Pacific Journal of Mathematics

TWISTED COHOMOLOGY THEORIES AND THE SINGLE OBSTRUCTION TO LIFTING

LAWRENCE LOUIS LARMORE

Vol. 41, No. 3 BadMonth 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 3, 1972

TWISTED COHOMOLOGY THEORIES AND THE SINGLE OBSTRUCTION TO LIFTING

LAWRENCE L. LARMORE

Consider any fibration $p: E \to B$, any finite C.W. — pair (K, L), and any maps $f: K \to B$ and $h: L \to E$ such that $p \circ h = f \mid L$. A map $g: K \to E$ such that $p \circ g = f$ and $g \mid L = h$ we call a *lifting of f rel h*.

In this paper single obstruction $\Gamma(f) \in H'(K, L, f; \mathscr{E})$ is defined. $\mathscr{E}$ is a so-called B-spectrum, and $H^*(\ ; \mathscr{E})$ is cohomology in that spectrum. If a lifting of f rel h exists, $\Gamma(f) = 0$; this condition is also sufficient if the fiber of p is k-connected and $\dim(K/L) \leqq 2k + 1$.

If g_0 and g_1 are liftings of f rel h, a single obstruction $\delta(g_0, g_1; h) \in H(K, L, f: \mathscr{E})$ is also defined; if g_0 and g_1 are connected by a homotopy of liftings of f rel h $\delta(g_0, g_1; h) = 0$; this condition is, also sufficient if p is k-connected and $\dim(K/L) \leqq 2k$.

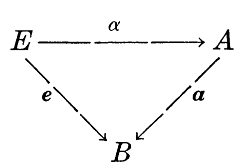
In § 4, a spectral sequence is constructed for cohomology in a B-spectrum, based on the Postnikov tower of that spectrum, and the relationship between the single obstruction and the classical obstructions is defined.

For similar treatments, see Becker [1], [2], and Meyer [5].

Throughout this paper, let (K, L) be a finite $C. W.$ pair, B any space, and $f: K \to B$ any map. All spaces and maps shall be in the category CG of compactly generated spaces and maps, as described by Steenrod [7], and all constructions (i.e., function spaces, quotient space, Cartesian products) shall be as defined in that paper. When possible without confusion, we shall allow $f \mid L$ and $f \mid K \cup L$ to be denoted simply as F. A map $\pi: X \to Y$ we call a *fibration* if it has a local product structure; the polyhedral covering homotopy extension property [4] is then satisfied.

2. Basic concepts.

We define a *B-bundle* to be an ordered pair (E, e) such that $e: E \to B$ is a fibration. A B-bundle map from a B-bundle $e = (E, e)$ to another B-bundle $a = (A, a)$ is defined to be a commutative diagram:



We denote this map $\alpha: e \to a$. A *pointed* B-bundle is an ordered triple $(E, \boldsymbol{e}, e')$ such that $\boldsymbol{e}: E \to B$ is a fibration and $e': B \to E$ is a *pointing*, i.e., $\boldsymbol{e} \circ e' = 1$, the identity on B. We call e' a pointing because it chooses a base-point for each fiber of $\boldsymbol{e}$. A *bi-pointed* B-bundle is an ordered quadruple $(E, \boldsymbol{e}, e', e'')$ such that (E, e) is a B-bundle and e' and e'' are both pointings. If $e = (E, \boldsymbol{e}, e')$ and $a = (A, \boldsymbol{a}, a')$ are pointed B-bundles, a B-bundle map $\alpha: e \to a$ is a *pointed* map if $\alpha \circ e' = a'$. Similarly, we can define bi-pointed maps between bi-pointed bundles. Two bundle maps (or pointed bundle maps, or bi-pointed bundle maps) are said to be homotopic if there exists a homotopy of bundle maps (or pointed bundle maps, or bi-pointed bundle maps) connecting them.

If $e = (E, \boldsymbol{e})$ is a B-bundle, $\boldsymbol{e}^{-1} b$ is called the fiber of e over b, for any $b \in B$. If $e = (E, \boldsymbol{e}, e')$ is a pointed B-bundle, each fiber, $(\boldsymbol{e}^{-1} b, e'b)$ is a pointed space. If $e = (E, \boldsymbol{e}, e', e'')$ is bi-pointed, we say that $e'b$ is the *South pole* of $\boldsymbol{e}^{-1} b$, while $e''b$ is the *North pole*.

Let $\mathscr{H}_B$ be the category of B-bundles and B-bundle maps. Let $\mathscr{H}_B{}^*$ and $\mathscr{H}_B{}^{**}$ be the categories of pointed and bi-pointed B-bundles and maps, respectively. We obviously have forgetful functors $\alpha: \mathscr{H}_B{}^{**} \to \mathscr{H}_B{}^*$ and $\beta: \mathscr{H}_B{}^* \to \mathscr{H}_B$ where $\alpha(E, \boldsymbol{e}, e', e'') = (E, \boldsymbol{e}, e')$ and $\beta(E, \boldsymbol{e}, e') = (E, \boldsymbol{e})$. We shall, whenever convenient, identify any object with its image under α, β, or $\beta \circ \alpha$. We also define functors as follows:

S: $\mathscr{H}_B \to \mathscr{H}_B{}^{**}$ two-point suspension

Σ: $\mathscr{H}_B{}^* \to \mathscr{H}_B{}^*$ one-point suspension

Ω: $\mathscr{H}_B{}^* \to \mathscr{H}_B{}^*$ looping

P: $\mathscr{H}_B{}^{**} \to \mathscr{H}_B$ paths from the South pole to the North pole

$S(E, e) = (S_B E, \boldsymbol{s}, s', s'')$ where $S_B E$ is the quotient space of $E \times I$ obtained by identifying $(x, 0)$ with $(y, 0)$ and $(x, 1)$ with $(y, 1)$ for any $x, y \in \boldsymbol{e}^{-1}b$ for any $b \in B$. For all $[x, t] \in S_B E$, $\boldsymbol{s}[x, t] = \boldsymbol{e}x$, while $s'b = [x, 0]$ and $s''b = [x, 1]$ for all $b \in B$, where x is any element in the fiber of e over b. $\Sigma(E, \boldsymbol{e}, e) = (\Sigma_B E, \boldsymbol{s}, s')$ where $\Sigma_B E$ is the quotient space of $E \times I$ obtained by identifying $(x, 0)$ with $((e' \circ \boldsymbol{e})x, t)$ $(x, 1)$ for any $x \in E$ and any $t \in I$. Then $\boldsymbol{s}[x, t] = \boldsymbol{e}x$ for all $[x, t] \in \Sigma_B E$ and $s'b = [e'b, 0]$ for any $b \in B$.

$\Omega(E, e, e') = (\Omega_B E, \sigma, \sigma')$ where $\Omega_B E$ is the space of all loops in E based on $e'(B)$ which lie in a single fiber of e; $\sigma\alpha = (\boldsymbol{e} \circ \alpha)(0)$ for all $\alpha \in \Omega_B E$, and $(\sigma'b)t = e'b$ for all $b \in B$, and all $t \in I$. $P(E, \boldsymbol{e}, e', e'') = (P_B E, \boldsymbol{p})$ where $P_B E$ is the space of all paths from $e'(B)$ to $e''(B)$ which lie in a single fiber, and $\boldsymbol{p}\alpha = (\boldsymbol{e} \circ \alpha)(0)$ for all $\alpha \in P_B E$.

We give two adjoint constructions. First, let $e = (E, \boldsymbol{e}, e')$ and $a = (A, \boldsymbol{a}, a')$ be two pointed B-bundles. If $\alpha: e \to \Omega a$ and β; $\Sigma e \to a$ are pointed B-bundle maps, we say that α and β are *adjoints* of each other if, for any $x \in E$ and any $t \in I$, $\beta[x, t] = (\alpha x)t$. Second, let $e = (E, \boldsymbol{e})$ be a B-bundle and $a = (A, \boldsymbol{a}, a', a'')$ a bi-pointed B-bundles. We say that maps $\alpha: e \to Pa$ and $\beta: Se \to a$ (where β is bi-pointed) are *adjoints* of each other if $\beta[x, t] = (\alpha x)t$ for all $x \in E$ and all $t \in I$.

Let $[K, L, h; e]_f$ denote the set of rel L fiber-homotopy classes of liftings of f to E rel h, where $e = (E, \boldsymbol{e})$ is a B-bundle and $h: L \to E$ is a lifting of $f \mid L$. If L is empty, write $[K: e]_f$. If $e = (E, \boldsymbol{e}, e')$ is pointed, write $[K, L; e]_f$ for $[K, L, e' \mid L; e]_f$. If $\alpha: e \to a$ is a B-bundle map, let $\alpha_\#: [K, L, h; e]_f \to [K, L, \alpha \circ h; a]_f$ be the function where $\alpha_\#[g] = [\alpha \circ g]$, where $[g]$ is the fiber-homotopy rel L class of any lifting g of f rel h. If $r: (K', L') \to (K, L)$ is a map of $C.W.$ pairs, let $r^*: [K, L, h; e]_f \to [K', L, h \circ r; e]_{f \circ r}$ be the function where $r^*[g] = [g \circ r]$. We omit the proof (based in part on the *PCHEP* of $\boldsymbol{e}$) of the following lemma:

LEMMA 2.1. *If $r: (K', L') \to (K, L)$ is a homotopy equivalence of pairs, then $r^*: [K, L, h; e]_f \cong [K', L', h \circ r; e]_{f \circ r}$.*

Let $e = (E, \boldsymbol{e})$ be a B-bundle. If each fiber of e is connected, we say that e is connected. Similarly, if each fiber of e is n-connected, or n-simple, for some integer $n \geqslant 1$, we say that e is n-connected, or n-simple. If e is n-simple, define $\pi_n e$ to be the local system of Abelian groups over B such that, for every $b \in B$, $(\pi_n e)b = \pi_n(\boldsymbol{e}^{-1}b)$. We call $\pi_n e$ the n^{th} homotopy group system of e. Similarly, if e is pointed, we can define $\pi_n e$ whether e is n-simple or not, since every fiber has a base-point. Note that e is n-connected if and only if e is connected and $\pi_k e = 0$ for all $k \leqslant n$. If $\alpha: e \to a$ is any B-bundle map, where e and a are both n-simple or both pointed (and α is pointed) or e is pointed and a is n-simple, α induces a homomorphism $\alpha_\#: \pi_n e \to \pi_n a$ in the obvious way.

Let $\alpha: e \to a$ be any B-bundle map, where $e = (E, \boldsymbol{e})$ and $a = (A, \boldsymbol{a}, a')$. We define the *fiber* of α to be the B-bundle $c = (C, \boldsymbol{c})$ where C is the space of all ordered pairs (x, σ) such that $x \in E$ and σ is a path in A such that $\sigma(0) \in a'(B)$, $\sigma(1) = \alpha x$, and $(\boldsymbol{a} \circ \sigma)t = \boldsymbol{e}x$ for all $t \in I$; and where $\boldsymbol{c}(x, \sigma) = \boldsymbol{e}x$ for all $(x, \sigma) \in C$. If $e = (E, \boldsymbol{e}, e')$ is pointed, then $c'b = (e'b, \sigma)$ gives a pointing of c, where $\sigma t = a'b$ for all $t \in I$. The reader will note that for any $b \in B$, $\boldsymbol{c}^{-1}b$ is precisely the fiber of $\alpha: \boldsymbol{e}^{-1}b \to \boldsymbol{a}^{-1}b$. The following sequence is thus exact, if $\alpha: e \to a$ is pointed:

$$\cdots \longrightarrow \pi_n(\Omega e) \xrightarrow{(\Omega\alpha)_\#} \pi_n(\Omega a) \xrightarrow{j_\#} \pi_n c \xrightarrow{i_\#} \pi_n e \xrightarrow{\alpha_\#} \pi_n a$$

where $i(x, \sigma) = \sigma(1)$ for all $(x, \sigma) \in C$, and $j(\tau) = (c'b, \tau)$ for all $\tau \in \Omega_B A$, where $b = (\boldsymbol{a}, \tau)(1)$.

Now if $\alpha: e \to a$ is a B-bundle map, we say that α is n-connected for any $n \geqslant 0$ if, for all $b \in B$ and $y \in \boldsymbol{a}^{-1}b$, the space

$$\{(x, \sigma) \in \boldsymbol{e}^{-1}b \times (\boldsymbol{a}^{-1}b)^I : \sigma(0) = y, \sigma(1) = \sigma x\}$$

is n-connected. If a is a connected pointed B-bundle, α is connected if and only if the fiber of α is n-connected.

Suppose now that $\alpha: e \to a$ is a B-bundle map. Consider

$$\alpha_\sharp : [K, L, h; e]_f \longrightarrow [K, L, \alpha \circ h; a]_f .$$

LEMMA 2.2. *Suppose α is n-connected for some $n \geqslant 0$. Then:* (i) *$\alpha_\sharp$ is onto if* $\dim(K/L) \leqslant n$. (ii) *$\alpha_\sharp$ is one-to-one if* $\dim(K/L) \leqslant n - 1$.

Proof. The connectivity of α equals the connectivity of the fiber of $\alpha: E \to A$, considered as a map of spaces. Simple application of ordinary obstruction theory enables us to complete the proof in a routine manner; we omit the details.

Suppose now that $g_0, g_1: K \to E$ are both liftings of f rel h.

LEMMA 2.3. *If α is n-connected for some $n \geqslant 1$, then g_0 and g_1 are homotopic* rel h *if and only if $\alpha \circ g_0$ and $\alpha \circ g_1$ are homotopic,* rel L; *provided* $\dim(K/L) \leqslant n - 1$.

Proof. We have a bi-pointed K-bundle map $f^{-1}\alpha: f^{-1}e \to f^{-1}a$, where $f^{-1}e = (f^{-1}E, f^{-1}\boldsymbol{e}, f^{-1}g_0, f^{-1}g_1)$ and

$$f^{-1}a = (f^{-1}A, f^{-1}\boldsymbol{a}, f^{-1}(\alpha \circ g^0), f^{-1}(\alpha \circ g_1)) ;$$

and $Pf^{-1}\alpha; Pf^{-1}e \to Pf^{-1}a$ is $(n-1)$-connected. A section of $Pf^{-1}e$ is equivalent to a fiber homotopy, rel L, of g_0 with g_1, while a section of $Pf^{-1}a$ is equivalent to a fiber homotopy, rel L, of $\alpha \circ g_0$ with $\alpha \circ g_1$. Apply Lemma 2.2, and we are done.

3. B-Spectra.

Suppose $e = (E, \boldsymbol{e}, e')$ is a pointed B-bundle. We define an operation "+" on $[K, L, \Omega e]_f$ as follows: for any two liftings of f rel $e' \mid L$, g and g', let $g + g': K \to \Omega_B E$ be the map where $((g+g')x)t = (gx)(2t)$ if $0 \leqq t \leqq 1/2$, $g'(x)(2t-1)$ if $1/2 \leqq t \leqq 1$, for all $x \in K$. Then $g + g'$ is also a lifting of f rel $e' \mid L$. We define $[g] + [g'] = [g+g]'$; it is trivial to verify that the operation is well-defined.

THEOREM 3.1. $[K, L; \Omega e]_f$ *is a group under the operation "+" with identity* $[e']$.

Proof. Let $[g]^{-1} = [g^{-1}]$ for any lifting g of f rel $e' \mid L$, where $(g^{-1}x)t = (gx)(1-t)$ for all $x \in K$ and all $t \in I$; it is routine to check that the group axioms are satisfied.

THEOREM 3.2. *$[K, L; \Omega^2 e]_f$ is an Abelian group.*

Proof. We omit the details; if g and g' are both liftings of f rel $e' \mid L$, a fiber homotopy rel L of $g + g'$ with $g' + g$ can easily be constructed in the same manner as the proof that $[X; \Omega^2 Y]$ is Abelian for pointed spaces X and Y, but the construction is done fiberwise over B.

DEFINITION 3.1. A B-spectrum is an ordered pair

$$\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\}_{i \geqq m})$$

for some integer m such that:

(i) For each $i \geqq m$, e_i is a pointed B-bundle.

(ii) For each $i \geqq m$, $\varepsilon_i \colon e_i \to e_{i+1}$ is a pointed B-bundle map.

Furthermore, we say that $\mathscr{E}$ is a Ω_B-spectrum if ε_i is a homotopy equivalence (in the category $\mathscr{K}_B{}^*$) for each i, and we say that ε is a weak Ω_B-spectrum if ε_i is infinitely connected for all $i \geqq m$. We say that ε is *stabilizing* if, for each integer n, there exists an integer $N \geqq m$ such that ε_i is $(n+i)$-connected for all $i \geqq N$. The e_i are called the elements of the spectrum, the ε_i are called the connection maps, and m is called the starting value. If the first finitely many elements of a spectrum are altered, no change occurs in cohomology with coefficients in that spectrum; in that sense, the starting value is arbitrary. We define the homotopy of a spectrum $\pi_n(\mathscr{E})$ for any integer n, to be the direct limit $\mathrm{Lim}_{i \to \infty} \pi_{n+i} e_i$, under the system of homomorphisms

$$(\varepsilon)_{i\#} \colon \pi_{n+i} e_i \longrightarrow \pi_{n+i} \Omega e_{i+1} \cong \pi_{n+i+1} e_{i+1}$$

thus $\pi_n(\mathscr{E})$ is a local system of Abelian groups on B. Note that $\pi_n(\mathscr{E})$ need not be zero for negative values of n.

Henceforth, we shall assume that $\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\}_{i \geqq m})$ is a B-spectrum.

DEFINITION 3.2. For any integer n, let $H^n(K, L, f; \mathscr{E})$ be the direct limit of the system of groups $\{[K, L; \Omega^{i-n} e_i]_f\}$ and homomorphisms $\{(\Omega^{i-n} \varepsilon_i)_\#\}$. (If L is empty, we write $H^n(K, f; \mathscr{E})$.) For any $i \geqq \min(n, m)$, let

$$[K, L; \Omega^{i-n} e_i]_f \longrightarrow H^n(K, L, f; \mathscr{E})$$

be called the representation. If $\mathscr{E}$ is stabilizing, the direct limit is achieved eventually, i.e., beyond some point, all representations are bijective; if $\mathscr{E}$ is a weak Ω_B-spectrum, the direct limit is achieved immediately, i.e., all representations are bijective. We call $H^*(K, L, f; \mathscr{E})$ the cohomology of the triple (K, L, f) with coefficients in the spectrum $\mathscr{E}$. If (K', L') is another $C.W.$ pair, and

$$r: (K', L') \longrightarrow (K, L)$$

is a map, an induced homomorphism

$$r^*: H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K', L', f \circ r; \mathscr{E})$$

can be defined in the obvious way.

Henceforth, let (K'', L'') be the pair $(K \times \{1\} \cup L \times I, L \times \{0\})$, and let p; $(K'', L'') \to (K, L)$ be projection onto the first factor. The reader can easily verify that p is a relative homotopy equivalence, and hence by the direct limit version of Lemma 2.1,

$$p^*: H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K'', L'', f \circ p; \mathscr{E})$$

is an isomorphism.

For any integer n, we define a connecting homomorphism

$$\delta: H^n(L, f; \mathscr{E}) \longrightarrow H^{n+1}(K, L, f; \mathscr{E})$$

as follows. For any $a \in H^n(L, f; \mathscr{E})$, pick $i \geqq m$ and $[g] \in [L; \Omega^{i-n} e_i]_f$ representing a. Consider $\Omega^{i-n} e_i = \Omega \Omega^{i-n-1} e_i$. Let $p^* \delta a$ be the image, in the direct limit, of $[G] \in [K'', L''; \Omega^{i-n-1} e_i]_{f \circ p}$, where $G(x, t) = (gx)t$ for all $x \in L$ and $t \in I$, and where $G(x, 1) = a'(fx)$ for all $x \in K$, where a' is the pointing of $\Omega^{i-n-1} e_i$; δa is well-defined since p^* is an isomorphisms.

The following remarks (analogous to some of the Eilenberg Steenrod axioms for a cohomology theory [3]) we state without proof:

REMARK 3.3. The following long sequence is exact, where i and j are inclusions:

$$\cdots \longrightarrow H^{n-1}(L, f; \mathscr{E}) \xrightarrow{\delta} H^n(K, L, f; \mathscr{E}) \xrightarrow{j^*} H^n(K, f; \mathscr{E})$$
$$\xrightarrow{i^*} H^n(L, f; \mathscr{E}) \xrightarrow{\delta} H^{n+1}(K, L, ; \mathscr{E}) \longrightarrow \cdots .$$

REMARK 3.5. If $r_t: (K', L') \to (K, L)$, $0 \leqq t \leqq 1$, is a homotopy of maps, where (K', L') is another $C.W.$ pair, such that $f \circ r_t = f \circ r_0$ for all t, then $r_1^* = r_0^*$.

Suppose now that $f_t: K \to B$, $0 \leqq t \leqq 1$, is a homotopy such that $f_0 = f$. Let $F: K \times I \to B$ be the map where $F(x, t) = f_t x$ for all

$(x, t) \in K \times I$. Let $i_0, i_1: (K, L) \to (K \times I, L \times I)$ be the inclusions along 0 and 1, respectively. According to Lemma 2.1., $(i_j)_\#$ is an isomorphism for $j = 0$ or 1. Let

$$F_\# = (i_1)_\# \circ (i_0)_\#^{-1}: H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K, L, f; \mathscr{E}) ,$$

clearly an isomorphism. Again without proof, we state:

REMARK 3.6. $F_\#$ depends only on the homotopy class of F, rel $K \times \{0, 1\}$.

REMARK 3.7. If G is a homotopy of f_1 with f_2, then

$$G_\# \circ F_\# = (F+G)_\#: H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K, L, f_2; \mathscr{E})$$

where $(F+G)(x, t) = F(x, 2t)$ if $0 \leqq t \leqq 1/2$; $G(x, 2t)$ if $1/2 \leqq t \leqq 1$, for all $x \in K$.

An immediate question one may ask is: if $f_1 = f$, is $F_\#$ the identity? The answer is generally no.

4. The associated spectrum and the single obstruction. Let $e = (E, e)$ be a B-bundle and $h: L \to E$ a lifting of $f \mid L$. Let

$$\mathscr{E} = \mathscr{E}(e) = (\{e_i\}_{i \geqq 1}, \{\varepsilon_i\}_{i \geqq 1})$$

be the B-spectrum where $e_i = \sum^{i-1} Se$ for all $i \geqq 1$, and $\varepsilon_i: e_i \to \Omega e_{i+1}$ is adjoint to the identity on $e_{i+1} = \sum e_i$. We call $\mathscr{E}$ the B-spectrum associated to e. We shall write $e_1 = Se = (S_B E, \mathbf{s}, s', s'')$.

Recall $(K'', L'') = (K \times \{1\} \cup L \times I, L \cup \{0\})$. We define $\Gamma(f; h) \in H^1(K, L, f; \mathscr{E})$ (or simply $\Gamma(f)$ when L is empty, or when h is understood), the *single obstruction to lifting f rel h*, to be $(p^*)^{-1}$ of the representation of $[H] \in [K'', L''; Se]_{f \circ p}$, where $H: K'' \to S_B E$ is the map such that $H(x, t) = [hx, t]$ for all $(x, t) \in L \times I$, and $H(x, 1) = (e'' \circ f)x$, the North pole of $\mathbf{e}^{-1} fx$, for all $x \in K$. We leave it to the reader to verify that if $f_t: K \to B$, for $0 \leqq t \leqq 1$, is a homotopy, and if $h_t: L \to E$ is a homotopy such that $\mathbf{e} \circ h_t = f_t \mid L$ for all t, and if $F(x, t) = f_t x$ for all $(x, t) \in K \times I$, then $F_\# \Gamma(f_0; h_0) = \Gamma(f_1; h_1)$; i.e., $\Gamma(f; h)$ is a homotopy invariant.

THEOREM 4.2. *If f has a lifting to E* rel h, $\Gamma(f; h) = 0$.

Proof. Let $g: K \to E$ be such a lifting. Let $H_u: K'' \to S_B E$, for $0 \leqq u \leqq 1$, be the rel L'' lifting of $f \circ p$ where $H_u(x, t) = [gx, tu]$ for all $0 \leqq t, u \leqq 1$. Then $H_1 = H$, while $H_0 = s' \circ f \circ p$, and we are done.

THEOREM 4.3. *If e is $(n-1)$-connected for some $n \geqq 1$, and if*

$\dim(K/L) \leqq 2n-1$, *then* f *has a lifting to* E rel h *if and only if* $\Gamma(f; h) = 0$.

Proof. "Only if" is the previous theorem. Suppose then that $\Gamma(f; h) = 0$. Without loss of generality, we may assume that L has empty interior, whence $\dim K'' \leqq 2n-1$. By a Serre spectral sequence argument, $(\Omega^{i-1}\varepsilon_i)$: $\Omega^{i-1}e_i \to \Omega^i e_{i+1}$ is $(2n+i-1)$-connected for all $i \geqq 1$, whence, by Lemma 2.2, the representation

$$[K'', L''; e_1]_{f\circ p} \longrightarrow H^1(K'', L'', f\circ p; \mathscr{E})$$

is one-to-one and onto. Thus $[H] = [s'\circ f\circ p]$. Let H_t: $K'' \to S_B E$ be a fiber-homotopy rel L'' such that $H_1 = H$ and $H_0 = s'\circ f\circ p$; define G: $K'' \to P_B S_B E$ to be the map where $(Gy)u = H_u y$ for all $y \in K''$. Let i: $e \to PSe$ be adjoint to the identity on $Se = e_1$. Again, by a Serre spectral sequence argument, i is $(2n-2)$-connected. Since $[K'', L'', i\circ h\colon PSe]_{f\circ p}$ is nonempty, $[K, L, h; e]_f$ is nonempty by Lemmas 2.1 and 2.2, and we are done.

Suppose now that f_0, g_1: $K \to E$ are liftings of f rel h. We define $\Delta(g_0, g_1; h) \in H^0(K, L, f; \mathscr{E})$, the single obstruction to fiber homotopy, rel L, of g_0 with g_1, to be $(p^*)^{-1}$ of the representation in $H^0(K'', L'', f\circ p; \mathscr{E})$ of $[G] \in [K'', L''; \Omega Se]_{f\circ p}$, where for all (x, t) K'' and all $0 \leqq u \leqq 1$:

$$G(x,t)u = \begin{cases} [g_1x, 2u] \text{ if } t = 0 \text{ and } 0 \leqq u \leqq 1/2 \\ [g_0x, 2-2u] \text{ if } t = 0 \text{ and } 1/2 \leqq u \leqq 1 \\ [hx, 2u(1-t)] \text{ if } x \in L \text{ and } 0 \leqq u \leqq 1/2 \\ [hx, (2-2u)(1-)] \text{ if } x \in L \text{ and } 1/2 \leqq u \leqq 1\,. \end{cases}$$

We leave it to the reader to check that $\Delta(g_0, g_1; h)$ is a homotopy invariant in the same sense that $\Gamma(f; h)$ is.

Hence forth, we shall write $\Omega Se = (\Omega_B S_B E, \boldsymbol{c}, c')$.

THEOREM 4.4. *If* g_0 *and* g_1 *are fiber-homotopic* rel h, *then* $\Delta(g_0, g_1; h) = 0$.

Proof. Let g_t be a fiber homotopy rel L. Let G_v: $K'' \to \Omega_B S_B E$, $0 \leqq v \leqq 1$, be the rel L'' fiber homotopy, where for all $0 \leqq u, v \leqq 1$:

$$G_v(x,t)u = \begin{cases} [g_{2v-1}x, 2u] \text{ if } t = 1,\ 0 \leqq u \leqq 1/2, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [g_0x, 2-2u] \text{ if } t = 1,\ 1/2 \leqq u \leqq 1, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [hx, 2u(1-t)] \text{ if } x \in L,\ 0 \leqq u \leqq 1/2, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [hx, (2-2u)(1-t)] \text{ if } x \in L,\ 1/2 \leqq u \leqq 1, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [g_0x, 4uv(1-t)] \text{ if } 0 \leqq u \leqq 1/2 \text{ and } 0 \leqq v \leqq 1/2\,. \\ [g_0x, 4(1-u)v(1-t)] \text{ if } 1/2 \leqq u \leqq 1 \text{ and } 0 \leqq v \leqq 1/2\,. \end{cases}$$

Note that $G_1 = G$ and $G_0 = c' \circ f \circ p$, and we are done.

THEOREM 4.5. *If e is $(n-1)$-connected for some $n \geqq 1$, and if* $\dim(K/L) \leqq 2n-2$, *then g_0 and g_1 are fiber homotopic if and only if* $\Delta(g_0, g_1; h) = 0$.

Proof. "Only if" is the previous theorem. Suppose, then, that $\Delta(g_0, g_1; h) = 0$. Then G is fiber homotopic, rel L'', to c', since by Lemma 2.2, $[K'', L''; \Omega Se]_{f \circ p} \to H^0(K'', L'', f \circ p; \mathscr{E})$ is onto. A routine argument using Lemma 2.1 then shows that $i \circ g_0$ is fiber homotopic, rel $i \circ h$, to $i \circ g_1$, where $i: e \to PSe$ is adjoint to the identity on Se. Our result follows immediately from Lemma 2.3.

THEOREM 4.6. *If g is any lifting of f* rel h, *and if* $d \in H^0(K, L, f; \mathscr{E})$, *then there exists some lifting g' of f* rel h, *such that $\Delta(g, g'; h) = d$, provided e is $(n-1)$-connected for some $n \geqq 1$ and* $\dim(K/L) \leqq 2n - 1$.

Proof. The representation $[K, L; \Omega Se]_f \to H^0(K, L, f; \mathscr{E})$ is onto by Lemma 2.2; pick a lifting, H, of f rel $c^0 \circ f \mid L$ which represents d. Let s be the lifting of f to $P_B S_B E$:

$$(sx)t = \begin{cases} (Hx)(2t) & \text{if } 0 \leqq t \leqq 1/2 \\ ((i \circ g)x)(2t-1) & \text{if } 1/2 \leqq t \leqq 1 \end{cases}$$

where $i: e \to PSe$ is adjoint to the identity map of Se. Now by the *PCHEP* of PSe, s is fiber homotopic to a lifting s' where $s \mid L' = i \circ h$. Now $i_\sharp: [K, L, h; e]_f \to [K, L, i \circ h; PSe]_f$ is onto by Lemma 2.2. Choose g' to be any rel h lifting of f such that $i_\sharp[g'] = [s']$. We leave it to the reader to verify that $\Delta(g, g'; h) = d$.

The proof of the next theorem we omit; it is a routine homotopy argument of the type the reader should by now be familiar with.

THEOREM 4.7. *If g_0, g_1, and g_2 are liftings of f* rel h, *then*

$$\Delta(g_0, g_2; h) = \Delta(g_0, g_1; h) + \Delta(g_1, g_2; h) .$$

COROLLARY 4.8. (Becker) *If e is $(n-1)$-connected for some $n \geqq 1$, and if $\dim(K/L) \leqq 2n - 2$, then $[K, L, h; e]_f$ has the structure of an affine group, and, if nonempty, is isomorphic to $H^0(K, L, f; \mathscr{E})$.*

Proof. See Becker [1] for the definition of an affine group. Pick any $[g_0] \in [K, L, h; e]_f$. Let $\iota: [K, L, h; e]_f \to H^0(K, L, f; \mathscr{E})$ be given by $\iota[g] = \Delta(g_0, g; h)$. This function is well-defined, one-to-one, and onto, and induces an affine group structure on $[K, L, h; e]_f$ which is independent of the choice of g_0, by Theorems 4.4, 4.5, 4.6, and 4.7. We leave the details to the reader.

5. B-spectrum maps and a spectral sequence for $H^*(K, L, f; \mathscr{E})$. Let $\mathscr{E} = (\{e_i\}_{i \geqslant m}, \{\varepsilon_i\})$ and $\mathscr{A} = (\{a_i\}_{i \geqslant n}, \{\alpha_i\})$ be B-spectra. We define a *B-spectrum map* $\mathscr{f}\colon \mathscr{E} \to \mathscr{A}$ *of degree* d to be an indexed collection $\{f_i\}_{i \geqslant p}$ of pointed B-bundle maps, where $p \geqslant \max(m, n-d)$, such that for any $i \geqslant p$, $f_i\colon e_i \to a_{i+d}$ and the following diagram is commutative:

$$\begin{array}{ccc} e_i & \xrightarrow{\quad \varepsilon_i \quad} & e_{i+1} \\ {\scriptstyle f_i}\Big\downarrow & & \Big\downarrow{\scriptstyle f_{i+1}} \\ a_{i+d} & \xrightarrow{\quad \alpha_{i+d} \quad} & a_{i+d+1} \end{array} .$$

We can define $\mathscr{f}_*\colon H^k(K, L, f; \mathscr{E}) \to H^{k+d}(K, L, f; \mathscr{A})$ for any integer k to be the direct limit of the $(f_i)_*$; similarly we can define

$$\mathscr{f}_*\colon \pi_k(\mathscr{E}) \longrightarrow \pi_{k-d}(\mathscr{A})$$

for any integer k.

Let $\mathscr{D} = (\{d_i\}_{i \geqslant p}, \{\delta_i\})$ be the *fiber* of $\mathscr{f}$, defined as follows. For any $i \geqslant p$, $d_i = (D_i, \boldsymbol{d}_i, d'_i)$ where

$$\begin{aligned} D_i = \{(x, \sigma) \in E_i \times A^I_{i+d}\colon \sigma(0) &= (a'_{i+d} \circ e_i)x, \sigma(1) \\ &= f_i x, \ \& \ a_{i+d}(\sigma t) = e_i x \text{ for all } t \in I\} , \end{aligned}$$

$d_i(x, \sigma) = e_i x$ for all $(x, \sigma) \in D_i$ and $d'_i b = (e'_i b, \langle b \rangle)$ for all $b \in B$, where $\langle b \rangle t = a'_{i+d} b$ for all $t \in I$. Let $\delta_i\colon d^i \to \Omega d_{i+1}$ be defined as follows: For any $(x, \sigma) \in D_i$ and any $t \in I$, $(\delta_i(x, \sigma))t = ((\varepsilon_i x)t, \tau)$, where $\tau u = (\alpha_{i+d}(\sigma u))t$ for all $u \in I$. Consider the sequence of B-spectra and B-spectrum maps (called the fibration sequence of $\mathscr{f}$):

$$\mathscr{A} \xrightarrow{\ \mathscr{h}\ } \mathscr{D} \xrightarrow{\ \mathscr{g}\ } \mathscr{E} \xrightarrow{\ \mathscr{f}\ } \mathscr{A} \tag{5-1}$$

where $\mathscr{g} = \{g_i\}_{i \geqq p}$ has degree 0 and $\mathscr{h} = \{h_i\}_{i \geqq p+d-1}$ has degree $-d+1$; defined as follows: For any $(x, \sigma) \in D_i$, $h_i(x, \sigma) = x$; and for any $y \in A_i$, $g_i y = ((e'_{i-d+1} \circ a_i)y, \alpha_i y)$. The sequence (5-1) is analogous to the fibration sequence for any map of pointed spaces (where F is the fiber of f):

$$Y \longrightarrow F \longrightarrow X \xrightarrow{\ f\ } Y .$$

As in that case, we may, in a straightforward manner, verify the exactness of the long sequences:

$$\cdots \longrightarrow \pi_{k-d+1}(\mathscr{A}) \xrightarrow{\mathscr{h}_\sharp} \pi_k(\mathscr{D}) \xrightarrow{\mathscr{g}_\sharp} \pi_k(\mathscr{E}) \xrightarrow{\mathscr{f}_\sharp} \pi_{k-d}(\mathscr{A}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{k+d-1}(K, L, f; \mathscr{A}) \xrightarrow{\mathscr{h}_\sharp} H^k(K, L, f; \mathscr{D}) \xrightarrow{\mathscr{g}_\sharp} H^k(K, L, f; \mathscr{E})$$
$$\xrightarrow{\mathscr{f}_\sharp} H^{k+d}(K, L, f; \mathscr{A}) \longrightarrow \cdots .$$

We say that $\mathscr{f}\colon \mathscr{E} \to \mathscr{A}$ is k-connected if $\mathscr{D}$ is k-connected, and we say that $\mathscr{f}$ is k-coconnected if $\mathscr{D}$ is k-coconnected, i.e., $\pi_r(\mathscr{D}) = 0$ for all $r \geqq k$.

Henceforth in this section, let $\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\})$ be a B-spectrum. We define a *resolution* of $\mathscr{E}$ to be a commutative diagram of B-spectra, where each map has degree 0:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathscr{E}_{k+1} & \xrightarrow{\mathscr{g}_{k+1}} & \mathscr{E}_k & \xrightarrow{\mathscr{g}_k} & \mathscr{E}_{k-1} & \longrightarrow \cdots \\ & \nwarrow{\scriptstyle \mathscr{h}_{k+1}} & & \uparrow{\scriptstyle \mathscr{h}_k} & & \nearrow{\scriptstyle \mathscr{h}_{k-1}} & \\ & & & \mathscr{E} & & & \end{array}$$

such that for any integer r, there exists an integer N such that $\mathscr{h}_k$ is r-connected for all $k \geqq N$, and an integer M such that $\mathscr{E}_k$ is r-coconnected for all $k \leqq M$. We are thus assured that $H^*(K, L, f\colon \mathscr{E})$ is isomorphic to the inverse limit $\mathrm{Lim}_{k\to\infty}\, H(K, L, f; \mathscr{E}_k)$ under the homomorphisms $(\mathscr{g}_k)_\sharp$. An important special case of a resolution of $\mathscr{E}$ is a Postnikov resolution: that is where $(\mathscr{h}_k)_\sharp\colon \pi_r(\mathscr{E}) \to \pi_r(\mathscr{E}_k)$ is an isomorphism for all $r \leqq k$, and where each $\mathscr{E}_k$ is $(k+1)$-coconnected. In § 6, we shall show that every B-spectrum has a Postnikov resolution.

Using a resolution of $\mathscr{E}$, (5-2), we construct a spectral sequence for $H^*(K, L, f; \mathscr{E})$. For any integer r, we have a filtration of $H^r(K, L, f; \mathscr{E})$:

$$0 \subset \cdots \subset G^{r+q,q} \subset G^{r+q-1,q-1} \subset \cdots H^r(K, L, f; \mathscr{E})$$

where $G^{p,q}$ is the kernel of

$$(\mathscr{h}_q)_\sharp\colon H^{p-q}(K, L, f; \mathscr{E}) \longrightarrow H^{p-q}(K, L, f\colon \mathscr{E}_q) .$$

(The conditions that $\mathscr{h}_k$ is highly connected for large k and $\mathscr{E}_k$ is highly coconnected for small k insures that the filtration has only finitely many distinct terms.) For any k, consider the fibration sequence of $\mathscr{g}_k$:

$$\mathscr{E}_{k-1} \xrightarrow{\mathscr{z}_k} \mathscr{K}_k \xrightarrow{\mathscr{d}_k} \mathscr{E}_k \xrightarrow{\mathscr{p}_k} \mathscr{E}_{k-1} .$$

Recall that $\mathscr{d}_k$ and $\mathscr{p}_k$ have degree 0, and $\mathscr{z}_k$ has degree 1. For any integers p and q, define $E_2^{p,q} = H^{p-q}(K, L, f; \mathscr{K}_q)$ and

$$D_2^{p,q} = H^{p-q}(K, L, f; \mathscr{E}_q) .$$

Let $(\mathscr{p}_q)_\# = i_2\colon D_2^{p,q} \to D_2^{p-1,q-1}$, $(\mathscr{z}_{q+1})_\# = j_2\colon D_2^{p,q} \to E_2^{p+2,q+1}$, and

$$(\mathscr{d}_q)_\# = k_2\colon E_2^{p,q} = \longrightarrow D_2^{p,q} .$$

Using general spectral sequence arguments, we can verify that

$$d_r\colon E_2^{p,q} \longrightarrow E_2^{p+r,q+r-1} \qquad \text{for all } r \geqq 2 ,$$

and that $E_\infty^{p,q} = G^{p-1,q-1}/G^{p,q}$ for all p and q.

In the special case that (5-2) is a Postnikov resolution, we can construct an E_1 term of the spectral sequence as follows. Let K^r be the r-skeleton of K, for any r: $K^r = \varnothing$ if $r < 0$. For any p and q, let $D_1^{p,q} = H^{p,q}(K^p \cup L, f; \mathscr{E})$ and $E_1^{p,q} = C^p(K, L, f^{-1}\pi_q(\mathscr{E}))$, the group of cochains with coefficients in the local system $f^{-1}\pi_q(\mathscr{E})$ over K. Let $i_1\colon D_1^{p,q} \to D_1^{p-1,q-1}$ and $k_1\colon E_1^{p,q} \to D_1^{p,q}$ be the homomorphisms induced by the appropriate inclusions, and let $j_1\colon D_1^{p,q} \to E_1^{p+1,q}$ be the connecting homomorphism of the pair $(K^{p+1} \cup L, K^p \cup L)$. The differential $d_1\colon C^p(K, L; f^{-1}\pi_q(\mathscr{E})) \to C^{p+1}(K, L; f^{-1}\pi_q(\mathscr{E}))$ is then the usual coboundary on cochains with local coefficients, hence

$$E_2^{p,q} = H^p(K, L; f^{-1}\pi_q(\mathscr{E})) .$$

We leave the rather routine verification that the above E_1, D_1, i_1, j_1, and k_1 yield the correct E_2, D_2, etc., to the reader. (Hint: If $\mathscr{E}$ is k-connected, $H^p(K, L, f; \mathscr{E}) = 0$ for all $p \geqq n - k$, where $n = \dim(K/L)$.)

We now explore the relation between the single obstruction and the classical obstructions. Let us suppose that $e = (E, e)$ is a k-connected B-bundle, for some $k \geqq 1$, and that diagram (5-2) is a Postnikov system for $\mathscr{E} = \mathscr{E}(e)$. For any integer r, let $\iota_r\colon \pi_r e \to \pi_r(\mathscr{E})$ be the composition

$$\pi_r e \longrightarrow \pi_r PSe \cong \pi_r \Omega Se \cong \pi_{r+1} e_1 \longrightarrow \pi_r(\mathscr{E}) ,$$

an isomorphism if $r \leqq 2k$. Now suppose that $f \mid K^m \cap L$ has a rel h lifting, g^m, for some integer m. Then

$$i^* \Gamma(f, h) = \Gamma(f \mid K^m \cup L; h) = 0$$

by Theorem 4.2. Consider the commutative diagram of groups and homomorphisms:

$$\begin{array}{ccccc}
H^1(K, L, f; \mathscr{K}_m) & \xrightarrow{(\mathscr{i}_m)_\sharp} & H^1(K, L, f; \mathscr{E}_m) & \xrightarrow{(\mathscr{p}_m)_\sharp} & H^1(K, L, f; \mathscr{E}) \\
\updownarrow = & & \downarrow (\mathscr{q}_m)_\sharp & \swarrow (\mathscr{p}_{m-1})_\sharp & \\
H^{k+1}(K, L, f^{-1}\pi_m(\mathscr{E})) & & H^1(K, L, f; \mathscr{E}_{m-1}) & & \\
\uparrow (\ell_m)_\sharp & & & & \\
H^{k+1}(K, L; f^{-1}\pi_m e)\ . & & & &
\end{array}$$

Since $\mathscr{E}_{m-1}$ is m-coconnected,

$$i^*: H^1(K, L, f; \mathscr{E}_{m-1}) \longrightarrow H^1(K^m \cup L, L, f; \mathscr{E}_{m-1})$$

is an isomorphism. Thus $(\mathscr{p}_{m-1})_\sharp\Gamma(f; h) = 0$. Since $\mathscr{K}_m$ is the fiber of $\mathscr{q}_m$, $(\mathscr{p}_m)_\sharp\Gamma(f; h) \in (\mathscr{i}_m)_\sharp H^1(K, L, \mathscr{K}_m)$. The classical obstruction to extending g^m over $K^{m+1} \cup L$, $\gamma(g^m) \in H^{k+1}(K, L; f^{-1}\pi_m e)$ up to some indeterminacy. It is a routine matter of checking definitions to verify that $(\mathscr{i}_m)_\sharp(\ell_m)_\sharp\gamma(g^m) = (\mathscr{p}_m)_\sharp\Gamma(f; h)$.

6. Construction of the Postnikov resolution of $\mathscr{E}$. For every integer, n, we define a functor K_n: $\mathscr{H}_B{}^* \to \mathscr{H}_B{}^*$ as follows. If $n < 0$, let K_n be the identity. Otherwise, if $e = (E, \boldsymbol{e}, e')$ is a pointed B-bundle, let B^{n+1} be a (topological) $(n+1)$-ball with boundary S^n and basepoint $* \in S^n$. Let $E_B^{S^n}$ be the space of all continuous maps h: $S^n \to E$ such that $h(*) \in e'(B)$ and $\boldsymbol{e} \circ h$ is constant. Let ε: $E_B^{S^n} \to E$ be the evaluation map, and let $(K_n)_B E = E \cup_\varepsilon (E_B^{S^n} \times B^{n+1})$. We define $K_n e$ to be the pointed B-bundle $((K_n)_B E, \boldsymbol{k}, k')$, where $k' = e'$, $\boldsymbol{k} \mid E = \boldsymbol{e}$, and $\boldsymbol{k}(h, b) = (\boldsymbol{e} \circ h)(*)$ for all $(h, b) \in (E_B^{S^n} \times B^{n+1})$. If α: $e \to a$ is any pointed B-bundle map, we define $K_n\alpha$: $K_n e \to K_n a$ in the obvious way: $K_n\alpha \mid = \alpha$, and $(K_n\alpha)(h, b) = (\alpha \circ h, b)$ for all $(h, b) \in E_B^{S^n} \times B^{n+1}$. A very simple homotopy argument shows:

REMARK 6.1. (i) For all $k < n$, $i_\sharp$: $\pi_k e \to \pi_k(K_k e)$ is an isomorphism, where i: $e \to K_n e$ is the inclusion. (ii) $\pi_n(K_n e) = 0$.

We define functors K_n^r: $\mathscr{H}_B{}^* \to \mathscr{H}_B{}^*$ for all integers $n \leqq r$, inductively, as follows: $K_n^n = K_n$, and $K_n^{r+1} = K_{r+1}K_n^r$ for all $n \leqq r$. It is very simple to see that the "union" $\bigcup_{r=n}^\infty K_n^r$ is also a functor, which we call K_n^∞: $\mathscr{H}_B{}^* \to \mathscr{H}_B{}^*$. We call K_n, K_n^r, and K_n^∞ *homotopy-killing functors*. The following remark is an immediate Corollary of 6.1:

REMARK 6.2. (i) $i_\#: \pi_k e \to \pi_k(K_n^\infty e)$ is an isomorphism for all $k < n$, where $i: e \to K_n e$ is the inclusion. (ii) $\pi_k(K_n e) = 0$ for all $k \geqq n$.

Thus K_n^∞ is the analogue of the $(n-1)^{\text{th}}$ stage in the Postnikov tower of a space. In order to pass to spectra, we must examine the relationship between the homotopy-killing functors and the looping functor. We define a pointed B-bundle map $T_n: K_n \Omega e \to \Omega K_{n+1} e$ for all integers n as follows: If $n \leqq -2$, T_n is the identity. If $n = -1$, $T_n = \Omega i: \Omega e \to \Omega K_0 e$, where $i: e \to K_0 e$ is the inclusion. Otherwise, let $T_n: \Omega_B E \cup_\varepsilon((\Omega_B E)^{S^n} \times B^{n+1}) \to \Omega_B(E \cup_\varepsilon(E_B^{S^{n+1}} \times B^{n+1}))$ be the identity on $\Omega_B E$, and for any $(h, b) \in (\Omega_B E)_B^{S^n} \times B^{n+1}$, and any $t \in I$, let $(T_n(h, b))t = (h, [b, t])$. Note: $B^{n+2} = \sum B^{n+1}$ and $(\Omega_B E)_B^{S^n} = E_B^{S^{n+1}}$. We leave it to the reader to verify that $(T_n)_\#: \pi_k(K_n \Omega e) \to \pi_k(\Omega K_{n+1} e)$ is an isomorphism for all $k \leqq n$.

Similarly, we define $T_n^r: K_n^r \Omega e \to K_{n+1}^{r+1} e$ inductively for all $n \leqq r$ as follows: $T_n^n = T_n$, and $T_n^{r+1} = T_{r+1} \circ (K_{r+1} T_n^r)$ for all $r \geqq n$. In an obvious way we can then define $T_n: K_n^\infty \Omega e \to \Omega K_{n+1}^\infty e$. We leave the proof of the following to the reader:

REMARK 6.3. The B-bundle map $T_n: K_n^\infty \Omega e \to \Omega K_{n+1}^\infty e$ is a weak homotopy equivalence.

We are now ready to define the Postnikov resolution of B-spectrum $\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\})$. For each integer n, let

$$\mathscr{E}_n = (\{K_{n+i+1}^\infty e_i\}_{i \geqq m}, \{T_{n+i+1}^\infty \circ (K_{n+i+1} \varepsilon_i)\}) .$$

Let $p_n: \mathscr{E} \to \mathscr{E}_n = \{p_i\}_{i \geqq m}$, where $p_i: e_i \to K_{n+i+1} e_i$ is the inclusion, and let $q_n: \mathscr{E}_n \to \mathscr{E}_{n-1} = \{q_{n,i}\}_{i \geqq m}$, where $q_{n,i} = K_{n+i+1}^\infty j: K_{n+i+1}^\infty e_i \to K_{n-i+1}^\infty e_i$, where $j: e_i \to K_{n+i} e_i$ is the inclusion. The resolution of $\mathscr{E}$ described above (see diagram (5-2)) is a Postnikov resolution, by Remarks 6.2 and 6.3.

I wish to thank the referee for many helpful suggestions.

REFERENCES

1. J. C. Becker, *Cohomology and the classification of liftings*, Amer. Math. Soc. Trans., **133** (1968), 447-475.
2. ———, *Orientability and extension of cohomology theories*, Conference on Algebraic Topology, University of Illinois, Chicago Circle, (1968), 1-8.
3. S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, 1952.
4. S. T. Hu, *Homotopy Theory*, Academic Press, 1959.
5. J.-P. Meyer, *Relative Stable Homotopy*, Conference on Algebraic Topology, University of Illinois at Chicago Circle, 1968, 206-212.
6. N. E. Steenrod, *Topology of Fiber Bundles*, Princeton University Press, 1951.

7. ———, *A convenient category of topological spaces*, Mich. Math. J., **14** (1967), 133–152.

Received June 23, 1970 and in revised form April 8, 1971.

CALIFORNIA STATE COLLEGE, DOMINGUEZ HILLS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 41, No. 3 BadMonth, 1972