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ON COMMUTATIVE P. P. RINGS

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The purpose of this paper is to study further the ideal and module structure of a commutative ring with identity, in which every principal ideal is projective. Results concerning particular modules being projective are also obtained, e.g. if R is a commutative ring with identity, then $Z_R(R_R) = 0$ and every finitely generated nonsingular R -module is projective if and only if R is semihereditary and K , the classical ring of quotients of R , is selfinjective.

A ring R is said to be a right *P.P.* ring if every right principal ideal of R is projective. These rings have been considered by Hattori [6] and by Endo [4], [5].

If R is a commutative ring with identity it can be shown that R is a *P.P.* ring if and only if for each $x \in R$, $r(x) = \{t \in R \mid xt = 0\} = eR$ for some idempotent $e \in R$. This latter property was used by Kist [9] to define 'commutative Baer rings'. In this paper, however by a *Baer ring* we will mean a ring R , with identity, such that for each subset $S \subseteq R$, $r(S) = \{t \in R \mid St = 0\} = eR$, where e is an idempotent of R . This is the definition used by Kaplansky [8, p. 2].

1. Notations and terminologies. Throughout this paper, unless otherwise indicated, a ring R is an associative ring with identity; all modules are unitary.

Given a subset S of a module M we set, as usual, $r.\text{ann}_R(S) = \{x \in R \mid Sx = 0\}$ and we abbreviate this to $r(S)$ if no ambiguity arises. The notion $l.\text{ann}_R(S) = l(S)$ is similarly defined; over a commutative ring no distinction is made between $l(S)$ and $r(S)$. If N is a submodule of M we set $(N:M) = r.\text{ann}_R(M/N)$.

For all homological notions used in this paper, the reader is referred to [10].

Throughout this paper, K will denote the classical ring of quotients of a commutative ring R . $\text{Spec } R$ will denote the space of prime ideals of commutative ring R , while $\text{Minp } R$ will denote the space of minimal prime ideals of R . Details of $\text{Minp } R$ may be found in [7]. If R is a commutative *P.P.* ring, let e_x be the unique idempotent such that $r(x) = e_x R$.

By a *regular ring* we mean a von Neumann regular ring, that is a ring with the property that every finitely generated right (left) ideal is generated by an idempotent. Regular rings, thus are in particular *P.P.* rings.

2. Quasi-regular rings. Quasi-regular rings were first discussed by Endo in [5].

DEFINITION 2.1. *A commutative ring R is said to be quasi-regular if the classical quotient ring K of R is a regular ring.*

THEOREM 2.2. *For a commutative ring R , the following are equivalent:*

- (1) *For all $x \in R$ there exists $x' \in R$ such that $rr(x) = r(x')$ and R is a semiprime ring.*
- (2) *For all $x \in R$ there exists a nonzero divisor $d \in R$ such that $xd = x^2$.*
- (3) *R is a quasi-regular ring.*

Proof. (1) implies (2). It will first be shown that $x + x'$ is a nonzero divisor of R . If $(x + x')s = 0$, then $xs = -x's$ and hence $xs \in rr(x) \cap r(x)$. It is a consequence of R being a semiprime ring that $rr(x) \cap r(x) = 0$ and hence $s \in r(x)$. Similarly $s \in r(x') = rr(x)$ and so $s = 0$. The result follows by observing $x(x + x') = x^2$.

(2) implies (3). Let $xd^{-1} \in K$ where $x \in R$ and d is a nonzero divisor of R . By (2), there exists a nonzero divisor $u \in R$ such that $xu = x^2$. Hence $x^2(d^{-1})^2du^{-1} = xd^{-1}$, which implies K is a regular ring.

(3) implies (1). Let $x \in R \subseteq K$. Then, as K is a P.P. ring $r.\text{ann}_K(x) = (sd^{-1})K$ where sd^{-1} is an idempotent of K , $s \in R$ and d a nonzero divisor of R . Hence $r.\text{ann}_R(r.\text{ann}_R(x)) = r.\text{ann}_K(r.\text{ann}_K(x)) \cap R = r.\text{ann}_K(sd^{-1}) \cap R = r.\text{ann}_R(s)$. Finally R is semiprime as K is.

REMARK 1. Since every quasi-regular ring is semiprime, condition (2) expresses the fact that for each $x \in R$ there is a nonzero divisor $d \in R$ such that $x \leq d$, where \leq is the partial ordering defined on any semiprime ring by $x \leq y$ if and only if $xy = x^2$, [1].

REMARK 2. Condition (1) was introduced in Theorem 3.4 of [7]. If R is a semiprime ring this condition implies $\text{Minp } R$ is compact. It has been stated in the paper of Henriksen and Jerison [7] and later in the paper of Mewborn [12] that an example of a semiprime ring R with $\text{Minp } R$ compact, but which does not satisfy condition (1) of the Theorem, has not been found.

COROLLARY. *Every commutative P.P. ring is a quasi-regular ring.*

Proof. For each $x \in R$, $rr(x) = r(e_x)$ where $r(x) = e_x R$ and the other half of condition (1) of the Theorem 2.2 is proved in the following lemma.

LEMMA 2.3. *Let R be a commutative ring in which every principal ideal is flat. Then R is a semiprime ring.*

Proof. Let $x^2 = 0$. Consider the exact sequence $0 \rightarrow r(x) \rightarrow R \rightarrow xR \rightarrow 0$. As xR is flat $0 \rightarrow r(x) \otimes_R xR \rightarrow R \otimes_R xR \rightarrow xR \otimes_R xR$ is an exact sequence. Hence by Proposition 1 of §5.4 of [10] it follows that $0 \rightarrow r(x).xR \rightarrow xR \rightarrow x^2R \rightarrow 0$ is exact: i.e. $0 \rightarrow 0 \rightarrow xR \rightarrow x^2R \rightarrow 0$ is exact and as $x^2 = 0$, $x = 0$.

REMARK 3. The above Corollary is contained in Endo's Proposition 1 [4, p. 168], which we record here as we shall have occasion to remark on it again.

PROPOSITION 2.4. *If R is a commutative ring, then R is a P.P. ring if and only if K is regular and R_V is an integral domain for each maximal ideal V of R .*

Quasi-regular rings are analogous to distributive $*$ -lattices [16]. The following proposition has an analogue in distributive lattices. This has been given in [16] and so the proof will not be given here.

PROPOSITION 2.5. *If R is a semiprime ring, then the following are equivalent:*

- (1) *For all $x \in R$, there exists $x' \in R$ such that $rx(x') = r(x')$.*
- (2) *For all $x \in R$, there exists an $x' \in R$ such that $xx' = 0$ and $x + x'$ is a non zero divisor.*
- (3) *If P is a prime ideal of R , which contains only zero divisors then P is a minimal prime ideal.*

REMARK 4. The Baer extension [9, p. 46] of a quasi-regular ring is simply the ring generated by R and the idempotents of K .

3. Modules in which every cyclic submodule is projective. A right R module is said to be a C.P. module if every right cyclic submodule is projective.

PROPOSITION 3.1. *If R is a ring and A_R is a right R -module, then the following are equivalent:*

- (1) *A_R is a C.P. module.*
- (2) *For each $x \in A_R$, $r(x) = eR$ for some idempotent $e \in R$.*

Proof. (1) implies (2). Consider the exact sequence $0 \rightarrow r(x) \xrightarrow{i} R \xrightarrow{j} xR \rightarrow 0$, where i is the imbedding map and $j: a \rightarrow xa$. Then, as

xR is projective, the exact sequence splits and $r(x)$ is a direct summand of R .

(2) implies (1). Suppose xR is a cyclic submodule. Then $xR \cong R/r(x) = R/eR$. As $R/eR \cong (1-e)R$ by the correspondence $r/eR \leftrightarrow (1-e)r$, $xR \cong (1-e)R$. Hence xR is projective.

THEOREM 3.2. *For a ring R the following are equivalent:*

- (1) R is right P.P.
- (2) Every free right R -module F_R is a C.P. module.
- (3) Every projective right R -module P_R is a C.P. module.

Proof. (1) implies (2). It suffices to prove (2) in the case $F_R = R^{(n)} = \{(x_1, x_2 \cdots x_n) \mid x_i \in R\}$ for some positive integer n . Thus suppose $n > 1$ and let $xR \subseteq R^{(n)}$ where $x = (x_1, \dots, x_n) \in R^{(n)}$. Let $\pi: R^{(n)} \rightarrow R$ be the map given by $\pi(r_1, \dots, r_n) = r_1$ (i.e. the projection on to the first component of $R^{(n)}$) and let $\bar{\pi} = \pi|_{xR}$. Then the exact sequence $0 \rightarrow \ker \bar{\pi} \xrightarrow{i} xR \xrightarrow{\bar{\pi}} \text{Im } \bar{\pi} \rightarrow 0$ splits as $\text{Im } \bar{\pi}$ is a principal right ideal of R . It follows that $xR \cong \ker \bar{\pi} \oplus \text{Im } \bar{\pi}$, where $\ker \bar{\pi}$ is a cyclic submodule of R^{n-1} and so projective by the induction assumption.

xR is a projective R -module as it is a direct sum of two projective right R -modules.

(2) implies (3). Trivial.

(3) implies (1). Trivial.

Next in this section we obtain a characterization of commutative P.P. rings. The following lemma will be required for the proof of this characterization. For modules $N \subseteq M$ we write $N \subseteq' M$ when M is an essential extension of N . For the definition of essential the reader is referred to the book of Lambek [10, p. 90].

LEMMA 3.3. *Let $R \subseteq S$ be rings (with the same identity) such that $R_R \subseteq' S_R$. Then for each $s \in S$, $r.\text{ann}_R(s)$ is generated by an idempotent of R if and only if $r.\text{ann}_S(s)$ is generated by an idempotent of R .*

Proof. Let $s \in S$ and suppose $r.\text{ann}_R(s) = eR$, where $e^2 = e \in R$. Since $r.\text{ann}_R(s) \subseteq' r.\text{ann}_S(s)$ (as R -modules) it follows that $eS \subseteq' r.\text{ann}_S(s)$ (as S -modules) and so $r.\text{ann}_S(s) = eS$.

Next suppose $r.\text{ann}_S(s) = eS$ for some $e^2 = e \in R$. We then have $eR \subseteq' eS \cap R = r.\text{ann}_R(s)$ and so $eR = r.\text{ann}_R(s)$.

THEOREM 3.4. *For a commutative ring R with total quotient ring K the following are equivalent:*

- (1) R is a P.P. ring.
- (2) K is a P.P. ring and every idempotent of K is an element

of R .

(3) Any ring A such that $R \subseteq A \subseteq K$, is a P.P. ring.

[Note: All rings in (3) share the identity of R .]

Proof. (1) implies (2). By Proposition 3.1 it suffices to that for each $k \in K$, $r.\text{ann}_K(k) = eK$ where e is an idempotent element of R . Thus let $k \in K$ and so $k = ad^{-1}$ where $a \in R$ and d is a nonzero divisor of R . The map $kR \rightarrow R$ given by $kr \rightarrow dkr$ is a module imbedding and so kR is projective. It follows that the epimorphism $R \rightarrow kR$ given by $r \rightarrow kr$, splits and so $r.\text{ann}_K(k) = \ker(R \rightarrow kR) = eR$ for some $e^2 = e \in R$. Lemma 3.3 now gives $r.\text{ann}_K(k) = eK$.

(2) implies (3). This is a clear consequence of Lemma 3.3.

(3) implies (1). Trivial.

We now have a corollary to Proposition 2.4. [Endo].

COROLLARY. Suppose R is a quasi-regular ring. For any ring A such that $R \subseteq A \subseteq K$ the following are equivalent:

- (1) R_V is an integral domain for every maximal ideal V of R .
- (2) A_V is an integral domain for every maximal ideal V of A .

4. Baer ideals and torsion free R -modules. Throughout this section R is assumed to have a right ring of quotients as defined by Levy [11, p. 133]. Any commutative ring has such a ring of quotients.

If M is a right R -module, let $T(M) = \{m \in M \mid md = 0 \text{ for some nonzero divisor } d \text{ of } R\}$. $T(M)$ is a submodule of R [11, Theorem 1.4]. M is said to be *torsion-free* if $T(M) = 0$.

If J is a right ideal of R , let J_B be the right ideal of R such that $T(R/J) = J_B/J$. Ideals such that $J_B = J$, have previously been used by Cateforis and Sandomierski [2, p. 162].

PROPOSITION 4.2. Let R be a ring, S the right ring of quotients of R and J a right ideal of R . Then the following are equivalent:

- (1) $T(R/J) = 0$.
- (2) There exists a right ideal J' of S such that $J' \cap R = J$.
- (3) $J_B = J$.

Proof. (1) implies (2). Clearly $J \subseteq JS \cap R$. Conversely if $x \in JS \cap R$, then $x = asd^{-1}$, where $a \in J$, $s \in R$ and d is a nonzero divisor of R . Hence $xd = as \in J$. Thus as $T(R/J) = 0$, $x \in J$.

(2) implies (3). If $xd \in J$, then $xdd^{-1} \in JS \subseteq J'$ which implies $x \in J$; i.e. $J_B = J$.

(3) implies (1). If M is any right R -module $T(M/T(M)) = 0$. Hence $T(R/J_B) = 0$.

PROPOSITION 4.3. *Let R be a ring, J a right ideal of R . Then*

- (1) $J_B = R$ if and only if J contains a nonzero divisor.
- (2) $T(R/J_B) = 0$, and if $J \subseteq I$ a right ideal such that $T(R/I) = 0$, then $J_B \subseteq I$.

Proof (1). Since $T(R/J) = R/J$ if and only if J contains a nonzero divisor, the result follows.

(2) From Proposition 4.2. $T(R/J_B) = 0$. Now suppose $x \in J_B$. Then $xd \in J \subseteq I$ for some nonzero divisor d of R . Hence as $T(R/I) = 0$, $x \in I$.

We now look at ideals J of a ring R such that $T(R/J) = 0$ when R is a quasi-regular or commutative P.P. ring. Properties of these ideals in quasi-regular rings have been looked at by Endo [5, p. 111-112]. From Proposition 3.5 we have the following.

COROLLARY 4.4. *If R is a commutative semiprime ring then the following are equivalent:*

- (1) R is a quasi-regular ring.
- (2) If P is a prime ideal of R , $T(R/P) = 0$ if and only if it is a minimal prime ideal.

COROLLARY 4.5. *If R is a commutative quasi-regular ring then the following are equivalent:*

- (1) If J is an ideal of R , then R , then $T(R/J) = 0$.
- (2) R is a regular ring.

Proof. (1) implies (2). From Corollary 4.4 this means that every prime ideal of R is a minimal prime ideal, and hence maximal ideal of R . Hence R_V is a field for each maximal ideal V of R . The result follows from Theorem 1 of Endo [5].

(2) implies (1). Every nonzero divisor of a regular ring is a unit.

PROPOSITION 4.6. *Let R be a commutative P.P. ring, J an ideal of R then the following are equivalent:*

- (1) $T(R/J) = 0$.
- (2) If $x - y \in J$ then $e_x - e_y \in J$.
- (3) R/J is a flat R -module.
- (4) If $x \in J$ then $rr(x) \subseteq J$.
- (5) $J = \cap \{M \in \text{Min } R \mid J \subseteq M\}$.

Proof. (1) implies (2). It will first be shown that if $s, x \in R$, then $sx \in J$ if and only if $se_x - s \in J$. If $se_x - s \in J$ then $-xs = x(se_x - s) \in J$. Hence $xs \in J$.

Conversely if $xs \in J$, then $(se_x - s)(x + e_x) = -sx + se_x - se_x =$

$sx \in J$.

Now if $x - y \in J$, then $x + J = y + J$. From the above it can be seen that this implies $(e_x + J)R/J = r.\text{ann}_{R/J}(x/J) = r.\text{ann}_{R/J}(y/J) = (e_y + J)R/J$ i.e. $e_x - e_y \in J$.

(2) implies (3). Let $x \in J$. Then $1 - e_x \in J$ and $x(1 - e_x) = x$. Thus R/J is flat [11, Ex 3, p. 135].

(3) implies (1). If $xd \in J$, then by [11, Ex 3, p. 135] there exists a $c \in J$ such that $xdc = xd$, which implies $x = xc \in J$.

The equivalence of (2), (4) and (5) has been shown by T. P. Speed [17].

REMARK 1. Conditions (1), (4) and (5), of the above theorem, are equivalent for a quasi-regular ring. Furthermore it can be shown as a corollary of Theorem 3.1 of Mewborn [12], that conditions (4) and (5) are equivalent for a semiprime ring R if and only if $\text{Minp } R$ is compact.

In [18] ideals satisfying property (2) were called *Baer ideals*. We will continue to use this name.

REMARK 2. In [18] a Baer homomorphism between two commutative *P.P.* rings, R and R' was defined to be a ring homomorphism f satisfying the additional property $f(e_x) = e_{f(x)}$, where $r.\text{ann}_{R'}(f(x)) = e_{f(x)}R'$, for all $x \in R$. It was shown that an ideal J of R is the kernel of a Baer homomorphism if and only if J satisfied condition (2) of the above theorem.

The Baer ideals of R , a commutative *P.P.* ring, form a pseudo complemented lattice [17], which we will denote by $I^B(R)$. If S is a commutative ring, denote the lattice of ideals of S by $I(S)$ and let $B(S)$ denote the Boolean algebra of idempotents of S .

PROPOSITION 4.7. *Let R be a commutative *P.P.* ring. Then $I(K) \cong I(B(K)) \cong I(B(R)) \cong I^B(R)$, where the isomorphisms are lattice isomorphisms.*

Proof. It can be seen from Proposition 4.2 (2) that $I(K) \cong I^B(R)$. It is well known that $I(K) \cong I(B(K))$ and $I(B(K)) \cong I(B(R))$ as $B(K) = B(R)$ (Theorem 3.4).

Finally in this section we look at the decomposition of commutative *P.P.* rings.

A submodule M of a right R -module A_R is said to be *large* if $M \cap N \neq 0$ for each nonzero submodule N of A_R . A large ideal is a large submodule of R . If R is a commutative semiprime ring, J is a large ideal if and only if $r(J) = 0$. A right R -module A_R is said to be *nonsingular* if $Z_R(A_R) = 0$ where $Z_R(A_R) = \{x \in R \mid r(x) \text{ is a large}$

ideal of R).

THEOREM 4.8. *Over a commutative P.P. ring the following are equivalent:*

- (1) R has the a.c.c. on Baer ideals.
- (2) R is a finite direct sum of integral domains.
- (3) Every torsion-free R -module is a C.P. module.
- (4) Every torsion-free R -module is a nonsingular module.
- (5) $Z(M) = T(M)$ for every module M .
- (6) Every large ideal contains a nonzero divisor of R .

Proof. (1) implies (2). This follows from the lemma of Hattori [6, p. 156].

(2) implies (3). If A_R is a torsion-free R -module then each cyclic submodule of R is a torsion-free R -module. Hence if $x \in R$, $r(x)$ is a Baer ideal, (Proposition 4.6). It is a consequence of R being a finite direct sum of integral domains that all Baer ideals of R are idempotently generated.

(3) implies (4). Free modules over nonsingular rings are nonsingular modules.

(4) implies (5). If an ideal J contains a nonzero divisor d then $J \subseteq' R$ and so $T(M) \subseteq Z(M)$. Now $Z(M)/T(M) \subseteq Z(M/T(M))$ and since $T(M/T(M)) = 0$ we have $Z(M/T(M)) = 0$ or $Z(M) \subseteq T(M)$.

(5) implies (6). If J is a large ideal of R , then $Z(R/J) = R/J$ and so $T(R/J) = R/J$. Hence there is a nonzero divisor $d \in R$ such that $d(1 + J) = 0$ or $d \in J$.

(6) implies (1). If B is a large ideal of K then $B \cap R$ is a large ideal of R and so B contains a nonzero divisor of R . It follows that $B = K$, as d^{-1} exists in K , and so has no large ideals $\neq K$. K is thus artinian semisimple. Hence the ideals of K satisfy the a.c.c. It now follows from Proposition 4.7 that the Baer ideals of R satisfy the a.c.c.

COROLLARY 4.9. *If R is a commutative hereditary (semihereditary) ring R with identity, then R is a finite direct sum of Dedekind (Prüfer) domains if and only if R has the a.c.c. on Baer ideals.*

5. Finitely generated nonsingular R -modules of a commutative semiprime ring. In this section we introduce a third torsion theory. If A_R is an R -module of a commutative ring R , let $U(A) = \{x \in A \mid \text{lr}(x) = 0\}$. This is the definition of 'torsion submodule' used by Pierce [13, p. 80-83]. Note that if R is an integral domain $U(A) = 0$ if and only if $T(A) = 0$. Also, if R is a semiprime ring $U(A) = 0$ if and only if $Z(A) = 0$ [11, Ex. 1. p. 108].

It was mentioned in the introduction of this paper, that by a Baer ring, we will mean a ring R with identity, such that for each subset $S \subseteq R$, $r(S) = \{t \in R \mid St = 0\} = eR$, where e is an idempotent of R . A commutative P.P. ring is a Baer ring if and only if its Boolean algebra of idempotents is complete. We will need the following lemma of Sandomierski [15, p. 226]. For the definition of closed submodule the reader is referred to [15].

LEMMA 5.1. *If B_R is a submodule of an R -module A_R , such that $Z(A/B) = 0$, then B is closed in A .*

THEOREM 5.2. *Let R be a commutative ring. Then the following are equivalent:*

- (1) R is a Baer ring.
- (2) $Z_R(R) = 0$, and if A_R is a nonsingular R -module, then A_R is a C.P. module.
- (3) If A_R is an R module such that $U(A) = 0$, then A is a C.P. module.
- (4) If $S \subseteq R$, then $R/r(S)$ is a projective R -module.

Proof. (1) implies (2). Since R is a semiprime ring, the closed ideals of R are exactly the annihilator ideals of R . If A_R is a nonsingular R -module, then $x \in A_R$ implies $r(x)$ is an annihilator ideal of R (Lemma 5.1). The result now follows by Proposition 3.1 as R is a Baer ring.

(2) implies (3). Since $Z(R) = 0$, $U(A) = 0$ if and only if $Z(A) = 0$.

(3) implies (4). Let $S \subseteq R$, where S is an arbitrary subset of R . It will first be shown that $U(R/r(S)) = 0$. Suppose that $r(S) \neq R$. Then there is an $x \in R$, such that $x \notin r(S)$. Now $r(x + r(S)) = \{t \in R \mid xt \in r(S)\} = r(xS)$. Hence if $rr(x + r(S)) = 0$, then $r(xS) = R$, which implies $xS = 0$, i.e. $x \in r(S)$. Thus $U(R/r(S)) = 0$ for $r(S) \neq R$. Hence as $R/r(S)$ is a cyclic R -module it is projective.

(4) implies (1). Since $R/r(S)$ is projective the exact sequence $0 \rightarrow r(S) \rightarrow R \rightarrow R/r(S) \rightarrow 0$ splits. Hence $r(S)$ is a direct summand of R , i.e. $r(S) = eR$ for some idempotent $e \in R$.

K. M. Rangaswamy and N. Vanaja have also considered the condition that every cyclic nonsingular R -module is projective. The above result generalised Proposition 1 of [14] for the commutative case.

The following proposition will be used in the proof of Theorem 5.4.

PROPOSITION 5.3. *Let J be a finitely generated ideal of a commutative quasi-regular ring R . Then if J does not contain a nonzero*

divisor, $r(J) \neq 0$.

Proof. Let $J = \sum_{i=1}^n a_i R$ and suppose $Jx = 0$ implies $x = 0$. Then J is a large ideal of R and so JK is a large ideal of K . Since K is regular and JK is a finitely generated K -ideal (generated by the a_i) it follows that $JK = K$. Thus $1 \in JK$ and $1 = \sum_{i=1}^n x_i k_i = \sum_{i=1}^n x_i (a_i d_i^{-1})$ where $x_i \in J$, d_i is a nonzero divisor of R . Now by [12, Lemma 1.3] there exist $b_i \in R$ and a nonzero divisor $d \in R$ such that $a_i d_i^{-1} = b_i d^{-1}$ and so $1 = \sum x_i b_i d^{-1}$. It follows that $d \in J$, a contradiction.

Let Q denote the complete ring of quotients of R [10, p. 40]. Cateforis and Sandomierski have introduced the condition $Z_R(Q \otimes_R Q) = 0$. [2, p. 151]. The next proposition gives an equivalent condition to this when R is a quasi-regular ring.

THEOREM 5.4. *Let R be a quasi-regular ring, Q the complete ring of quotients of R , then the following are equivalent:*

- (1) $Z_R(Q \otimes_R Q) = 0$.
- (2) $Q = K$, the classical ring of quotients of R .

Proof. (1) implies (2). Q is a flat R -module as $\text{Minp } R$ is compact [12, Thm 3.1]. As R is also semiprime it is possible to use Theorem 1.6 of [3]. Hence $Z(Q \otimes_R Q)$ is equivalent to the condition that for each $q \in Q$, $(R :_R q) = \{r \in R \mid rq \in R\}$ contains a finitely generated large submodule, J . Q is an essential extension of R and therefore J is a large ideal of R . Now if J is a finitely generated large ideal of R , $r(J) = 0$ and hence by Proposition 5.3 J must contain a nonzero divisor d . Finally, if $q \in Q$ there exists a nonzero divisor d of R such that $dq \in R \subseteq K$ which implies $q \in K$.

(2) implies (1). As $K \otimes_R K \cong K$ it follows that $Z_R(Q \otimes_R Q) = 0$.

If R is a semiprime ring with $\text{Minp } R$ compact and $Z_R(Q \otimes_R Q) = 0$, where Q is the complete ring of quotients of R , then $Q = \bar{K}$, the classical ring of quotients of the Baer extension of R . The Baer extension has been introduced in [9].

The following Proposition is due to Cateforis [3].

PROPOSITION 5.5. *For a ring R , the following are equivalent:*

- (1) $Z(R_R) = 0$ and every finitely generated nonsingular right R -module is projective.
- (2) R is semihereditary, Q_R is flat and $Z(Q \otimes_R Q) = 0$.

COROLLARY 5.6. *Let R be a commutative ring. Then the following are equivalent:*

(1) If A_R is a finitely generated R -module and $U(A) = 0$, then A_R is projective.

(2) $Z_R(R_R) = 0$ and every finitely generated nonsingular R -module is projective.

(3) R is semihereditary and K the classical quotient ring of R is self injective.

Proof. The proof is derived from Theorem 5.2, Theorem 5.4 and Proposition 5.5.

Theorem 24.5 of Pierce [13] can be obtained directly from this corollary. For if R is a regular ring, R is semihereditary and $R = K$.

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