SQUARES IN SOME RECURRENT SEQUENCES

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Elementary methods are used to locate the perfect squares in certain sequences of integers defined by three term recurrence relations.

We consider for \( n \geq 0 \) the polynomials \( P_n(x) \), \( Q_n(x) \), \( p_n(x) \) and \( q_n(x) \) defined by

\begin{align*}
1) \quad & P_0(x) = p_0(x) = 0; \quad P_1(x) = p_1(x) = 1 \\
2) \quad & Q_0(x) = q_0(x) = 2; \quad Q_1(x) = q_1(x) = x \\
3) \quad & P_{n+2}(x) = xP_{n+1}(x) + P_n(x) \\
4) \quad & Q_{n+2}(x) = xQ_{n+1}(x) + Q_n(x) \\
5) \quad & p_{n+2}(x) = xp_{n+1}(x) - p_n(x) \\
6) \quad & q_{n+2}(x) = xq_{n+1}(x) - q_n(x).
\end{align*}

These polynomials arose in a natural way in the course of previous work [2, 3] and using the result of [1] the complete solutions of the Diophantine equations \( y^2 = P_n(x) \), \( 2y^2 = P_n(x) \) and the six similar ones obtained by substituting \( Q_n(x) \), \( p_n(x) \) and \( q_n(x) \) for \( P_n(x) \) in positive integers \( x \), \( y \) and \( n \), with \( x \) restricted to odd values, have been found. The method, although fairly long, was elementary.

The same problems for even values of \( x \) seem to be far harder, although in certain cases they may be trivial. For \( x = 2 \), the only significant problem is \( y^2 = P_n(2) \). Ljunggren [5] has shown that \( n = 0, 1, 7 \) yield the only solutions in this case, but the method is non-elementary and involves much computation. It is unlikely that method could be applied to provide a complete solution in \( n \) and \( x \). The main object of the present note is to consider an infinite set of even values of \( x \) for which an elementary method is available for the determination of \( n \). Use is then made of these results to prove some theorems on Diophantine equations of the form \( X^2 = DY^4 \pm 1 \), \( X^2 = DY^4 \pm 4 \).

Using (1)–(6) we find easily that

\begin{align*}
7) \quad & P_n(x) = \left( \frac{x + (x^2 + 4)^{1/2}}{2} \right)^n - \left( \frac{x - (x^2 + 4)^{1/2}}{2} \right)^n \\
8) \quad & Q_n(x) = \left( \frac{x + (x^2 + 4)^{1/2}}{2} \right)^n + \left( \frac{x - (x^2 + 4)^{1/2}}{2} \right)^n
\end{align*}
For convenience we may use (3) and (5) to extend the definitions of \( P_n(x) \) and \( p_n(x) \) to negative values of \( n \), yielding

\[
(11) \quad P_{-n}(x) = (-1)^{n-1}P_n(x) \\
(12) \quad p_{-n}(x) = -p_n(x).
\]

We also obtain

\[
(13) \quad Q_n^2(x) - (x^2 + 4)P_n^2(x) = (-1)^n 4 \\
(14) \quad q_n^2(x) - (x^2 - 4)p_n^2(x) = 4
\]

whence

\[
(15) \quad (Q_n(x), P_n(x)) = 1 \text{ or } 2 \\
(16) \quad (q_n(x), p_n(x)) = 1 \text{ or } 2.
\]

Also using (7)–(10) with (13) and (14) we obtain

\[
(17) \quad \text{if } m \text{ is odd, } \quad P_n(Q_m(a)) = \frac{P_{mn}(a)}{P_m(a)}, \quad Q_n(Q_m(a)) = Q_{mn}(a) \\
(18) \quad \text{if } m \text{ is even, } \quad p_n(Q_m(a)) = \frac{P_{mn}(a)}{P_m(a)}, \quad q_n(Q_m(a)) = Q_{mn}(a) \\
(19) \quad p_n(q_m(a)) = \frac{P_{mn}(a)}{P_m(a)}, \quad q_n(q_m(a)) = q_{mn}(a).
\]

Now suppose that \( m \equiv 3 \mod 6 \) and that \( x = Q_m(a) \) with \( a \) odd. Then from (17) we see that \( Q_n(x) = Q_{mn}(a) \) and so using [2; Theorem 7] we find that \( y^2 = Q_n(x) \) is possible only for \( mn = 3 \), with \( a = 1 \) or 3. This the only solutions are provided by \( n = 1 \), with \( x = 4 \) or 36. Similarly \( 2y^2 = Q_n(x) \) gives \( mn = 0 \), or \( mn = 6 \) with \( a = 1 \) or 5 (in view of [1]) or \( m = 3, n = 2, x = 4 \) or 140. Thus we have proved

**Theorem 1.** If \( x = Q_m(a) \) with \( a \) odd, \( m \equiv 3 \mod 6 \), then \( y^2 = Q_n(x) \) is possible only for \( n = 1 \) with \( x = 4 \) or 36, and \( 2y^2 = Q_n(x) \) is possible only for \( n = 0 \), and for \( n = 2 \) with \( x = 4 \) or 140.

We next consider \( P_n(x) \) under the same conditions. We have \( P_1(x) = 1 \), and if \( n \equiv 1 \mod 4 \), \( n \neq 1 \), we write \( n = 1 + 2hk \), where
$k = 2^r, r \geq 1$ and $h$ is odd. Then using [2; (22)] we obtain by (17)
\[
P_m(a)P_n(x) = P_{mn}(a)
\equiv (-1)^{mh}P_m(a) \pmod{Q_h(a)}
\equiv -P_m(a) \pmod{Q_h(a)},
\]
since $mh$ is odd. Now it is easily verified that $P_m(a)$ and $Q_h(a)$ have no factor in common and so we obtain
\[
P_n(x) \equiv -1 \pmod{Q_h(a)}
\]
from which it follows that $P_n(x) \neq y^2$, since $Q_h(a) \equiv 3 \pmod{4}$ in virtue of [2; (16)]. Since, by (11), for $n$ odd $P_n(x) = P_{-n}(x)$ it follows that $P_n(x) = y^2$ is possible with $n$ odd, $n > 0$ only for $n = 1$.

Now for $n$ even we have using (7) and (8) that
\[
P_n(x) = P_{(1/2)n}(x)Q_{(1/2)n}(x)
\]
and so in view of (15) $y^2 = P_n(x)$ implies
\begin{align*}
either & Q_{(1/2)n}(x) = y_1^2; P_{(1/2)n}(x) = y_2^2; 
& the former implies 1/2n = 1 with x = 4 or 36, both of which satisfy the latter, 
or & Q_{(1/2)n}(x) = 2y_1^2; P_{(1/2)n}(x) = 2y_2^2; 
& the former implies 1/2n = 0 which satisfies the latter, or 1/2n = 2 with x = 4 or 140, but neither of these satisfies the later.
\end{align*}
Finally, considering $2y^2 = P_n(x)$ we see easily that since $x$ is even, $n$ must also be even, and we obtain as before $Q_{(1/2)n}(x) = y_1^2$ or $2y_2^2$, yielding $n = 0$ or $n = 4$, $x = 4$. Thus we have

**Theorem 2.** If $x = Q_m(a)$ with a odd, $m \equiv 3 \pmod{6}$, then $y^2 = P_n(x)$ possible only for $n = 0$ and $n = 1$ and for $n = 2$ with $x = 4$ or $36$; $2y^2 = P_n(x)$ is possible only for $n = 0$ and for $n = 4$ with $x = 4$.

An exactly parallel treatment for $x = q_m(a)$ with $3|\,m$ leads to the following results, whose proofs are omitted.

**Theorem 3.** If $x = q_m(a)$ with a odd, $3|\,m$, then $y^2 = q_n(x)$ is impossible, and $2y^2 = q_n(x)$ is possible only for $n = 0$, and for $n = 1$ with $x = 18$ or $19,602$.

**Theorem 4.** If $x = q_m(a)$ with a odd, $3|\,m$, then $y^2 = p_n(x)$ is possible only for $n = 0$ and $1$, and $2y^2 = p_n(x)$ is possible only for $n = 0$, and for $n = 2$ with $x = 18$ or $19,602$.

We now prove

**Theorem 5.** The equation $y^2 = P_m(a)P_n(a)$ where a is odd and
$m \geq n > 0$ has only the trivial solution $m = n$, except for $a = A^2$; $m = 2$, $n = 1$; $a = 1$, $m = 12$, $n = 1$; $a = 1$, $m = 12$, $n = 2$; $a = 1$, $m = 6$, $n = 3$; $a = 3$, $m = 6$, $n = 3$.

Proof. Let $r = (m, n)$. Then as is well known

$$(P_m(a), P_n(a)) = P_r(a)$$

and so if $m = Mr$, $n = Nr$ we must have

$$y_i^2 = \frac{P_M(a)}{P_r(a)} \quad ; \quad y_i^2 = \frac{P_N(a)}{P_r(a)} .$$

We consider four cases:

(a). $2 \nmid r$, $3 \nmid r$; then using (17) we have $y_i^2 = P_M(Q_r(a))$. Since $Q_r(a)$ is odd, we have using [2; Theorem 5] that $M = 1$ or 2 or 12. Now $M = 1$ always satisfies this; $M = 2$ implies $y_i^2 = Q_r(a)$ and so $r = 1$, $a = y_i^2$; $M = 12$ implies $1 = Q_r(a)$ or $r = a = 1$.

(b). $2 | r$, $3 \nmid r$; then using (18) $y_i^2 = P_M(Q_r(a))$. Since $Q_r(a)$ is odd, we have using [3; Theorem 5] that $M = 1$ or 2 or 6. $M = 1$ always satisfies this; $M = 2$ implies $y_i^2 = Q_r(a)$ which is impossible for $r$ even; $M = 6$ implies $3 = Q_r(a)$ and so $r = 2$, $a = 1$.

(c). $2 | r$, $3 | r$; then $y_i^2 = P_M(Q_r(a))$ and so Theorem 2 is applicable yielding $M = 1$ and $M = 2$ with $r = 3$ and $a = 1$ or 3.

(d). $6 | r$; then $y_i^2 = P_M(Q_r(a)) = P_M(x)$ where $x = Q_r(a) = q_{(1/2)}(Q_2(a))$ using (18). Now $Q_2(a)$ is odd, and so using Theorem 4 we obtain only $M = 1$.

Combining the four cases we find that $M = 1$, except if

$$r = 1, a = y_i^2, M = 2$$
$$r = 1, a = 1, M = 12$$
$$r = 3, a = 1, M = 2$$
$$r = 3, a = 1, M = 2$$
$$r = 2, a = 1, M = 6 .$$

Similar results hold for $N$, and so we obtain $M = N = 1$, or $m = n$, except for

$$r = 1, a = y_i^2, M = 2, N = 1 \quad \text{i.e.} \quad m = 2, n = 1$$
$$r = 1, a = 1, M = 12, N = 1 \quad \text{i.e.} \quad m = 12, n = 1$$
$$r = 3, a = 1, M = 2, N = 1 \quad \text{i.e.} \quad m = 6, n = 3$$
Theorem 6. The equation $2y^2 = P_m(a)P_n(a)$, where $a$ is odd and $m > n > 0$ has no solutions, the following cases only excepted,

- $a = 1$, with $m, n = 3, 2; 3, 1; 6, 1; 6, 2; 12, 3$ or $12, 6$
- $a = 5$, with $m, n = 12, 6$
- $a \neq 1$, with $a^2 = 2A^2 - 1$ and $m, n = 3, 1$.

Proof. As in the proof of the previous theorem let $r = (m, n)$, $m = Mr$, $n = Nr$ and we find that

$$y_r^2 = \frac{P_{Mr}(a)}{P_r(a)}; \quad 2y_r^2 = \frac{P_{NR}(a)}{P_r(a)},$$

or vice-versa. The former yields (since $M \neq 0$) $M = 1$, except if $a = 1$ when also $r = 2, M = 6$ or $r = 1$ and $M = 2$ or $12$, and if $a = 3$ when also $r = 3, M = 2$ and if $a = A^2$ with $r = 1, M = 2$.

Consider now the latter with $N \neq 0$. As before we distinguish four cases.

(a). $2 \nmid r$, $3 \nmid r$; then $2y_r^2 = P_N(Q_r(a))$. Since $Q_r(a)$ is odd, we may use [2; Theorem 6] and we see that the only possibilities are $N = 6$ with $Q_r(a) = 1$, i.e. $r = a = 1$, and perhaps $N = 3$. But $N = 3$ would require $2y_r^2 = (Q_r(a))^2 + 1$, and we shall show that this is impossible except for $r = 1$.

Since $r$ is odd, it follows from [2; (11)] that we require $Q_{sr}(a) = 2y_r^2 + 1$. If we allow the possibility of negative $r$, we can assume that $r \equiv 1 \pmod{4}$, since we can show just as in (11) that $Q_{-n}(x) = (-1)^nQ_n(x)$. Then if $r \neq 1$, let $r = 1 + hk$, where $h$ is odd and $k = 2^s$, with $R \geq 2$. Thus

$$2y_r^2 + 1 = Q_{sr}(a)$$

$$= Q_{2+2hk}(a)$$

$$\equiv -Q_2(a) \quad (\text{mod } Q_2(a)) \quad \text{using [2; (23)]}$$

$$\equiv -(a^2 + 2) \quad (\text{mod } Q_2(a)).$$

From [2; (16), (17)] we see that $Q_k(a) \equiv 7 \pmod{8}$ since $R \geq 2$, and so we should have to have

$$1 = (y_r^2 | Q_k(a))$$

$$= (-2 | Q_k(a)) \left( \frac{a^2 + 3}{4} \right) | Q_k(a))$$
\[ = -1 \] in view of [2; (27), (28)] since \( Q_k(a) \equiv 7 \pmod{8} \),

and this contradiction shows that we can have only \( r = 1 \).

For this to occur we must have \( r = 1, N = 3, a^2 = 2y_i^2 - 1 \).

(b) \( 2 \mid r, 3 \nmid r; \) then \( 2y_i^2 = P_N(Q_r(a)) \) with \( Q_r(a) \) odd. Thus using [3; Theorem 6] we see that the only possibility is \( N = 3 \), whence \( 2y_i^2 = (Q_r(a))^2 - 1 \), or since \( r \) is even, we have with \( b = Q_{(1/2) r}(a) \), \( 2y_i^2 = (b^2 + 2)^2 - 1 \), or \( 2y_i^2 = (b^2 + 1)(b^2 + 3) \). It is easily seen that the only possibility for these last equations is \( b = 1 = Q_{(1/2)r}(a) \), i.e. \( a = 1, r = 2, N = 3 \).

(c) \( 2 \nmid r, 3 \mid r; \) then \( 2y_i^2 = P_N(Q_r(a)) \), where now \( Q_r(a) \) is even. Thus Theorem 2 applies and we find that we can only have \( N = 4, Q_r(a) = 4 \), i.e. \( a = 1, r = 3, N = 4 \).

(d) \( 6 \mid r; \) then \( 2y_i^2 = P_N(Q_r(a)) = p_N(x) \) where \( x = Q_r(a) = q_{(1/2)r}(Q_4(a)) \) as before. Thus Theorem 4 may be used, and we find that we can have only \( N = 2 \) with \( x = Q_r(a) = 18 \) or 19,602, i.e. \( r = 6 \) with \( a = 1 \) or 5.

Thus in all we have the following solutions to our equation:—

If \( a = 1 \). Then \( r = 1 \) gives \( N = 3, M = 2 \) or \( N = 6, M = 1 \); \( r = 2 \) gives \( N = 3, M = 1 \); \( r = 3 \) gives \( N = 4, M = 1 \), and \( r = 6 \) gives \( N = 2, M = 1 \).

If \( a = 5 \). Then \( M = 1, N = 2, r = 6 \).

If \( a \neq 1, a^2 = 2y_i^2 - 1 \), then \( r = 1, N = 3, M = 1 \). The other case does not occur since it would require \( a^2 = 2y_i^2 - 1, a = y_i \). But this is impossible for we should have to have \( (y_i^2 - 1)^2 = y_i - y_i^2 \), and this cannot occur if \( a \neq 1 \).

This concludes the proof of the theorem.

**Theorem 7.** Let \( D = dN^2 \) where \( d \) is such that \( X^2 - dY^2 = -4 \) possesses solutions with both \( X \) and \( Y \) odd; then no one of the four equations \( X^2 = DY^4 \pm 1 \), \( X^2 = DY^4 \pm 4 \) possesses more than one solution in positive integers, and between them they have at most two such solutions, the following cases only excepted

(i) \( D = 5 \) when there are in all five solutions, viz. \( Y = 1 \) for \( X^2 = 5Y^4 - 1 \), \( X^2 = 5Y^4 \pm 4 \), \( Y = 2 \) for \( X^2 = 5Y^4 + 1 \), \( Y = 12 \) for \( X^2 = 5Y^4 + 4 \)

(ii) \( D = 20 \) when there are in all three solutions, viz. \( Y = 1 \) for \( X^2 = 20Y^4 - 4 \), \( Y = 2 \) for \( X^2 = 20Y^4 + 4 \) and \( Y = 6 \) for \( X^2 = 20Y^4 + 1 \).

**Proof.** We are given that \( X^2 - dY^2 = -4 \), possesses solutions with both \( X \) and \( Y \) odd, and so if \( X = a, Y = b \) is the fundamental solution it is easily seen that both \( a \) and \( b \) must be odd, for the
general solution is given by $X + Yd^{1/2} = 2[(a + bd^{1/2})/2]^n-1$. Then we
find without difficulty that, considering only positive values, the general solution of $X^2 - dY^2 = -4$ is $Y = bP_{2n-1}(a)$, the general solution of $X^2 - dY^2 = 4$ is $Y = bP_{2n}(a)$, the general solution of $X^2 - dY^2 = -1$ is $Y = (1/2)bP_{2n-3}(a)$, the general solution of $X^2 - dY^2 = 1$ is $Y = (1/2)bP_{2n}(a)$.

Consider first $X^2 - DY^4 = -4$; by the above remarks, we see that for a solution we must have $NY^2 = bP_{2n-1}(a)$, and so if there were two solutions we should have, with $m \neq n$, $P_{2m-1}(a)P_{2n-1}(a) = y^2$, but that is impossible by Theorem 5. The same applies to the equation $X^2 = DY^4 - 1$.

Similarly for $X^2 - dY^4 = 4$ we find that for a solution we must have $NY^2 = bP_{2n}(a)$, and so two different solutions require $m \neq n$ and $P_{2m}(a)P_{2n}(a) = y^2$. Theorem 5 shows that this can occur only for $a = 1, 2m = 12, 2n = 2$, from which we find $d = 5, NY^2 = 1$ and 144 and so we get only $D = 5, Y = 1$ and 12. Similarly we find that $X^2 = DY^4 + 1$ never has more than one solution.

This shows that no one of the equations has more than one solution ($D \neq 5$); to complete the proof we must consider how often two different equations of the set can have solutions. Whenever this occurs we find that $P_r(a)P_s(a) = y^2$ or $2y^2$. These cases are all easily identified using Theorems 5 and 6, and we obtain the required result; for we see that unless $a = 1$, there are in all at most two solutions and examination of $a = 1$ yields all the exceptional cases.

This concludes the proof. In just the same way as above, we may prove the following three results, the proofs of which are omitted.

**Theorem 8.** The equation $y^2 = p_m(a)p_n(a)$ where $a$ is odd, $a \geq 3$ and $m \geq n > 0$ has only the trivial solution $m = n$ except for $a = 3, m = 6, n = 1$ and for $a = A^2, m = 2, n = 1$.

**Theorem 9.** The equation $2y^2 = p_m(a)p_n(a)$ where $a$ is odd, $a \geq 3$, and $m > n > 0$ has no solutions except for the following cases

$a = 3, m = 6, n = 3; a = 27, m = 6, n = 3$ and $a^2 = 2A^2 + 1, m = 3, n = 1$.

**Theorem 10.** Let $D = dN^2$ where $d$ is such that $X^2 - dY^2 = 4$ possesses solutions with both $X$ and $Y$ odd, although the equation $X^2 - dY^2 = -4$ does not; then the equations $X^2 = DY^4 + 1$ and $X^2 = DY^4 + 4$ possess between them at most two solutions in positive integers, the former having at most one such solution.

It may be seen from the last theorem, that the equation $X^2 =$
189 \( Y^4 + 1 \) possesses only the solution \( X = 55, Y = 2 \) in positive integers, although 189 is not a value to which the methods of [2] or [3] apply; similarly for \( D = 325 \), using Theorem 7, we find that \( X^2 = 325 Y^4 + 1 \) has only the solution \( Y = 6 \), and \( X^2 = 325 Y^4 - 1 \) has only the solution \( Y = 1 \), while \( X^2 = 325 Y^4 \pm 4 \) have no positive solutions, although again 325 is not a value of \( D \) to which the methods of [2] or [3] apply.

We now prove similar results for \( Q_n(a) \) and \( q_n(a) \), where we suppose throughout that \( a \) is odd, and in the case of the latter that \( a \neq 3 \).

We recall that in the reference [2] we designated \( Q_n(a) \) by \( v_n(a) \), and in [3] we designated \( q_n(a) \) by \( v_n \). Where no confusion arises, we shall write simply \( Q_n \) and \( q_n \).

**Lemma 1.** \((Q_m, Q_n) = 2^{i}x\), where

\[
x = Q_r \quad \text{if} \quad r = (m, n) \text{ and } m/r, n/r \text{ are both odd ,}
\]

\[
= 1 \quad \text{otherwise ;}
\]

\[
\text{and } \quad i = 0 \quad \text{unless } \quad x = 1, 3| r
\]

\[
= 1 \quad \text{if } \quad x = 1, 3| r
\]

**Proof.** If \( X = (Q_m, Q_n) \) then since \( P_{2t} = P_t Q_t \), we find that \( X \) divides \((P_{2m}, P_{2n}) = P_{2(m, 2n)} = P_{2r} = P_r Q_r \). Now \( P_r | P_m \) and so no odd factor of \( P_r \) divides \( Q_m \) in view of (15). Also, if \( m/r \) is even we find in view of [2; (19)] that \( 2Q_m \equiv \pm 4 \text{ (mod } Q_r) \), and so no odd factor of \( Q_r \) divides \( Q_m \). Similarly if \( n/r \) is even. On the other hand if \( M = m/r \) is odd, then \( Q_n(a) = Q_n(Q_r(a)) \) by (17) if \( r \) is odd, and \( Q_m(a) = q_n(Q_r(a)) \) if \( r \) is even by (18), and in either case, \( Q_m(a) \) is divisible by \( Q_r(a) \). Thus if we define \( x \) as in the statement of the lemma, we find that \( X = 2^i x \) for some suitable \( i \). If \( 3 \nmid r \), then \( 2 \nmid X \) and \( i = 0 \). If \( 6| r \) then \( 2| Q_m, 2\| Q_n \) and \( 2\| Q_r \) and so \( i = 0 \) if \( x = Q_r \) and \( i = 1 \) if \( x = 1 \). If \( r \equiv 3 \text{ (mod } 6) \), then if \( x \neq 1, 2^2| Q_r, 2^2| Q_m, 2^2| Q_n \) and \( i = 0 \), whereas if \( x = 1 \), then one of \( m \) and \( n \) must be even, and again \( i = 1 \).

In exactly the same way we may prove

**Lemma 2.** \((q_m, q_n) = 2^{i}x\) where

\[
x = q_r \quad \text{if} \quad r = (m, n) \text{ and } m/r, n/r \text{ are both odd ,}
\]

\[
= 1 \quad \text{otherwise ;}
\]

\[
\text{and } \quad i = 0 \quad \text{unless } \quad x = 1, 3| r
\]

\[
= 1 \quad \text{if } \quad x = 1, 3| r
\]

The proof is exactly similar, and is omitted.
**Lemma 3.** \( Q_n = ay^2 \) implies \( n = 1 \), except for \( a = 1, n = 3 \).

**Proof.** By [2] \( a = 1 \) occurs only for \( n = 1, 3 \). In what follows we suppose that \( a > 1 \). Then \( a|Q_n \) implies that \( n \) is odd.

(i) Suppose \( n \equiv 1 \pmod{4}, n \neq 1 \). Then we may write \( n = 1 + 2hk \), where \( h \) is odd, and \( k = 2^r, R \geq 1 \). Thus using [2; (23)] we obtain from the equation,

\[
Q_1y^2 = ay^2 = Q_n = Q_{1+2hk} \\
\equiv -Q_1 \pmod{Q_k}.
\]

Thus in view of Lemma 1, we see that we should have \( y^2 \equiv -1 \pmod{Q_k} \) which is impossible, since by [2; (16)] \( Q_k \equiv 3 \pmod{4} \).

(ii) Suppose \( n \equiv 3 \pmod{4} \). Then \( n = 3 \) would give \( y^2 = a^2 + 3 \), impossible if \( a \neq 1 \), while if \( n \neq 3 \) we write \( n = 3 + 2hk \) as before, and obtain

\[
ay^2 = Q_3 = Q_n \\
\equiv -Q_3 \pmod{Q_k},
\]

whence \( (a|Q_k) = -(Q_3|Q_k) \), which is impossible in view of [2; (27), (28)]. This concludes the proof.

**Lemma 4.** \( q_n = ay^2 \) implies \( n = \pm 1 \).

**Proof.** As before \( n \) must be odd. If \( n \equiv 1 \pmod{4} \) and \( n \neq 1 \) then \( n = 1 + 2hk \) gives as before

\[
ay^2 = q_n \equiv -q_1 = -a \pmod{q_k}
\]

which is impossible.

If \( n \equiv 3 \pmod{4} \), then \( q_{-n} = q_n \) in view of [3; (7)] and \( -n \equiv 1 \pmod{4} \), and the result follows.

**Lemma 5.** \( Q_n = 2ay^2 \) is impossible, except for \( a = 1 \) with \( n = 0, n = 6 \).

**Proof.** By [2], \( a = 1 \) gives only \( n = 0, n = 6 \) and so we suppose that \( a > 1 \). As before \( a|Q_n \) then implies that \( n \) is odd, and \( 2|Q_n \) implies that \( 3|n \). Thus \( n \equiv 3 \pmod{6} \) from which we find that \( Q_n \equiv 4 \pmod{8} \), which makes \( 2ay^2 = Q_n \) impossible.

**Lemma 6.** \( q_n = 2ay^2 \) is impossible for \( a > 1 \).
Proof. As before we find $n = 3N$ with $N$ odd, and so

$$2y^2 = \frac{1}{a} q_n \equiv \frac{1}{a} q_3 \pmod{8}$$

$$\equiv 6 \pmod{8},$$

using [3; (17)], and this is impossible.

**Theorem 11.** The equation $y^2 = Q_m(a)Q_n(a)$ where $a$ is odd and $m \geq n \geq 0$ has only the trivial solution $m = n$, except for $a = 1, m = 6, n = 0; a = 1, m = 3, n = 1$ and $a = 5, m = 6, n = 0$.

Proof. In view of Lemma 1, we find three possibilities, where $r = (m, n)$:

(a) $Q_m(a) = y_i^c; Q_n(a) = y_i^c$;
(b) $Q_m(a) = 2y_i^c; Q_n(a) = 2y_i^c$;
(c) $Q_m(a) = Q_r(a) y_i^c; Q_n(a) = Q_r(a) y_i^c$.

Cases (a) and (b) are easily dealt with, using [2], and we find just the three exceptions stated in the statement of the theorem. Consider case (c).

(i) If $r \equiv \pm 1 \pmod{6}$, then write $A = Q_r(a)$ where $A$ is odd, and then in view of (17) we find, where $M = m/r$, $Ay_i^c = Q_M(A)$. Using Lemma 3, we find that we must have $M = 1$, or $m = r = n$ (similarly) except if $A = 1$, when we find also $m = 3r, n = r$ with $A = 1 = Q_r(a)$. But this is possible only for $a = r = 1$, a case we have dealt with already.

(ii) If $r \equiv \pm 2 \pmod{6}$, then similarly $A = Q_r(a)$ is odd and using (18) we find $Ay_i^c = q_m(A)$ which in view of Lemma 4 yields only $m = r = n$.

(iii) If $3 | r$, then $M = m/r$ is odd. Suppose first that $M \equiv 1 \pmod{4}$. Then if $M \neq 1$, we find that $m = r + 2hk$ where $h$ is odd and $k = 2^k$. Thus as before we find

$$Q_r(a)y_i^c = Q_m(a) \equiv -Q_r(a) \pmod{Q_{2k}(a)}.$$

But by Lemma 1, $(Q_r, Q_{2k}) = 1$, and again we see that this is impossible.

If $r$ is even, and $M \equiv 3 \pmod{4}$, we find that $m$ is even and then in view of [2; (7)] $Q_{-m}(a) = Q_m(a)$ where now $-m/r \equiv 1 \pmod{4}$, and the result follows from the last part.

Finally, if $r$ is odd, $3 | r$ and $M \equiv 3 \pmod{4}$, we find if $X = Q_r(a)$ that $4 | X$. But then $xy_i^c = Q_u(X)$, and then using (8) we obtain
Thus $y_i^2 \equiv 3 \pmod{4}$, clearly impossible.
This concludes the proof of the theorem.

**Theorem 12.** The equation $2y^2 = Q_m(a)Q_n(a)$ where $a$ is odd and $m > n \geq 0$, has no solutions, except for

\[ a = 1 \quad \text{with} \quad m, n = 3, 0 \text{ or } 6, 1 \text{ or } 6, 3; \quad \text{or} \quad 1, 0 ; \]
\[ a = 3 \quad \text{with} \quad m, n = 3, 0 \]
\[ a = A^2 \quad \text{with} \quad m, n = 1, 0 . \]

**Proof.** In view of Lemma 1, $2y^2 = Q_m(a)Q_n(a)$ implies

\[ \text{either } Q_m(a) = y_i^2; \quad Q_n(a) = 2y_i^2, \quad \text{or vice-versa;} \]
\[ \text{or } \quad Q_m(a) = Q_r(a)y_i^2; \quad Q_n(a) = 2Q_r(a)y_i^2 \quad \text{or vice-versa .} \]

The former gives the exceptions of the theorem, using [2] with [1]. We consider therefore the latter.

As we have seen in the proof of the last theorem, $Q_m(a) = Q_r(a)y_i^2$ is possible only for $m = r$, except for $r = a = 1$, $m = 3$ and again this gives only some of the exceptions found already.

Consider therefore $Q_n(a) = 2Q_r(a)y_i^2$, where $N = n/r$ is odd, $Q_r(a) \neq 1$.

(i) If $r \equiv \pm 1 \pmod{6}$, then $A = Q_r(a)$ yields as before $q_N(A) = 2Ay_i^2$, impossible by Lemma 5, since $A = Q_r(a) \neq 1$.

(ii) If $r \equiv \pm 2 \pmod{6}$, then $A = Q_r(a)$ yields as before $q_N(A) = 2Ay_i^2$, impossible in view of Lemma 6.

(iii) If $3 | r$, then we find since $N = n/r$ is odd that $Q_r(a)$ and $Q_n(a)$ are divisible by the same power of 2, and so $Q_n(a) = 2Q_r(a)y_i^2$ is impossible in this case.

This concludes the proof.

**Theorem 13.** Let $d$ be such that $X^2 - dY^2 = -4$ has solutions with both $X$ and $Y$ odd. Then for any positive integer $N$, the four equations $N^2X^2 - dY^2 = \pm 1, \pm 4$ have between them at most one solution in positive integers $X, Y$, with the two exceptions

(i) $d = 5, N = 1$ when we obtain precisely three solutions, viz.
$X = 1$ or $2$ for $X^4 - 5Y^2 = -4$ and $X = 3$ for $X^4 - 5Y^2 = 1$

(ii) $d = 5$, $N = 2$ when we obtain precisely two solutions, viz. $X = 1$ for $4X^4 - 5Y^2 = -1$ and $X = 3$ for $4X^4 - 5Y^2 = 4$.

Proof. Since $X^2 - dY^2 = -4$ has solutions with both $X$ and $Y$ odd, it follows that $d \equiv 5 \pmod{8}$, and that every factor of $d \equiv 1 \pmod{4}$. Thus $d$ has at least one prime factor $p$, with $p \equiv 5 \pmod{8}$. If $p \nmid N$, then clearly no one of the equations $N^2X^4 - dY^2 = \pm 1, \pm 4$ has a solution. If $p \mid N$, then since both $-1$ and $4$ are quartic-non-residues modulo $p$ we see that it is impossible that one equation of the pair $N^2X^4 - dY^2 = 1, -4$ and one of the pair $N^2X^4 - dY^2 = -1, 4$ should have solutions.

As in the proof of Theorem 7, we find that the general solution of $X^2 - dY^2 = 4$ is given by $X = Q_{2n}(a)$, $Y = bP_{2n}(a)$ (with analogous results for $X^2 - dY^2 = -4, 1, -1$), and so if any one of the four equations had more than one solution we should obtain $Q_m(a)Q_n(a) = \gamma^2$ with $m > n > 0$, if we restrict our attention to positive solutions for both $X$ and $Y$. In view of Theorem 11, this cannot occur, with the sole exception of $a = 1, m = 3, n = 1$, when we find $d = 5, N = 1$ with $X = 1$ or $2$ satisfying $X^4 - 5Y^2 = -4$. Similarly, if both equations of a pair have solutions, then we should have $Q_m(a)Q_n(a) = 2\gamma^2$ with $m > n > 0$, and in view of Theorem 12, this occurs only for $a = 1$, with $m = 6$ and $n = 1$ or $3$. These easily yield the remaining exceptions, mentioned in the statement of the theorem. This concludes the proof.

In exactly the same way we may prove

Theorem 14. The equation $y^2 = q_m(a)q_n(a)$, where $a \geq 3$, and $a$ is odd, and $m \geq n \geq 0$ has only the trivial solution $m = n$, except for $a = 3$ or $27$ when also $m = 3, n = 0$.

Theorem 15. The equation $2y^2 = q_m(a)q_n(a)$, where $a \geq 3$, and $a$ is odd, and $m > n \geq 0$ has no solutions except in the case $a = A^2$, when only $m = 1, n = 0$.

Theorem 16. Suppose that $d$ is such that $X^2 - dY^2 = 4$ has a solution with both $X$ and $Y$ odd, but that $X^2 - dY^2 = -4$ does not; then for any positive integer $N$, the equations $N^2X^4 - dY^2 = 1$ and $N^2Y^4 - dY^2 = 4$ have between them at most one solution in positive integers.

The details of the proofs are similar to the previous ones, and
are omitted.

We now consider for a given odd $a$ and given $N$ the problem of determining all positive integers $n$ such that $P_n(a) = N y^2$. Without loss of generality we may assume that $N$ is square-free. The cases $N = 1, 2$ have been completely dealt with in [2] and so we assume that $N \geq 3$. In view of Theorem 5 we see that there is at most one such value of $n$, with the sole exception $N = 10, a = 3$ when we can have $n = 3$ or $n = 6$. In other cases the problem of determining the single value of $n$, if it exists, remains. For convenience we treat separately $P_n(a) = N y^2$ and $P_n(a) = 2 N y^2$ where $N$ is odd, square-free, and $N \neq 1$.

We see that in view of (3) the residues modulo $N$ of the sequence $P_n(a)$ form a periodic sequence (with period $\leq N^2$) and since $P_0(a) = 0$ there exists a least positive integer $\rho = \rho(N, a)$, say, such that $N \mid P_\rho(a)$. It is then easily seen that $N \mid P_\rho(a)$ if and only if $\rho \mid n$.

(a) Suppose $\rho \equiv \pm 1 \pmod{6}$.

We have using (13) that with $d = (a^2 + 4)N^2$, the equation $X^2 - dy^2 = -4$ is satisfied by $X = A = Q_\rho(a)$ and $Y = B = N^{-1} P_\rho(a)$. Since $3 \nmid \rho$, both $A$ and $B$ are odd and since the general solution of $X^2 - (a^2 + 4)Y^2 = -4$ is given by $X = Q_{2n-1}(a)$, $Y = P_{2n-1}(a)$, it is clear that $A + Bd^{1/2}$ is the fundamental solution of $X^2 - dY^2 = -4$. Thus the methods of [2] apply for this value of $d$, and we find in the notation employed there that, in view of (7) and (8)

$$d^{1/2} u_r = \left\{ \frac{A + Bd^{1/2}}{2} \right\}^r - \left\{ \frac{A - Bd^{1/2}}{2} \right\}^r$$

$$= \left\{ \frac{Q_\rho(a) + (a^2 + 4)^{1/2} P_\rho(a)}{2} \right\}^r - \left\{ \frac{Q_\rho(a) - (a^2 + 4)^{1/2} P_\rho(a)}{2} \right\}^r$$

$$= \left\{ \frac{a + (a^2 + 4)^{1/2}}{2} \right\}^r - \left\{ \frac{a - (a^2 + 4)^{1/2}}{2} \right\}^r$$

$$= (a^2 + 4)^{1/2} P_{r\rho}(a).$$

Thus $P_{r\rho}(a) = N u_r$. Accordingly we see that $P_{r\rho}(a) = N y^2$ implies $u_r = y^2$, and using [2; Theorem 3] this is possible for positive $r$ only with $r = 1, 2$ and for $d = 5$ with $r = 12$. But $d = 5$ is impossible since $N \neq 1$. Also $r = 2$ would require $A = Q_\rho(a)$ to be a square, and using [2; Theorem 7] this would require $\rho \leq 3$, that is $\rho = 1$. But $\rho = 1$ is impossible, since then $N \nmid P_\rho(a)$.

Similarly $P_n(a) = 2 N y^2$ implies $n = r\rho$ with $u_r = 2y^2$. Using [2; Theorem 4] we see that since $d \neq 5$, we need consider only $r = 3$. But this too is impossible, for we should obtain $2y^2 = B(A^2 + 1)$. Since $A^2 - dB^2 = -4, A^2 + 1 \equiv -3 \pmod{B}$ and so since $3 \nmid (A^2 + 1)$ we
should have \( A^2 + 1 = 2y_1^2 \); \( B = y_2^2 \). But then \( P_\rho(a) = NB = N\eta_\rho \), whence \( P_\rho(a)P_{3\rho}(a) = 2y_3^2 \), impossible in view of Theorem 6, since in this case \( \rho \geq 5 \).

Thus in case (a) \( P_n(a) = Ny^2 \) can occur only for \( n = \rho \); \( P_n(a) = 2Ny^2 \) cannot occur at all for \( n > 0 \).

(b) Suppose \( \rho \equiv \pm 2 \) (mod 6).

We now find in analogous fashion that if \( d = (a^2 + 4)N^2 \), then \( X^2 - dY^2 = -4 \) has no solution, but that the fundamental solution of \( X^2 - dY^2 = 4 \) is \( A = Q_\rho(a), B = N^{-1}P_\rho(a) \) with both \( A \) and \( B \) odd. Thus we use the notation and methods of [3], finding as before that \( P_\rho(a) = Nu_\rho \) and so \( P_{\rho\rho}(a) = Ny^2 \) implies \( u_\rho = y^2 \). For positive \( r \) this can occur [3; Theorem 3] only for \( r = 1, 2 \) or \( 3 \). But \( r = 2 \) is impossible for it would require \( y^2 = u_\rho = B(A^2 - 1) \), whence \( B = 3y_3^2; A^2 - 1 = 3y_3^2 \). Now since \( A \) is odd, \( A^2 - 1 \equiv 0 \) (mod 8) and so we must have \( A^2 \equiv 1 \) (mod 16). Thus \( A = \pm 1 \) (mod 8) and this leads to \( \rho \equiv 0 \) (mod 4). Thus if \( c = Q_{(1/4)\rho}(a) \) we find using [2; (11)] that

\[
3y_3^2 = (Q_{(1/2)\rho}(a))^2 - 2]^2 - 1 \\
= (Q_{(1/2)\rho}^2 - 1)(Q_{(1/2)\rho}^2 - 3) \\
= ((c^2 \pm 2)^2 - 1)((c^2 \pm 2)^2 - 3) \\
= (c^4 \pm 4c^2 + 3)(c^4 \pm 4c^2 + 1)
\]

where \( c \) is odd. Now both expressions in brackets are positive except for \( c = 1 \); otherwise since \( c^4 \pm 4c^2 + 1 \equiv 6 \) (mod 8) we must have

\[
c^4 \pm 4c^2 + 1 = 6y_3^2 \\
c^4 \pm 4c^2 + 3 = 2y_3^2.
\]

Now we reject the lower sign since \( 3 | (c^4 - 4c^2) \) for every \( c \), contradicting the former. The upper sign gives

\[
\frac{c^2 + 1}{2}(c^2 + 3) = y_3^2.
\]

This requires \( c^2 + 3 = y_3^2 \), and this is possible only for \( c = 1 \). But \( c = 1 = Q_{(1/4)\rho}(a) \) can occur only for \( a = 1, \rho = 4 \). But this would require \( N = 3 \), since \( P_4(1) = 3 \), but \( P_{12}(1) = 144 \neq 3y^2 \).

Finally, \( P_{\rho\rho}(a) = 2Ny^2 \) implies \( u_\rho = 2y^2 \), possible in view of [3; Theorem 4] only for \( r = 3 \), with \( B = y^3 \). But then \( P_\rho(a) = NB = Ny_\rho \).

Thus \( P_\rho(a)P_{3\rho}(a) = 2y_3^2 \), possible in view of Theorem 6 only for \( a = 1, \rho = 2 \). But again this cannot occur since \( P_3(1) = 1 \).
Thus in case (b), $P_n(a) = Ny^2$ can occur only for $n = \rho$;
$P_n(a) = 2Ny^2$ is impossible for $n > 0$.

(c) Suppose $\rho \equiv 3 \pmod{6}$.
Then $P_{r\rho}(a) = Ny^2$ compels $r$ to be even. For if $r$ is odd, then $2|P_{r\rho}(a)$, $4 \not| P_{r\rho}(a)$. Thus we write $r = 2s$, and then

$$y^2 = N^{-1}P_{2s\rho}(a) = \{N^{-1}P_{s\rho}(a)\}Q_{s\rho}(a)\}.$$

Thus in view of (15) we have

either $P_{s\rho}(a) = Ny_1^2$; $Q_{s\rho}(a) = y_2^2$
or $P_{s\rho}(a) = 2Ny_1^2$; $Q_{s\rho}(a) = 2y_2^2$.

Now using [2; Theorem 7] we find that the former requires $s = 3$, with $a = 1$ or 3, but then $P_{s\rho}(a) = 2$ or 10, neither of which gives a value for $N$. Using [2; Theorem 8], with $s\rho = 6$ with $a = 1$ or 5, whence $2Ny_1^2 = 8$ or 3640. The former gives no value, the latter $\rho = 3, r = 4, a = 5, N = 455$; but $455 \not| P_3(5)$ and so we find that this cannot occur.

Thus in case (c), $P_n(a) = Ny^2$ cannot occur for $n > 0$.

Unfortunately, there does not seem to be a similar method available for handling $P_n(a) = 2Ny^2$ in this case.

(d) Suppose $\rho \equiv 0 \pmod{6}$.
This case is slightly more complicated; suppose $2^t || \rho$. Then it may be shown that $2^{t+2} || P_{r\rho}(a)$ and so if $t$ is odd, we find that $Ny^2 = P_n(a)$ implies $n = r\rho$ with $r$ even, and then just as in the above case we find no value for $n > 0$, except in the case $a = 5, \rho = 6, N = 455, n = 12$. On the other hand, if $t$ is even, we find that $2Ny^2 = P_n(a)$ implies $n = r\rho$ with $r$ even, and then there is no value for $n > 0$.

Thus in case (d), if $2^t || \rho$, then $P_n(a) = 2Ny^2$
has no solution, and if $2^{t+1} || \rho$, then

$P_n(a) = Ny^2$ has no solution, except in the single case

$a = 5, N = 455, n = 12$, all for $n > 0$.

We see however, that in the cases in which $3 | \rho(N, a)$, we have not succeeded in determining possible values of $n$. This problem remains open. A similar situation exists for equations of the type $p_n(a) = Ny^2$.

In conclusion, we observe that as far as Theorems 1-4 are concerned, although the method applies to infinite sets of values of $x$ in each case, many values are not covered; thus considering values < 6,000 the only values covered are 4, 36, 76, 140, 364, 756, 1364, 2236, 3420
and 4964 in the case of Theorems 1 and 2, and 18, 110, 322, 702, 1298, 2158, 3330, 4862 and 5778 in the case of Theorems 3 and 4. For such values it is also clear that a method similar to that used in [4] will be available for handling any sequence of integers satisfying a recurrence relationship of the form (3) or (5) respectively.

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