_1, U_2, \dots, U_n$  be all the elements of  $\mathscr{U}$  which intersect  $B_0$ . If  $V'_1 = U_1 \cap B_0$ ,  $V'_i = (U_i - \bigcup_{j < i} U_j) \cap B_0$ , then  $\bigcup_{i=1}^n V'_i = B_0$ , and  $V'_i$  is open in  $B_0$ . For each point  $(\alpha_{i_\beta}, \Omega_1) \in V'_i$ , there exists  $\overline{\gamma}_{i_\beta} \in [0, \Omega_1)$  such that for all  $\gamma \geq \overline{\gamma}_{i_\beta}, (\alpha_{i_\beta}, \gamma) \in U_i$ . If  $\overline{\gamma}_i =$ 

 $\sup \{\overline{\gamma}_{i_{\beta}} | (\alpha_{i_{\beta}}, \Omega_1) \in V'_i\}$ , then  $V'_i \times [\overline{\gamma}_i, \Omega_1] \subseteq U_i$ . If  $\delta \ge \max \{\overline{\gamma}_i | i=1, \dots, n\}$ , subject to the condition that  $\delta$  is not a limit ordinal, then  $V'_i \times [\delta, \Omega_1] \subseteq U_i$ ,  $i = 1, \dots, n$ . Let  $W_i = V'_i \times [\delta, \Omega_1]$ , and we note that  $W_i$  is an open subset of  $u_i$ . Now  $[0, \Omega_0] \times [0, \delta - 1]$  is a compact subset of  $Y_1$ , which is also open in Y. Therefore there exists a finite 0-dimensional refinement  $\mathscr{U}'$  of  $\mathscr{U}$ , covering this set. Now the collection of members of  $\mathscr{U}'$  together with the  $\{W_i\}_{i=1}^n$ , forms a 0-dimensional finite open cover of  $Y_1 - (\{\Omega_0\} \times [\delta, \Omega_1))$ . We will add enough members to this collection in such a way that an at most 1-dimensional cover will result.

If  $C = \{\Omega_0\} \times [\delta, \Omega_1)$ , then a finite number of the elements of  $\mathcal{U}_i$ , say  $U_1'', \dots, U_k''$ , cover C. Thus if  $D_i = U_i'' \cap C$ , then  $\{D_i\}_{i=1}^k$  is a finite open cover of C in C. Since C is 0-dimensional, there exists a finite 0-dimensional refinement, say  $E_1 \cdots E_s$ . Now  $[0, \Omega_0] \times E_i$  is open in  $Y_1$ . If  $N_i = ([0, \Omega_0] \times E_i) \cap U_{E_i}$ , where  $U_{E_i}$  is any element of  $\{U_i\}_{i=1}^n$  which contains  $E_i$ , then the collection  $\{N_i\}_{i=1}^s$  is 0-dimensional collection.

Let  $\mathscr{Q}$  be the finite collection of open sets consisting of the members of  $\mathscr{U}'$ , the members of  $\{W_i\}_{i=1}^n$ , and the members of  $\{N_i\}_{i=1}^s$ . Then  $\mathscr{Q}$  is an at most 1-dimensional cover of  $Y_i$ , for the only possible intersections are between an  $N_i$  and either an element of  $\mathscr{U}'$  or of  $\{W_i\}_{i=1}^n$ .

It follows that  $Y_1$  is 1-dimensional.

THEOREM 2.2.  $Y_n$  has dimension  $\geq n$ .

*Proof.* The proof is by induction on the method of proof of Theorem 2.1. Let  $\mathscr{U}: U_0, U_1, \dots, U_n$  be an open cover of  $Y_n$  defined by  $U_i = Y_n - \bigcup_{j \neq i} B_{n,j}$ , where  $B_{n,j} = \{(x_0, x_1, \dots, x_j, \dots, x_n) \in Y_n | x_j \in [0, \Omega_j) \text{ and } x_k = \Omega_k \text{ for } k \neq j\}$ . Let  $\mathscr{V}$  be any finite refinement of  $\mathscr{U}$ . By Lemma 2.1, for each  $i, 0 \leq i \leq n$ , there exists  $V_i \in \mathscr{V}$  such that  $V_i$  contains a cofinal subset of  $B_{n,i}$ . We will show that  $\bigcap_{i=0}^n V_i \neq \emptyset$ , by showing that  $\bigcap_{i=0}^{n-1} V_i$  has an end segment of  $B_{n,n}$  as limit points. We note that  $Y_{n-1}$  is homeomorphic to

$$\{(x_0, x_1, \cdots, x_{n-1}, \Omega_n) \in Y_n | (x_0, x_1, \cdots, x_{n-1}) \in Y_{n-1}\} \subseteq Y_n$$
,

and that  $B_{n-1,i}$  is homeomorphic to  $B_{n,i}$  for  $0 \leq i \leq n-1$ . We identify these in the remainder of the proof, without explicitly referring to the homeomorphism between them.

We assume, as an inductive hypothesis, that for any collection of  $V_i$ 's chosen as above,  $\bigcap_{i=0}^{n-1} V_i \neq \emptyset$ . For n = 2, this is proved in Theorem 2.1. Let  $V'_i = Y_{n-1} \cap V_i$ , let  $V' = \bigcap_{i=0}^{n-1} V'_i$ , and let  $V = \bigcap_{i=0}^{n-1} V_i$ .

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Now card  $Y_{n-1} < \operatorname{card} \Omega_n$ , and therefore card  $V' < \operatorname{card} \Omega_n$ . For  $x \in V'$ , let  $\alpha_x \in [0, \Omega_n)$  be an ordinal such that if  $\alpha > \alpha_x$ , then  $(x, \alpha) \in V$ . We know that  $\alpha_x$  exists since V is open in  $Y_n$  containing the point  $(x, \Omega_n)$ . Card  $\{\alpha_x | x \in V'\} < \operatorname{card} \Omega_n$ . Therefore if  $\overline{\alpha} = \sup \{\alpha_x\}$ , then  $\overline{\alpha} < \Omega_n$  and  $V' \times [\overline{\alpha}, \Omega_n] \subseteq V$ .

Now let  $y = (\Omega_0, \Omega_1, \dots, \Omega_{n-1}, \beta) \in B_{n,n}$  such that  $\beta \ge \overline{\alpha}$ . We wish to show that y is a limit point of V. We accomplish this by showing that if  $(\beta_0, \beta_1, \dots, \beta_{n-1}, \beta) \in Y_n, \beta_i < \Omega_i$ , then there exist  $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$  such that  $\gamma_i > \beta_i$  and  $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \beta) \in V$ . To this end, let  $\eta_i$  be a non-limit ordinal  $> \beta_i$  such that  $\eta_i \in V_i \cap B_{n,i}$ , let

$$W_i = \left( \left[ \eta_i, \, arOmega_i 
ight] imes \prod_{j=0 \ j 
eq i}^n x_j 
ight) \cap \, V_i \;\;\; ext{for}$$

 $0 \leq i \leq n-1$ , and let  $W'_i = W_i \cap Y_{n-1}$ . Then  $W'_i$  is a neighborhood of a cofinal set in  $B_{n,i} \subseteq Y_{n-1}$ , and therefore, by the inductive hypothesis,  $W'_i = \bigcap_{i=0}^{n-1} W'_i \neq \emptyset$ . We note that  $W' \subseteq V'$ , and therefore there exists a point

$$(\gamma_0, \gamma_1, \cdots, \gamma_{n-1}) \in W' \subseteq V'$$

Then  $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \beta) \in V' \times [\bar{\alpha}, \Omega_n] \subseteq V.$ 

Thus we have shown that each point of  $\{(\Omega_0, \Omega_1, \dots, \Omega_{n-1})\} \times [\overline{\alpha}, \Omega_n)$ is a limit point of V. Therefore any neighborhood  $V_n$  of a cofinal set in  $B_{n,n}$  must meet V and  $\bigcap_{i=0}^n V_i \neq \emptyset$ . It follows that dim  $Y_n \ge n$ .

THEOREM 2.3.  $Y_n$  has dimension  $\leq n$ .

*Proof.* Let  $\mathscr{U}$  be a finite open cover of  $Y_n$ . We wish to find a finite refinement of  $\mathscr{U}$  which is an at most *n*-dimensional cover of  $Y_n$ . Let  $B_{n,i}$  be defined as in the proof of Theorem 2.2, and again we don't distinguish in notation between  $B_{n-1,i}$  in  $Y_{n-1}$ , and its homeomorphic copy  $B_{n,i}$  in  $Y_n$  for  $1 \leq i \leq n-1$ . We also identify  $Y_i$  with  $(Y_i \times \prod_{j=i+1}^n \{\Omega_j\})$  in  $Y_n$ .

We assume, as an inductive hypothesis, that  $Y_{n-1}$  has an at most (n-1) dimensional refinement of any given cover. Let  $\mathscr{U}'$  be the elements of  $\mathscr{U}$  intersected with  $Y_{n-1}$ . Then  $\mathscr{U}'$  is a cover of  $Y_{n-1}$ , and by hypothesis, has an at most (n-1)-dimensional refinement. Let  $\mathscr{V}': V'_1, \dots, V'_k$  be such a refinement of  $\mathscr{U}'$ . Let  $V'_i \subseteq U'_i$ . Note that it is possible that  $U'_i = U'_j$  for  $i \neq j$ . Note also that card  $Y_{n-1} < \operatorname{card} \Omega_n$ . Fix *i*. For each  $x \in V'_i$ , there exists  $\alpha_{x_i}$  such that for  $\alpha > \alpha_{x_i}$ ,  $(x, \alpha) \in U_i$ . Let  $\overline{\alpha}_i = \sup \{\alpha_{x_i} | x \in V'_i\}$  and we see that  $\overline{\alpha}_i < \Omega_n$ . Let  $\overline{\alpha} \geq \sup \{\overline{\alpha}_i | i = 1, \dots, k\}$  such that  $\overline{\alpha}$  is a non-limit ordinal. Then  $V_i = V'_i \times [\overline{\alpha}, \Omega_n] \subseteq U_i \in \mathscr{U}$ , for each  $i, 1 \leq i \leq k$ . Thus  $Y_{n-1} \times [\overline{\alpha}, \Omega_n]$  is a subset of  $Y_n$  which is covered by an at most (n-1)-dimen-

sional finite collection,  $\mathscr{V}: V_1, \dots, V_k$ , which refines  $\mathscr{U}$ . The points of  $[B_{n,n} \cup (Y_{n-1} \times [0, \overline{\alpha}))]$  are the only points not yet covered. Now  $Z_{n-1} \times [0, \overline{\alpha})$  is a zero dimensional clopen subset of  $Y_n$ , and therefore there exists a finite, zero dimensional refinement of  $\mathscr{U}$  covering it, say  $\mathscr{Q}$ .

It remains to cover the set of points of  $\{(\Omega_0, \Omega_1, \dots, \Omega_{n-1})\} \times [\overline{\alpha}, \Omega_n\}$ . Call this set A. The elements of  $\mathscr{U}$  intersected with A is a cover of A, say  $\mathscr{U}''$ , and A is zero dimensional. Thus there exists a zero dimensional finite refinement of  $\mathscr{U}''$  which covers A, say  $\mathscr{W}: W_1$ ,  $W_2, \dots, W_s$ . Now fix i. For each  $x \in W_i \subseteq U_i$ , there exists a clopen product neighborhood,  $H_{i_x} = H_0 \times H_1 \cdots \times H_{n-1} \times W_{i_x}$ , where  $W_{i_x} \subseteq W_i$  and such that  $H_{i_x} \subseteq U_i$ . Let  $H'_i = \bigcup_{y \in W_i} H_{i_y}$ . Then  $x \in H'_i, H'_i \subseteq U_i$ , and  $H'_i$  is open in  $Y_n$ . Further  $H'_i \cap H'_j = \emptyset$  for  $i \neq j$ , and  $H'_i \cap O_j = \emptyset$  for any  $O_j \in \mathscr{C}$ . Let  $\mathscr{M} = \{H'_1, H'_2, \dots, H'_s\}$ . Then since at most n elements of  $\mathscr{V}$  have  $\neq \emptyset$  intersection, the addition of  $\mathscr{M}$  to  $\mathscr{V}$  will make at most n + 1 elements intersect. Thus the cover defined by the elements of  $\mathscr{V}$ ,  $\mathscr{Q}$ , and  $\mathscr{M}$  is an at most n-dimensional cover of  $Y_n$ .

It follows that  $Y_n$  is at most *n*-dimensional.

THEOREM 2.4.  $Y_n$  has dimension n.

Proof. Clear from Theorem 2.2 and 2.3.

THEOREM 2.5. Y is infinite dimensional and contains subsets of dimension n, for every positive integer n.

*Proof.* It is clear that Y contains n-dimensional subsets for all n, since  $Y_n$  is homeomorphic to  $\{(x_0, x_1, \dots, x_n, \dots) \in Y | x_i = \Omega_i \text{ for all } i > n\} \subseteq Y$ . We identify this set with  $Y_n$  in our notation. Since  $Y_n$  is closed in Y, it follows that dim  $Y = \infty$ .

THEOREM 2.6. X is a 0-dimensional, compact, Hausdroff space, and contains subsets of dimension n, for  $1 \leq n \leq \infty$ .

*Proof.* By Theorem 2.5, Y is  $\infty$ -dimensional and contains subsets of dimension n for  $1 \leq n \leq \infty$ . But  $Y \subseteq X$ . The theorem follows.

COROLLARY 2.6.1. For each  $n, 1 \leq n \leq \infty$ , there exists an n-dimensional continuum with subsets of dimension  $k, 1 \leq k \leq \infty$ .

*Proof.*  $X_i$  is a totally ordered, totally disconnected compact Hausdorff space, and can be imbedded in a generalized arc,  $A_i$ , preserving the order of  $X_i$ , in such a way that if x < y in  $X_i$  and there does not exist z in  $X_i$  with x < z < y, then  $A_i$  contains a real arc

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from x to y.

Now  $X = \prod_{i=0}^{\infty} X_i$  and is a compact totally disconnected Hausdorff space with a lattice-like structure, intuitively like the ordered  $\infty$ tuples of integers in  $\mathbb{R}^{\infty}$ . We fill in the lattice structure, to make a 1-dimensional grid, in the following way. Let  $M = \prod_{i=1}^{\infty} A_i$  and let  $M_1$  be the subset of M defined by  $M_1 = \{(\alpha_1, \alpha_2, \dots, \alpha_i, \dots) \in M |$  there exists at most one coordinate  $\alpha_k$  such that  $\alpha_k \in A_k - X_k\}$ . We show that dim  $M_1 = 1$ . Clearly ind  $M_1 = 1$ , since, if  $x \in M_1$  and U is open with  $x \in U$ , then there exists a neighborhood V of x such that  $V \subseteq$ U, and BdV is homeomorphic to a subset of  $\prod_{i=0}^{\infty} X_i$ . Thus BdV is 0-dimensional, and ind  $M_1 \leq 1$ . But  $M_1$  contains arcs. Thus ind  $M_1 =$ 1. Now, by Theorem B page 198 of [10], since  $M_1$  is compact Hausdorff, dim  $M_1 \leq$  ind  $M_1$ . It follows that dim  $M_1 = 1$ , since clearly dim  $M_1 > 0$ .

Let  $I_i = [0, 1]$  and let  $M_n = M_1 \times \prod_{i=1}^{n-1} I_i$ . By Theorem 4.7 of [2] the product of an *n*-dimensional compact space with the unit interval has dimension  $\leq n + 1$ . Now  $M_1$  contains an interval, so that dim  $M_2 = 2$ , and inductively, dim  $M_n = n$ . Since  $X \subseteq M_n$ , the corollary follows.

3. On compactifications of  $Y_n$  and Y. In [4, pg. 124], it is shown that the one point compactification of  $Y_1 = \beta Y_1 = Z_1$ . In this section, we show that the one points compactification of  $Y_n = \beta Y_n = Z_n$ , for  $1 \leq n < \infty$ . Thus for each  $n, 1 \leq n < \infty$ , we have an example of a space of covering dimension n, whose Stone-Čech compactification is of dimension 0. On pages 244-245 of [4], it is proved that dim  $X = \dim \beta X$ , if X is normal. Thus we see the necessity of the hypothesis of normality. We note that in [4, pg. 248], it is pointed out that  $Y_1$  is a space of *positive* dimension whose Stone-Čech compactification has dimension 0.

LEMMA 3.1. Let f be continuous from  $X_n$  (or  $X_n - \{\Omega_n\}$ ) to the reals,  $n \ge 1$ . Then there exists an end segment  $E = [\alpha, \Omega_n]$  (or  $[\alpha, \Omega_n]$ ) such that f is constant on E.

*Proof.* This is proved for  $X_1$  on page 75 of [4]. It is clear that the same proof works for  $X_n$ .

THEOREM 3.1. The Stone-Čech compactification of  $Y_n$  is  $Z_n$ ,  $n \ge 1$ . Proof. For n = 1, this is proved on page 124 of [4].

In general,  $Y_n = \prod_{i=0}^n X_i - \{(\Omega_0, \Omega_1, \dots, \Omega_n)\}$ . By Theorems 6.4, 6.5 of [4] it is sufficient to show that every (bounded) continuous

function from  $Y_n$  to the reals, has a continuous bounded extension to  $Z_n$ . It follows from the argument below that any continuous real function on  $Y_n$  is necessarily bounded.

We assume, as an inductive hypothesis, that every real continuous function on  $Y_{n-1}$  can be extended to  $Z_{n-1}$ . Let  $f: Y_n \to R$  be continuous. For each  $x \in Y_{n-1}$ ,  $W_x = \{(x, \alpha) \mid \alpha \in X_n\}$  is homeomorphic to  $X_n$ , and therefore  $f \mid W_x$  is constant on an end segment, by Lemma 3.1; that is, there exists  $\alpha_x$  such that  $f(x, \alpha) = f(x, \alpha_x)$  for all  $\alpha \ge \alpha_x$ . Further, if  $W = \{(\Omega_0, \Omega_1, \dots, \Omega_{n-1}, \alpha) \mid \alpha \in [0, \Omega_n)\}$ , then f is constant on an end segment of W, also, say for  $\alpha \ge \alpha_0$ . Now card  $Z_{n-1} < \operatorname{card} \Omega_n$ . Let  $A = [\{\alpha_0\} \cup \{\alpha_x \mid x \in Y_{n-1}\}]$ , and it follows that card  $A < \operatorname{card} \Omega_n$ . Therefore  $\sup A < \Omega_n$ . Let  $\overline{\alpha} \ge \sup A$  such that  $\overline{\alpha}$  is a non-limit ordinal. Then  $[Z_{n-1} \times (\overline{\alpha}, \Omega_n]] - \{(\Omega_0, \Omega_1, \dots, \Omega_n)\}$  has the property that on each  $\{x\} \times (\overline{\alpha}, \Omega_n]$ , f is constant. It follows that for  $x \in Y_{n-1}$ ,  $f((x, \Omega_n)) =$  $f((x, \beta))$ , for each  $\beta \ge \overline{\alpha}$ . Now, since  $f \mid Y_{n-1} \times \{\Omega_n\}$  can be extended to  $(\Omega_0, \dots, \Omega_{n-1}, \Omega_n)$ , say it has value k, then since this extension is unique, it has same value as f on  $(\Omega_0, \dots, \Omega_{n-1}, \beta)$  for each  $\beta \ge \overline{\alpha}$ . Thus we can extend f to  $Z_n$ , by defining  $f((\Omega_0, \dots, \Omega_n)) = k$ .

It follows that  $Z_n$  is the Stone-Čech compactification of  $Y_n$ .

QUESTION. Is X the Stone-Čech compactification of Y?

THEOREM 3.2. For each  $n, 1 \leq n < \infty$ , there exists a (non-normal) space of dimension n, whose Stone-Čech compactification has dimension 0.

*Proof.* Our spaces  $Y_n$  satisfy the conditions of the theorem, by Theorems 2.4, 2.5, and 3.1.

4. The non-monotony of large inductive dimension. In §2, we proved that dim  $Y_n = n, 1 \leq n \leq \infty$ . In this section, we show that Ind  $Y_n \geq n, 1 < n < \infty$ , Ind  $Y_1 = 1$ , and Ind  $Y_{\infty} = \infty$ . Thus Ind X = 0, but X contains subsets  $Y_n$  whose large inductive dimension is at least  $n, 1 \leq n \leq \infty$ .

LEMMA 4.1. If  $F_1$  and  $F_2$  are disjoint closed sets in  $[0, \Omega_n), 0 < n < \infty$ , then one of them is bounded.

*Proof.* (See 5.12 (b) p. 74 of [4]. Proof is included here for completeness.) Suppose neither is bounded. Let  $\{\alpha_i\}_{i=1}^{\infty}$  be a sequence of elements of  $[0, \Omega_n)$  chosen inductively so that (1)  $\alpha_i < \alpha_{i+1}$ , (2)  $\alpha_i \in F_1$  for *i* odd, and (3)  $\alpha_i \in F_2$  for *i* even. If  $\gamma = \text{lub} \{\alpha_i\}$  then  $\gamma < \Omega_n$  and  $\gamma$  is a limit point of both  $F_1$  and  $F_2$ . Therefore  $\gamma \in F_1 \cap F_2$ , and this is a contradiction.

LEMMA 4.2. Ind  $[0, \Omega_n) = 0, 0 \leq n < \infty$ .

*Proof.* Case 1. n > 0. Let F be closed in  $[0, \Omega_n)$ , and let U be a neighborhood of F. We wish to find an open set V such that  $F \subseteq V \subseteq U$  and Bd  $V = \emptyset$ .

Suppose F is bounded, there exists a clopen cover of F by subsets  $V_{\alpha}$  of U. Since F is bounded then it must be compact, and therefore there exist  $V_{\alpha_1}, \dots, V_{\alpha_k}$  contained in U such that  $F \subseteq V = \bigcup_{i=1}^k V_{\alpha_i}$ . Then V is open,  $F \subseteq V \subseteq U$ , and Bd  $V = \emptyset$ .

If F is not bounded then, by Lemma 4.1, C(U) is bounded, and by a similar argument, we find the required set V.

Thus, in either event, there exists an open set V such that  $F \subseteq V \subseteq U$  and Bd  $V = \emptyset$ .

Case 2. n = 0. Clear, since the nonnegative integers are discrete.

LEMMA 4.3. If  $\mathscr{U}$  is a finite open cover of  $[0, \Omega_n), 1 \leq n < \infty$ , then some element of  $\mathscr{U}$  contains an end segment of  $[0, \Omega_n)$ .

*Proof.* Suppose no element of  $\mathscr{U}$  contains an end segment. Then at least two elements of  $\mathscr{U}$  must contain disjoint cofinal sets in  $[0, \Omega_n)$ . Let  $U_1, \dots, U_k$  be those elements of  $\mathscr{U}$  which contain pairwise disjoint cofinal sets  $A_1, \dots, A_k$  respectively, in  $[0, \Omega_n)$  with  $A_i \cap U_j = \emptyset, j \neq i, i, j = 1, \dots, k$ , and let  $a_{i,1}, a_{2,1}, \dots, a_{k,1}$  be elements of  $A_1 \dots, A_k$  respectively, such that  $a_{i,1} > x$  for all  $x \in \bigcup_{j=1}^k U_j$ . Let  $b_1 \in [0, \Omega_n)$  such that  $b_1 > a_{i,1}, 1 \leq i \leq k$ , let  $a_{1,2}, a_{2,2}, a_{3,2}, \dots, a_{k,2}$  elements of  $A_1, \dots, A_k$  respectively, such that  $a_{i,2} > b_1$ , and let  $b_2 \in [0, \Omega_n)$ such that  $b_2 > a_{i,2}, 1 \leq i \leq k$ .

Continue the above process inductively, and let  $b = \sup \{b_i\}$ . Then  $b < \Omega_n$  and b is a limit point of  $A_i$ ,  $1 \leq i \leq k$ . Further, if  $b \in U_j$  then  $j \in \{1, 2, \dots, k\}$  and  $U_j$  contains points of each  $A_i$ ,  $1 \leq i \leq k$ . But  $A_i \cap U_j = \emptyset$  for  $i \neq j$  when  $1 \leq i \leq k$ , and this is a contradiction.

LEMMA 4.4. If  $F_1$  and  $F_2$  are disjoint closed sets in  $\prod_{i=0}^{n-1} [0, \Omega_i] \times [0, \Omega_n)$ , then at least one of  $F_1$  and  $F_2$  is bounded away from the corner point of  $Y_n$  in the n<sup>th</sup> direction; that is, there exist non-limit ordinals  $\alpha_0, \alpha_1 \cdots, \alpha_n$  such that  $\prod_{i=0}^{n-1} [\alpha_i, \Omega_i] \times [\alpha_n, \Omega_n)$  misses, say,  $F_2$ . (The corner point of  $Y_n$  is  $(\Omega_0, \cdots, \Omega_n)$  even though this point does not belong to  $Y_n$ .)

*Proof.* Let  $B_n = \{(\Omega_0, \dots, \Omega_{n-1}, x) | x \in [0, \Omega_n)\}$  and we see by Lemma 4.1, not both  $F_1 \cap B_n$  and  $F_2 \cap B_n$  can be unbounded in  $B_n$ . Thus we assume that there exists a non-limit ordinal  $\alpha_n$  such that if  $\alpha \ge \alpha_n$  then  $(\Omega_0, \Omega_1, \dots, \Omega_{n-1}, \alpha) \notin F_2$ .

Now, suppose the lemma is false. Then for each point  $(\beta_0, \beta_1, \dots, \beta_n)$  with  $\beta_n > \alpha_n$ , and  $\beta_i < \Omega_i$  for all *i*, there exists another point  $(\beta'_0, \beta'_1, \dots, \beta'_n)$  in  $F_2$  such that  $\beta_i < \beta'_i \leq \Omega_i$  for  $0 \leq i \leq n-1$  and  $\beta_n < \beta'_n < \Omega_n$ . Thus after a countable number of choices, with each point closer, in at least one coordinate to  $(\Omega_0, \dots, \Omega_n)$  than the preceding point (closer in 0<sup>th</sup> coordinate whenever possible), we exhaust all possibilities for  $\beta_0$ . This is true because there are only a countable number of points in  $X_0$ . Then if  $\beta_{i,1}$  denotes  $\sup \{\beta'_i|$  for each choice of  $\beta'_0 \leq \Omega_0$  it follows that  $(\Omega_0, \beta_{1,1}, \beta_{2,1}, \dots, \beta_{n,1})$  is a point or limit point of  $F_2$  and therefore must be in  $F_2$ . Further, each  $\beta_{i,1} \leq \Omega_i$ ,  $1 \leq i < n$ , and  $\beta_{n,1} < \Omega_n$ .

We continue the process inductively and note that after at most  $|\Omega_1|$  choices, we obtain a point  $(\Omega_0, \Omega_1, \beta_{2,2} \cdots, \beta_{n,2})$  as a point or limit point of  $F_2$ , and therefore in  $F_2$ , and such that  $\beta_{i,2} \leq \Omega_i, 2 \leq i < n$ , with  $\beta_{n,2} < \Omega_n$ . After *n* steps, we obtain a point  $(\Omega_0, \Omega_1, \cdots, \Omega_{n-1}, \beta_{n,n})$  in  $B_n$  as a point or limit point of  $F_2$  and therefore in  $F_2$ . But  $\beta_{n,n} > \alpha_n$  and this is a contradiction. Thus  $F_2$  must be bounded away from the corner point of  $Y_n$  in the  $n^{\text{th}}$  direction.

LEMMA 4.5. If  $B \subseteq A$  and  $U \subseteq A$ , then  $\operatorname{Bd}_B(U \cap B) \subseteq \operatorname{Bd}_A(U)$ .

*Proof.* Let  $x \in Bd_B(U \cap B)$ . Then each neighborhood of x contains points of  $U \cap B$  and of  $C(U) \cap B$  and therefore contains points of both U and C(U). It follows that  $x \in Bd_A(U)$ .

REMARK. The following two lemmas are well known among dimension theorists. However, we include them here since they may not have been explicitly stated in other published work. The proof is the same as the proof for the metric case [10, II. 1.A].

LEMMA 4.6. If Ind A = 0 and B is a closed subset of A, then Ind B = 0.

LEMMA 4.7. If Ind A = k and B is a closed subset of A, then Ind  $B \leq k$ .

REMARK. The next proposition is not used in the rest of the paper, but is included for its (possible) interest. Also, the proof of this proposition is referred to in Theorem 4.1.

PROPOSITION 4.1.<sup>1</sup> If  $S = (\prod_{i=0}^{n-1} X_i) \times [0, \Omega_n)$ , then dim S = Ind S = 0.

<sup>&</sup>lt;sup>1</sup> The referee has pointed out that this theorem follows easily from a theorem of Morita (*Topogical completeness and M-spaces*, Sci. Reports Tokyo Kyoiku Daigaku. Sec. A **10** (1970) 271-288) that dim  $X \times Y \leq \dim X + \dim Y$  if X is paracompact, locally compact, and Y is pseudoparacompact.

Proof. Let  $\mathscr{U}$  be a finite open cover of S. Each  $\{x\} \times [0, \Omega_n)$  is homeomorphic to  $[0, \Omega_n)$ , and therefore, by Lemma 4.3, some end segment of  $\{x\} \times [0, \Omega_n)$ , say  $\{x\} \times [\alpha_x, \Omega_n)$ , is a subset of every element of  $\mathscr{U}$  which it meets. Since card  $(\prod_{i=0}^{n-1} X_i) < \text{card } [0, \Omega_n)$ , let  $\alpha' =$  $\sup \{\alpha_x | x \in \prod_{i=0}^{n-1} X_i\}$  and let  $\alpha > \alpha'$  be a non-limit ordinal. Then for each x, there is a j such that  $(\{x\} \times [\alpha, \Omega_n)) \subseteq U_j$  Further, if  $\{x\} \times [\alpha, \Omega_n)$  meets several  $U_j$ 's, then it is a subset of each of them. Consider  $(\prod_{i=0}^{n-1} X_i) \times \{\alpha\}$ . This is a compact zero dimensional space, and  $\mathscr{U}' =$  $\{U_j \cap (\prod_{i=0}^{n-1} X_i \times \{\alpha\}) | U_j \in \mathscr{U}\}$  is a finite open cover of this space. Thus there exists a zero dimensional refinement  $\mathscr{V}'$  of  $\mathscr{U}'$ . For each  $V' \in \mathscr{V}', V' \times [\alpha, \Omega_n] \subseteq$  some  $U_j$ , and the collection  $\mathscr{V} = \{V' \times [\alpha, \Omega_n) | V' \in$  $\mathscr{V}'\}$  is a zero dimensional cover of  $(\prod_{i=0}^{n-1} X_i) \times [\alpha, \Omega_n)$  refining  $\mathscr{U}$ .

Now since  $\prod_{i=0}^{n-1} X_i \times [0, \alpha - 1]$  is a zero dimensional compact Hausdorff space, there exists a finite zero dimensional refinement  $\mathscr{W}$ of  $\mathscr{U}$  covering this set. Then the collection  $\mathscr{V} \cup \mathscr{W}$  is a finite zero dimensional refinement of  $\mathscr{U}$ . It follows that dim S = 0.

Now in the appendix of [7], it is shown that for any space S, dim S = 0 iff Ind S = 0. Thus the lemma is proved.

THEOREM 4.1. Ind  $Y_1 = 1$ .

*Proof.* If  $B_0 = \{(x, \Omega_1) | x \in [0, \Omega_0)\}, B_1 = \{(\Omega_0, x) | x \in [0, \Omega_1)\}$ , and  $U = Y_1 - B_1$ , then  $B_0$  is a closed subset of  $Y_1$  and  $B_0 \subseteq U$ , with U open in  $Y_1$ . As in the proof of Theorem 2.1, we see that for any open set V such that  $B_0 \subseteq V \subseteq U$ , Bd V must contain  $a \neq \emptyset$  end segment of  $B_1$ . Thus Ind  $Y_1 \ge 1$ .

We now show Ind  $Y_1 \leq 1$ . Let F be closed in  $Y_1$  and let U be and neighborhood of F. F and Bd U are disjoint closed sets in  $Y_1$ .

If F contains a cofinal set in  $B_1$ , then by Lemma 4.1, Bd U misses an end segment of  $B_1$  and by Lemma 4.4, there exist non-limit ordinals  $\alpha_0$  and  $\alpha_1$  such that  $[\alpha_0, \Omega_0] \times [\alpha_1, \Omega_1)$  misses Bd U. Thus Bd  $U \subseteq K \cup$  $([\alpha_0, \Omega_0] \times \{\Omega_1\})$ , where K is the totally disconnected compact open subset of  $Y_1$  defined by  $K = Z_1 - [\alpha_0, \Omega_0] \times [\alpha_1, \Omega_1]$ . Then  $K \cap \text{Bd } U$ can be covered by a finite number of clopen sets  $V_1, \dots, V_s$ , each of which is a subset of K and misses F, so that  $V = U - \bigcup_{i=1}^s V_i$  is open, contains F, and Bd  $V = \emptyset$  or Bd  $V \subseteq [\alpha_0, \Omega_0] \times {\Omega_1}$ .

If F contains no cofinal set in  $B_1$ , then by an argument like that of the above paragraph,  $F \subseteq L \cup ([\beta_0, \Omega_0) \times \{\Omega_1\})$  where L is a totally disconnected compact open subset of  $Y_1$  defined by  $L = Z_1 - [\beta_0, \Omega_0] \times$  $[\beta_1, \Omega_1]$ , and  $\beta_0$  and  $\beta_1$  are some non-limit ordinals. Then  $L \cap F$  can be covered by a finite number of clopen sets  $W_1, \dots, W_t$ , each of which is a subset of  $U \cap L$ , so that if  $W = \bigcup_{i=1}^t W_i$ , then  $\operatorname{Bd} W = \emptyset$ . From this we see that if  $F \subseteq L$ , there exists an open set W such that  $F \subseteq W \subseteq U$  and Bd  $W = \emptyset$ . Otherwise  $F \cap (Y_1 - L) \subseteq [\beta_0, \Omega_0] \times \{\Omega_1\}$ . Let  $F' = F \cap (Y_1 - L)$  and let  $U' = U \cap (Y_1 - L)$ . It suffices to show that there exists W' open such that  $F' \subseteq W' \subseteq U'$  and Bd W' is at most 0-dimensional (in the large inductive sense).

To this end, we note that U' is a neighborhood of  $F', F' \subseteq ([\beta_0, \Omega_0) \times \{\Omega_1\})$  and, since there are only countably many elements in F', by the method used in Proposition 4.1, there exists a non-limit ordinal  $\gamma$  such that if  $W' = ((F' \cap B_0) \times [\gamma, \Omega_1])$ , then  $W' \subseteq U'$ . Since  $[0, \Omega_0)$  is discrete, W' is open, so that W' is open, contains F' and  $\operatorname{Bd} W' \subseteq B_1$  or  $\operatorname{Bd} W' = \emptyset$ . Thus  $\operatorname{Bd} W'$  is at most zero dimensional. The theorem follows.

COROLLARY 4.1.1. If 
$$A = [\alpha, \Omega_0] \times [\beta, \Omega_1] - \{(\Omega_0, \Omega_1)\}$$
 then Ind  $A = 1$ .

Proof. Clear from the proof of Theorem 4.1.

COROLLARY 4.1.2. Let  $Y'_1 = [0, \Omega_1] \times [0, \Omega_2] - \{(\Omega_1, \Omega_2)\}$  and let  $Y'_1(\alpha, \beta) = [\alpha, \Omega_1] \times [\beta, \Omega_2] - (\Omega_1, \Omega_2)$ . Then  $\operatorname{Ind} Y'_1 = 1$  and  $\operatorname{Ind} Y'_1(\alpha, \beta) = 1$ .

*Proof.* A simple modification of the proof of Theorem 4.1 shows this.

DEFINITION. Let  $Y'_{n-1} = \prod_{i=1}^{n} [0, \Omega_i] - \{(\Omega_1, \Omega_2, \dots, \Omega_n)\}$ . By a corner of  $Y'_{n-1}$  we mean a set of the form  $\prod_{i=1}^{n} [\alpha_i, \Omega_i] - \{(\Omega_1, \dots, \Omega_n)\}$ , where  $\alpha_i$  is a non-limit ordinal,  $i = 1, \dots, n$ . In the proof below, we identify  $Y'_{n-1}$  with the set of points of  $Y_n$  whose first coordinate is  $\Omega_0$ .

THEOREM 4.2. Ind  $Y_n \ge n, 1 \le n < \infty$ .

*Proof.* If 
$$B_0 = \{(x, \Omega_1, \Omega_2, \dots, \Omega_n) \in Y_n | x \in [0, \Omega_0)\}$$
 and

$$B_i = \{ (\Omega_0, \cdots, \Omega_{i-1}, x, \Omega_{i+1}, \cdots, \Omega_n) \in Y_n | x \in [0, \Omega_i) \}$$

then  $B_0$  is closed and  $U = Y_n - Y'_{n-1}$  is a neighborhood of  $B_0$ . We assume as an inductive hypothesis that any corner of  $Y'_{n-1}$  has large inductive dimension at least n-1, and note that for n=2, this is just Corollary 4.1.2. We will show that any neighborhood V of  $B_0$ such that  $V \subseteq U$  must contain a corner of  $Y'_{n-1}$  on its boundary. As in the proofs of §2, we do not distinguish in notation between  $Y'_{n-1}$  and its copy in  $Y_n$ .

To this end, we first prove that if V is any neighborhood of  $B_0$ such that  $V \subseteq U$ , then there exist non-limit ordinals  $\alpha_1, \alpha_2, \dots, \alpha_n$ such that  $B_0 \times \prod_{i=1}^n [\alpha_i, \Omega_i] \subseteq V$ . By an analogous argument to the proof of Theorem 2.1, there is a non-limit ordinal  $\alpha_1$  such that  $B_0 \times$   $[\alpha_1, \Omega_1] \subseteq U$ . Inductively assume that there exist non-limit ordinals  $\alpha_1, \dots, \alpha_{k-1}$  such that  $B_0 \times \prod_{i=1}^{k-1} [\alpha_i, \Omega_i] \subseteq V$ . Now each point of  $B_0 \times \prod_{i=1}^{k-1} [\alpha_i, \Omega_i]$  is the last point of a "vertical set" over it in  $Y_n$  with the vertical set homeomorphic to  $[0, \Omega_k]$ , and card  $(B_0 \times \prod_{i=1}^{k-1} [\alpha_i, \Omega_i]) < \Omega_k$ . Thus for each of these "vertical sets", there exists  $\beta < \Omega_k$  such that the end segment below  $\beta$  lies in V, since V is a neighborhood of the last point of this segment. If we let  $\alpha_k$  be greater than or equal to  $\sup \{\beta \mid \beta \text{ is obtained as described for each point of } B_0 \times \prod_{i=1}^{k-1} [\alpha_{i_k}, \Omega_i]\}$  with  $\alpha_k$  a non-limit ordinal, then  $\alpha_k < \Omega_k$  and  $B_0 \times \prod_{i=1}^{k-1} [\alpha_i, \Omega_i] \subseteq V$ . Our assertion about the existence of  $\alpha_1, \dots, \alpha_n$  follows.

Thus we see that each point of

$$A = [\{\Omega_0\} \times \prod_{i=1}^n [\alpha_i, \Omega_i] - \{(\Omega_0, \Omega_1, \cdots, \Omega_n)\}]$$

is a limit point of  $B_0 \times \prod_{n=1}^{n} [\alpha_i, \Omega_i]$ , so that  $A \subseteq \operatorname{Bd} V$ . But  $\operatorname{Ind} A = n-1$ , and it follows that any neighborhood V of  $B_0$  such that  $V \subseteq U$  has at least (n-1) dimensional (Ind) set on its boundary. Thus  $\operatorname{Ind} Y_n \ge n$ .

COROLLARY 4.2.1. Let  $T = \prod_{\substack{i=0 \ i \neq i_0}}^n [0, \Omega_i] - \{(\Omega_0, \dots, \widehat{\Omega_{i_0}}, \dots, \Omega_n)\}$  and let  $T' = \prod_{\substack{i=0 \ i \neq i_0}}^n [\alpha_i, \Omega_i] - \{(\Omega_0, \dots, \widehat{\Omega_{i_0}}, \dots, \Omega_n)\}$ . Then each of T and T'has large inductive dimension at least n - 1.

*Proof.* A modification of the proof of Theorem 4.2 works.

THEOREM 4.3. Ind  $Y = \infty$ .

*Proof.*  $Y_n \subseteq Y$  homeomorphically and we identify  $Y_n$  with its image in Y. Since Ind  $Y_n \ge n$ , and  $Y_n$  is closed in Y, Ind  $Y = \infty$ .

THEOREM 4.4. X is a totally disconnected compact Hausdorff space with Ind X = 0, but X contains subsets  $Y_n$ ,  $1 \le n \le \infty$ , with Ind  $Y_n \ge n$ ; Ind  $Y_1 = 1$ , and Ind  $Y_{\infty} = \infty$ .

*Proof.* Follows from Theorems 4.2 and 4.3.

QUESTION. Is Ind  $Y_n = n$  for  $1 < n < \infty$ ?

#### References

1. P. S. Aleksandrov, *The present status of the theory of dimension*, Amer. Math. Soc. Translations, Series 2, 1, (1955), 1-26. Original article in Russian (1951).

2. P. Alexandroff, On the dimension of normal spaces, Proc. Royal Soc. London, Series A, **189** (1947), 11-39.

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3. C. H. Dowker, Local dimension of normal spaces, Quart. J. of Math., Oxford (2) 6 (1955), 101-120.

4. Gillman and Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, N. J., 1960.

5. P. R. Halmos, Naive Set Theory, Van Nostrand, Princeton, N. J., 1960.

6. D. W. Henderson, An infinite dimensional compactum with no positive dimensional compact subsets – a simpler construction, Amer. J. of Math., **89** (1967), 105–121.

7. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Univ. Press, Princeton, New Jersey, 1948.

8. J. R. Isbell, *Uniform Spaces*, Amer. Math. Soc., Providence, Rhode Island, (1964), 102-103.

9. J. L. Kelley, General Topology, Van Nostand, Princeton, N. J., 1955.

10. J. Nagata, *Modern Dimension Theory*, Interscience Publishers of John Wiley and Sons, New York, 1965.

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