TWISTED COHOMOLOGY THEORIES AND THE SINGLE OBSTRUCTION TO LIFTING

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Consider any fibration \( p: E \to B \), any finite C.W. — pair \((K, L)\), and any maps \( f: K \to B \) and \( h: L \to E \) such that \( p \circ h = f \mid L \). A map \( g: K \to E \) such that \( p \circ g = f \) and \( g \mid L = h \) we call a lifting of \( f \) rel \( h \).

In this paper single obstruction \( \Gamma(f) \in H^*(K, L; \mathcal{E}) \) is defined. \( \mathcal{E} \) is a so-called \( B \)-spectrum, and \( H^*(\cdot; \mathcal{E}) \) is cohomology in that spectrum. If a lifting of \( f \) rel \( h \) exists, \( \Gamma(f) = 0 \); this condition is also sufficient if the fiber of \( p \) is \( k \)-connected and \( \dim(K/L) \leq 2k + 1 \).

If \( g_0 \) and \( g_1 \) are liftings of \( f \) rel \( h \), a single obstruction \( \delta(g_0, g_1; h) \in H(K, L; f; \mathcal{E}) \) is also defined; if \( g_0 \) and \( g_1 \) are connected by a homotopy of liftings of \( f \) rel \( h \) \( \delta(g_0, g_1; h) = 0 \); this condition is also sufficient if \( p \) is \( k \)-connected and \( \dim(K/L) \leq 2k \).

In § 4, a spectral sequence is constructed for cohomology in a \( B \)-spectrum, based on the Postnikov tower of that spectrum, and the relationship between the single obstruction and the classical obstructions is defined.

For similar treatments, see Becker [1], [2], and Meyer [5].

Throughout this paper, let \((K, L)\) be a finite C. W. pair, \( B \) any space, and \( f: K \to B \) any map. All spaces and maps shall be in the category \( CG \) of compactly generated spaces and maps, as described by Steenrod [7], and all constructions (i.e., function spaces, quotient space, Cartesian products) shall be as defined in that paper. When possible without confusion, we shall allow \( f \mid L \) and \( f \mid K \cup L \) to be denoted simply as \( F \). A map \( \pi: X \to Y \) we call a fibration if it has a local product structure; the polyhedral covering homotopy extension property [4] is then satisfied.

2. Basic concepts. We define a \( B \)-bundle to be an ordered pair \((E, e)\) such that \( e: E \to B \) is a fibration. A \( B \)-bundle map from a \( B \)-bundle \( e = (E, e) \) to another \( B \)-bundle \( a = (A, a) \) is defined to be a commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & A \\
\downarrow{e} & & \downarrow{a} \\
B & \xleftarrow{} & \phantom{A}
\end{array}
\]
We denote this map $\alpha: e \to a$. A pointed $B$-bundle is an ordered triple $(E, e, e')$ such that $e: E \to B$ is a fibration and $e': B \to E$ is a pointing, i.e., $e \circ e' = 1$, the identity on $B$. We call $e'$ a pointing because it chooses a base-point for each fiber of $e$. A bi-pointed $B$-bundle is an ordered quadruple $(E, e, e', e'')$ such that $(E, e)$ is a $B$-bundle and $e'$ and $e''$ are both pointings. If $e = (E, e, e')$ and $a = (A, a, a')$ are pointed $B$-bundles, a $B$-bundle map $\alpha: e \to a$ is a pointed map if $\alpha \circ e' = \alpha'$. Similarly, we can define bi-pointed maps between bi-pointed bundles. Two bundle maps (or pointed bundle maps, or bi-pointed bundle maps) are said to be homotopic if there exists a homotopy of bundle maps (or pointed bundle maps, or bi-pointed bundle maps) connecting them.

If $e = (E, e)$ is a $B$-bundle, $e^{-1} b$ is called the fiber of $e$ over $b$, for any $b \in B$. If $e = (E, e, e')$ is a pointed $B$-bundle, each fiber, $(e^{-1} b, e'b)$ is a pointed space. If $e = (E, e, e', e'')$ is bi-pointed, we say that $e'b$ is the South pole of $e^{-1} b$, while $e''b$ is the North pole.

Let $\mathcal{B}_b$ be the category of $B$-bundles and $B$-bundle maps. Let $\mathcal{B}_b^*$ and $\mathcal{B}_b^{**}$ be the categories of pointed and bi-pointed $B$-bundles and maps, respectively. We obviously have forgetful functors $\alpha: \mathcal{B}_b^{**} \to \mathcal{B}_b^*$ and $\beta: \mathcal{B}_b^* \to \mathcal{B}_b$ where $\alpha(E, e, e', e'') = (E, e, e')$ and $\beta(E, e, e') = (E, e)$. We shall, whenever convenient, identify any object with its image under $\alpha$, $\beta$, or $\beta \circ \alpha$. We also define functors as follows:

- $S: \mathcal{B}_b \to \mathcal{B}_b^{**}$ two-point suspension
- $\Sigma: \mathcal{B}_b^* \to \mathcal{B}_b^*$ one-point suspension
- $\Omega: \mathcal{B}_b^* \to \mathcal{B}_b^{**}$ looping
- $P: \mathcal{B}_b^{**} \to \mathcal{B}_b$ paths from the South pole to the North pole

$S(E, e) = (S_B E, s, s', s'')$ where $S_B E$ is the quotient space of $E \times I$ obtained by identifying $(x, 0)$ with $(y, 0)$ and $(x, 1)$ with $(y, 1)$ for any $x, y \in e^{-1} b$ for any $b \in B$. For all $[x, t] \in S_B E$, $s[x, t] = ex$, while $s'b = [x, 0]$ and $s''b = [x, 1]$ for all $b \in B$, where $x$ is any element in the fiber of $e$ over $b$. $\Sigma(E, e, e) = (\Sigma_B E, s, s')$ where $\Sigma_B E$ is the quotient space of $E \times I$ obtained by identifying $(x, 0)$ with $((e' \circ e)x, t)$ $(x, 1)$ for any $x \in E$ and any $t \in I$. Then $s[x, t] = ex$ for all $[x, t] \in \Sigma_B E$ and $s'b = [e'b, 0]$ for any $b \in B$.

$\Omega(E, e, e') = (\Omega_B E, \sigma, \sigma')$ where $\Omega_B E$ is the space of all loops in $E$ based on $e'(B)$ which lie in a single fiber of $e$; $\sigma \alpha = (e \circ \alpha)(0)$ for all $\alpha \in \Omega_B E$, and $(\sigma'b)t = e'b$ for all $b \in B$, and all $t \in I$. $P(E, e, e', e'') = (P_B E, p)$ where $P_B E$ is the space of all paths from $e'(B)$ to $e''(B)$ which lie in a single fiber, and $p \alpha = (e \circ \alpha)(0)$ for all $\alpha \in P_B E$.

We give two adjoint constructions. First, let $e = (E, e, e')$ and $a = (A, a, a')$ be two pointed $B$-bundles. If $\alpha: e \to \Omega a$ and $\beta; \Sigma e \to a$ are pointed $B$-bundle maps, we say that $\alpha$ and $\beta$ are adjoints of each
other if, for any \( x \in E \) and any \( t \in I \), \( \beta[x, t] = (\alpha x) t \). Second, let \( e = (E, e) \) be a \( B \)-bundle and \( a = (A, a, a', a'') \) a bi-pointed \( B \)-bundles. We say that maps \( \alpha: e \to Pa \) and \( \beta: Se \to a \) (where \( \beta \) is bi-pointed) are adjoints of each other if \( \beta[x, t] = (\alpha x) t \) for all \( x \in E \) and all \( t \in I \).

Let \([K, L, h; e]_f\) denote the set of rel \( L \) fiber-homotopy classes of liftings of \( f \) to \( E \) rel \( h \), where \( e = (E, e) \) is a \( B \)-bundle and \( a = (A, a, a', a'') \) a bi-pointed \( B \)-bundle. We say that maps \( a: e \to Pa \) and \( \beta: Se \to a \) (where \( \beta \) is bi-pointed) are adjoints of each other if \( \beta[x, t] = (\alpha x) t \) for all \( x \in E \) and all \( t \in I \).

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Let \( e = (E, e) \) be a \( B \)-bundle. If each fiber of \( e \) is connected, we say that \( e \) is connected. Similarly, if each fiber of \( e \) is \( n \)-connected, or \( n \)-simple, for some integer \( n \geq 1 \), we say that \( e \) is \( n \)-connected, or \( n \)-simple. If \( e \) is \( n \)-simple, define \( \pi_n e \) to be the local system of Abelian groups over \( B \) such that, for every \( b \in B \), \( \pi_n e(b) = \pi_n(e^{-1} b) \). We call \( \pi_n e \) the \( n \)th homotopy group system of \( e \). Similarly, if \( e \) is pointed, we can define \( \pi_n e \) whether \( e \) is \( \pi \)-simple or not, since every fiber has a base-point. Note that \( e \) is \( n \)-connected if and only if \( e \) is connected and \( \pi_k e = 0 \) for all \( k < n \). If \( \alpha: e \to a \) is any \( B \)-bundle map, where \( e \) and \( a \) are both \( \pi \)-simple or both pointed (and \( \alpha \) is pointed) or \( e \) is pointed and \( a \) is \( \pi \)-simple, \( \alpha \) induces a homomorphism \( \alpha_\pi: \pi_n e \to \pi_n a \) in the obvious way.

Let \( \alpha: e \to a \) be any \( B \)-bundle map, where \( e = (E, e) \) and \( a = (A, a, a') \). We define the fiber of \( \alpha \) to be the \( B \)-bundle \( c = (C, c) \) where \( C \) is the space of all ordered pairs \((x, \sigma)\) such that \( x \in E \) and \( \sigma \) is a path in \( A \) such that \( \sigma(0) \in a'(B) \), \( \sigma(1) = \alpha x \), and \( (a \circ \sigma) t = \alpha x \) for all \( t \in I \); and where \( c(x, \sigma) = \alpha x \) for all \((x, \sigma) \in C \). If \( e = (E, e, e') \) is pointed, then \( e'b = (e'b, \sigma) \) gives a pointing of \( c \), where \( \sigma t = \alpha' b \) for all \( t \in I \). The reader will note that for any \( b \in B \), \( e^{-1} b \) is precisely the fiber of \( \alpha: e^{-1} b \to a^{-1} b \). The following sequence is thus exact, if \( \alpha: e \to a \) is pointed:

\[
\cdots \to \pi_n(\Omega e) \xrightarrow{(\Omega \alpha)_*} \pi_n(\Omega a) \xrightarrow{j_*} \pi_n e \xrightarrow{i_*} \pi_n e \xrightarrow{\alpha_*} \pi_n a
\]

where \( i(x, \sigma) = \sigma(1) \) for all \((x, \sigma) \in C \), and \( j(\tau) = (c'b, \tau) \) for all \( \tau \in \Omega_B A \), where \( b = (a, \tau)(1) \).
Now if \( \alpha: e \to a \) is a \( B \)-bundle map, we say that \( \alpha \) is \( n \)-connected for any \( n \geq 0 \) if, for all \( b \in B \) and \( y \in a^{-1}b \), the space
\[
\{(x, \sigma) \in e^{-1}b \times (a^{-1}b)' : \sigma(0) = y, \sigma(1) = \sigma x\}
\]
is \( n \)-connected. If \( a \) is a connected pointed \( B \)-bundle, \( \alpha \) is connected if and only if the fiber of \( \alpha \) is \( n \)-connected.

Suppose now that \( \alpha: e \to a \) is a \( B \)-bundle map. Consider
\[
\alpha_\#: [K, L, h; e]_f \to [K, L, \alpha \circ h; a]_f.
\]

**Lemma 2.2.** Suppose \( \alpha \) is \( n \)-connected for some \( n \geq 0 \). Then:
(i) \( \alpha_\# \) is onto if \( \dim(K/L) \leq n \). (ii) \( \alpha_\# \) is one-to-one if \( \dim(K/L) \leq n - 1 \).

**Proof.** The connectivity of \( \alpha \) equals the connectivity of the fiber of \( \alpha: E \to A \), considered as a map of spaces. Simple application of ordinary obstruction theory enables us to complete the proof in a routine manner; we omit the details.

Suppose now that \( g_0, g_1: K \to E \) are both liftings of \( f \) rel \( h \).

**Lemma 2.3.** If \( \alpha \) is \( n \)-connected for some \( n \geq 1 \), then \( g_0 \) and \( g_1 \) are homotopic rel \( h \) if and only if \( \alpha \circ g_0 \) and \( \alpha \circ g_1 \) are homotopic, rel \( L \); provided \( \dim(K/L) \leq n - 1 \).

**Proof.** We have a bi-pointed \( K \)-bundle map \( f^{-1} \alpha: f^{-1} e \to f^{-1} a \), where \( f^{-1} e = (f^{-1} E, f^{-1}e, f^{-1}g_0, f^{-1}g_1) \) and
\[
f^{-1}a = (f^{-1}A, f^{-1}a, f^{-1}(\alpha \circ g_0), f^{-1}(\alpha \circ g_1));
\]
and \( Pf^{-1}\alpha; Pf^{-1}e \to Pf^{-1}a \) is \( (n-1) \)-connected. A section of \( Pf^{-1}e \) is equivalent to a fiber homotopy, rel \( L \), of \( g_0 \) with \( g_1 \), while a section of \( Pf^{-1}a \) is equivalent to a fiber homotopy, rel \( L \), of \( \alpha \circ g_0 \) with \( \alpha \circ g_1 \). Apply Lemma 2.2, and we are done.

3. \( B \)-Spectra. Suppose \( e = (E, e, e') \) is a pointed \( B \)-bundle. We define an operation “+” on \([K, L; \Omega e]_f \) as follows: for any two liftings of \( f \) rel \( e' \mid L \), \( g \) and \( g' \), let \( g + g': K \to \Omega B E \) be the map where
\[
((g + g')x)t = (gx)(2t) \text{ if } 0 \leq t \leq 1/2, \quad g'(x)(2t - 1) \text{ if } 1/2 \leq t \leq 1, \text{ for all } x \in K.
\]
Then \( g + g' \) is also a lifting of \( f \) rel \( e' \mid L \). We define
\[
[g] + [g'] = [g + g']; \text{ it is trivial to verify that the operation is well-defined.}
\]

**Theorem 3.1.** \([K, L; \Omega e]_f \) is a group under the operation “+” with identity \([e']\).
**Proof.** Let $[g]^{-1} = [g^{-1}]$ for any lifting $g$ of $f$ rel $e' \mid L$, where $(g^{-1}x)t = (gx)(1-t)$ for all $x \in K$ and all $t \in I$; it is routine to check that the group axioms are satisfied.

**Theorem 3.2.** $[K, L; \Omega^e]_f$ is an Abelian group.

**Proof.** We omit the details; if $g$ and $g'$ are both liftings of $f$ rel $e' \mid L$, a fiber homotopy rel $L$ of $g + g'$ with $g' + g$ can easily be constructed in the same manner as the proof that $[X; \Omega^2Y]$ is Abelian for pointed spaces $X$ and $Y$, but the construction is done fiberwise over $B$.

**Definition 3.1.** A $B$-spectrum is an ordered pair

$$\mathcal{E} = ([e_i]_{i \geq m}, [\varepsilon_i]_{i \geq m})$$

for some integer $m$ such that:

(i) For each $i \geq m$, $e_i$ is a pointed $B$-bundle.

(ii) For each $i \geq m$, $\varepsilon_i : e_i \to e_{i+1}$ is a pointed $B$-bundle map.

Furthermore, we say that $\mathcal{E}$ is a $\Omega^n$-spectrum if $\varepsilon_i$ is a homotopy equivalence (in the category $\mathcal{E}^*_B$) for each $i$, and we say that $\varepsilon$ is a weak $\Omega^n$-spectrum if $\varepsilon_i$ is infinitely connected for all $i \geq m$. We say that $\varepsilon$ is stabilizing if, for each integer $n$, there exists an integer $N \geq m$ such that $\varepsilon_i$ is $(n+i)$-connected for all $i \geq N$. The $e_i$ are called the elements of the spectrum, the $\varepsilon_i$ are called the connection maps, and $m$ is called the starting value. If the first finitely many elements of a spectrum are altered, no change occurs in cohomology with coefficients in that spectrum; in that sense, the starting value is arbitrary. We define the homotopy of a spectrum $\pi_n(\mathcal{E})$ for any integer $n$, to be the direct limit $\text{Lim}_{i \to \infty} \pi_{n+i}e_i$, under the system of homomorphisms

$$(\varepsilon)_i : \pi_{n+i}e_i \to \pi_{n+i}\Omega e_{i+1} \cong \pi_{n+i+1}e_{i+1}$$

thus $\pi_n(\mathcal{E})$ is a local system of Abelian groups on $B$. Note that $\pi_n(\mathcal{E})$ need not be zero for negative values of $n$.

Henceforth, we shall assume that $\mathcal{E} = ([e_i]_{i \geq m}, [\varepsilon_i]_{i \geq m})$ is a $B$-spectrum.

**Definition 3.2.** For any integer $n$, let $H^n(K, L, f; \mathcal{E})$ be the direct limit of the system of groups $[[K, L; \Omega^{-n}e_i]]$ and homomorphisms $\{(\Omega^{i-n}\varepsilon_i)\}$. (If $L$ is empty, we write $H^n(K, f; \mathcal{E})$.) For any $i \geq \min(n, m)$, let

$$[K, L; \Omega^{-n}e_i]_f \to H^n(K, L, f; \mathcal{E})$$
be called the representation. If \( \mathcal{E} \) is stabilizing, the direct limit is achieved eventually, i.e., beyond some point, all representations are bijective; if \( \mathcal{E} \) is a weak \( \Omega_p \)-spectrum, the direct limit is achieved immediately, i.e., all representations are bijective. We call \( H^*(K, L, f; \mathcal{E}) \) the cohomology of the triple \((K, L, f)\) with coefficients in the spectrum \( \mathcal{E} \). If \((K', L')\) is another C.W. pair, and

\[
r: (K', L') \longrightarrow (K, L)
\]

is a map, an induced homomorphism

\[
r^*: H^*(K, L, f; \mathcal{E}) \longrightarrow H^*(K', L', f \circ r; \mathcal{E})
\]

can be defined in the obvious way.

Henceforth, let \((K'', L'')\) be the pair \((K \times \{1\} \cup L \times I, L \times \{0\})\), and let \(p: (K'', L'') \to (K, L)\) be projection onto the first factor. The reader can easily verify that \(p\) is a relative homotopy equivalence, and hence by the direct limit version of Lemma 2.1,

\[
p^*: H^*(K, L, f; \mathcal{E}) \longrightarrow H^*(K'', L'', f \circ p; \mathcal{E})
\]

is an isomorphism.

For any integer \(n\), we define a connecting homomorphism

\[
\delta: H^n(L, f; \mathcal{E}) \longrightarrow H^{n+1}(K, L, f; \mathcal{E})
\]

as follows. For any \(a \in H^n(L, f; \mathcal{E})\), pick \(i \geq m\) and \([g] \in [L; \Omega^{i-n}e_i]_f\) representing \(a\). Consider \(\Omega^{i-n}e_i = \Omega^{i-n-1}e_i\). Let \(p^*\delta a\) be the image, in the direct limit, of \([G] \in [K'', L''; \Omega^{i-n-1}e_i], G(x, t) = (gx)t\) for all \(x \in L\) and \(t \in I\), and where \(G(x, 1) = a'(fx)\) for all \(x \in K\), where \(a'\) is the pointing of \(\Omega^{i-n-1}e_i\); \(\delta a\) is well-defined since \(p^*\) is an isomorphisms.

The following remarks (analogous to some of the Eilenberg Steenrod axioms for a cohomology theory [3]) we state without proof:

**Remark 3.3.** The following long sequence is exact, where \(i\) and \(j\) are inclusions:

\[
\cdots \longrightarrow H^{n-1}(L, f; \mathcal{E}) \longrightarrow H^n(K, L, f; \mathcal{E}) \longrightarrow H^*(K, f; \mathcal{E}) \longrightarrow H^*(K, L; \mathcal{E}) \longrightarrow \cdots
\]

**Remark 3.5.** If \(r_t: (K', L') \to (K, L), 0 \leq t \leq 1\), is a homotopy of maps, where \((K', L')\) is another C.W. pair, such that \(f \circ r_t = f \circ r_0\) for all \(t\), then \(r^*_t = r^*_0\).

Suppose now that \(f_t: K \to B, 0 \leq t \leq 1\), is a homotopy such that \(f_0 = f\). Let \(F: K \times I \to B\) be the map where \(F(x, t) = f_t(x)\) for all
(x, t) ∈ K × I. Let \( i_0, i_1 : (K, L) \to (K × I, L × I) \) be the inclusions along 0 and 1, respectively. According to Lemma 2.1., \((i_j)_*\) is an isomorphism for \( j = 0 \) or 1. Let

\[
F_* = (i_1)_* \circ (i_0)_*^{-1} : H^*(K, L, f; \mathcal{E}) \to H^*(K, L, f; \mathcal{E}),
\]

clearly an isomorphism. Again without proof, we state:

**Remark 3.6.** \( F_* \) depends only on the homotopy class of \( F \), rel \( K × \{0, 1\} \).

**Remark 3.7.** If \( G \) is a homotopy of \( f_i \) with \( f_2 \), then

\[
G_* \circ F_* = (F + G)_* : H^*(K, L, f; \mathcal{E}) \to H^*(K, L, f_2; \mathcal{E})
\]

where \((F + G)(x, t) = F(x, 2t)\) if \( 0 ≤ t ≤ 1/2 \); \( G(x, 2t)\) if \( 1/2 ≤ t ≤ 1 \), for all \( x \in K \).

An immediate question one may ask is: if \( f_t = f \), is \( F_* \) the identity? The answer is generally no.

4. The associated spectrum and the single obstruction. Let \( e = (E, e) \) be a \( B \)-bundle and \( h : L \to E \) a lifting of \( f \) \rel \( L \). Let

\[
\mathcal{E} = \mathcal{E}(e) = ([e_i]_{i ≥ 1}, \{ε_i\}_{i ≥ 1})
\]

be the \( B \)-spectrum where \( e_i = \sum^{i-1} Se \) for all \( i ≥ 1 \), and \( ε_i : e_i \to Ωe_{i+1} \) is adjoint to the identity on \( e_{i+1} = \sum e_i \). We call \( \mathcal{E} \) the \( B \)-spectrum associated to \( e \). We shall write \( e_i = Se = (S_B E, s, s', s'') \).

Recall \((K'', L'') = (K × \{1\} \cup L \times I, L \cup \{0\})\). We define \( Γ(f; h) \in H^1(K, L, f; \mathcal{E}) \) (or simply \( Γ(f) \) when \( L \) is empty, or when \( h \) is understood), the single obstruction to lifting \( f \) \rel \( h \), to be \((p^*)^{-1}\) of the representation of \([H] \in [K'', L''; S_B]_{f, p}\), where \( H : K'' \to S_B E \) is the map such that \( H(x, t) = [hx, t] \) for all \((x, t) \in L \times I \), and \( H(x, 1) = (e'' \circ f)x \), the North pole of \( e'' f x \), for all \( x \in K \). We leave it to the reader to verify that if \( f_i : K \to B \), for \( 0 ≤ t ≤ 1 \), is a homotopy, and if \( h_t : L \to E \) is a homotopy such that \( e \circ h_t = f \) \rel \( L \) for all \( t \), and if \( F(x, t) = f_t x \) for all \((x, t) \in K \times I \), then \( F_1 Γ(f_1; h_0) = Γ(f_1; h_1) \); i.e., \( Γ(f; h) \) is a homotopy invariant.

**Theorem 4.2.** If \( f \) has a lifting to \( E \) \rel \( h \), \( Γ(f; h) = 0 \).

**Proof.** Let \( g : K \to E \) be such a lifting. Let \( H_u : K'' \to S_B E \), for \( 0 ≤ u ≤ 1 \), be the rel \( L'' \) lifting of \( f \circ p \) where \( H_u(x, t) = [gx, tu] \) for all \( 0 ≤ t, u ≤ 1 \). Then \( H_1 = H \), while \( H_0 = s' \circ f \circ p \), and we are done.

**Theorem 4.3.** If \( e \) is \((n-1)\)-connected for some \( n ≥ 1 \), and if
dim \((K/L) \leq 2n - 1\), then \(f\) has a lifting to \(E\) rel \(h\) if and only if \(\Gamma(f; h) = 0\).

Proof. “Only if” is the previous theorem. Suppose then that \(\Gamma(f; h) = 0\). Without loss of generality, we may assume that \(L\) has empty interior, whence \(\dim K'' \leq 2n - 1\). By a Serre spectral sequence argument, \((\Omega^{2-i} \epsilon_i): \Omega^{2-i} \epsilon_i \to \Omega^i \epsilon_{i+1}\) is \((2n+i-1)\)-connected for all \(i \geq 1\), whence, by Lemma 2.2, the representation

\[
[K'', L''; e_f]_{f, p} \longrightarrow H^i(K'', L'', f \circ p; \mathcal{E})
\]

is one-to-one and onto. Thus \([H] = [s' \circ f \circ p]\). Let \(H_i: K'' \to \Omega \epsilon S \epsilon E\) be a fiber-homotopy rel \(L''\) such that \(H_1 = H\) and \(H_0 = s' \circ f \circ p\); define \(G: K'' \to P \epsilon \epsilon S \epsilon E\) to be the map where \((G y)u = H_u y\) for all \(y \in K''\). Let \(i: \epsilon \to P \epsilon S \epsilon E\) be adjoint to the identity on \(S \epsilon = e_\epsilon\). Again, by a Serre spectral sequence argument, \(i\) is \((2n-2)\)-connected. Since \([K'', L'', i \circ h: PS \epsilon E]_{f, p}\) is nonempty, \([K, L, h; e]_f\) is nonempty by Lemmas 2.1 and 2.2, and we are done.

Suppose now that \(f_0, g_1: K \to E\) are liftings of \(f\) rel \(h\). We define \(\Delta(g_0, g_1; h) \in H^i(K, L, f; \mathcal{E})\), the single obstruction to fiber homotopy, rel \(L\), of \(g_0\) with \(g_1\), to be \((p^*)^{-1}\) of the representation in \(H^i(K'', L'', f \circ p; \mathcal{E})\) of \([G] \in [K'', L'']; \Omega \epsilon S \epsilon E]_{f, p}\), where for all \((x, t) K''\) and all \(0 \leq u \leq 1\):

\[
G(x, t)u = \begin{cases} 
  [g_0 x, 2u] & \text{if } t = 0 \text{ and } 0 \leq u \leq 1/2 \\
  [g_0 x, 2-2u] & \text{if } t = 0 \text{ and } 1/2 \leq u \leq 1 \\
  [h x, 2u(1-t)] & \text{if } x \in L \text{ and } 0 \leq u \leq 1/2 \\
  [h x, (2-2u)(1-t)] & \text{if } x \in L \text{ and } 1/2 \leq u \leq 1 .
\end{cases}
\]

We leave it to the reader to check that \(\Delta(g_0, g_1; h)\) is a homotopy invariant in the same sense that \(\Gamma(f; h)\) is.

Hence forth, we shall write \(\Omega S \epsilon = \Omega \epsilon S \epsilon E, c, c'\).

**Theorem 4.4.** If \(g_0\) and \(g_1\) are fiber-homotopic rel \(h\), then \(\Delta(g_0, g_1; h) = 0\).

Proof. Let \(g_t\) be a fiber homotopy rel \(L\). Let \(G_t: K'' \to \Omega \epsilon S \epsilon E, 0 \leq v \leq 1\), be the rel \(L''\) fiber homotopy, where for all \(0 \leq u, v \leq 1\):

\[
G_v(x, t)u = \begin{cases} 
  [g_{v-t} x, 2u] & \text{if } t = 1, 0 \leq u \leq 1/2, \text{ and } 1/2 \leq v \leq 1 . \\
  [g_0 x, 2-2u] & \text{if } t = 1, 1/2 \leq u \leq 1, \text{ and } 1/2 \leq v \leq 1 . \\
  [h x, 2u(1-t)] & \text{if } x \in L, 0 \leq u \leq 1/2, \text{ and } 1/2 \leq v \leq 1 . \\
  [h x, (2-2u)(1-t)] & \text{if } x \in L, 1/2 \leq u \leq 1, \text{ and } 1/2 \leq v \leq 1 . \\
  [g_0 x, 4uv(1-t)] & \text{if } 0 \leq u \leq 1/2 \text{ and } 0 \leq v \leq 1/2 . \\
  [g_0 x, 4(1-u)v(1-t)] & \text{if } 1/2 \leq u \leq 1 \text{ and } 0 \leq v \leq 1/2 .
\end{cases}
\]
Note that $G_i = G$ and $G_0 = c \circ f \circ p$, and we are done.

**Theorem 4.5.** If $e$ is $(n-1)$-connected for some $n \geq 1$, and if \( \dim (K/L) \leq 2n - 2 \), then $g_0$ and $g_1$ are fiber homotopic if and only if \( \Delta(g_0, g_1; h) = 0 \).

**Proof.** "Only if" is the previous theorem. Suppose, then, that \( \Delta(g_0, g_1; h) = 0 \). Then $G$ is fiber homotopic, rel $L''$, to $c'$, since by Lemma 2.2, \([K'', L''; \Omega Se]_{f \circ p} \to H^0(K'', L'', f \circ p; \mathscr{E})\) is onto. A routine argument using Lemma 2.1 then shows that $i \circ g_0$ is fiber homotopic, rel $i \circ h$, to $i \circ g_1$, where $i : e \to PSe$ is adjoint to the identity on $Se$. Our result follows immediately from Lemma 2.3.

**Theorem 4.6.** If $g$ is any lifting of $f$ rel $h$, and if \( d \in H^0(K, L, f; \mathscr{E}) \), then there exists some lifting $g'$ of $f$ rel $h$, such that \( \Delta(g, g'; h) = d \), provided $e$ is $(n-1)$-connected for some $n \geq 1$ and \( \dim (K/L) \leq 2n - 1 \).

**Proof.** The representation \([K, L; \Omega Se], f \to H^0(K, L, f; \mathscr{E})\) is onto by Lemma 2.2; pick a lifting, $H$, of $f$ rel $c \circ f \mid L$ which represents $d$. Let $s$ be the lifting of $f$ to $P_bS_bE$:

\[
(sx)t = \begin{cases} 
(Hx)(2t) & \text{if } 0 \leq t \leq 1/2 \\
((i \circ g)x)(2t-1) & \text{if } 1/2 \leq t \leq 1
\end{cases}
\]

where $i : e \to PSe$ is adjoint to the identity map of $Se$. Now by the PCHEP of $PSe$, $s$ is fiber homotopic to a lifting $s'$ where $s \mid L' = i \circ h$. Now $i_4 : [K, L, h; e]_f \to [K, L, i \circ h; PSe]_f$ is onto by Lemma 2.2. Choose $g'$ to be any rel $h$ lifting of $f$ such that $i_4[g'] = [s']$. We leave it to the reader to verify that $\Delta(g, g'; h) = d$.

The proof of the next theorem we omit; it is a routine homotopy argument of the type the reader should by now be familiar with.

**Theorem 4.7.** If $g_0$, $g_1$, and $g_2$ are liftings of $f$ rel $h$, then

\[
\Delta(g_0, g_1; h) = \Delta(g_0, g_1; h) + \Delta(g_1, g_2; h).
\]

**Corollary 4.8.** (Becker) If $e$ is $(n-1)$-connected for some $n \geq 1$, and if \( \dim(K/L) \leq 2n - 2 \), then $[K, L, h; e]_f$ has the structure of an affine group, and, if nonempty, is isomorphic to $H^0(K, L, f; \mathscr{E})$.

**Proof.** See Becker [1] for the definition of an affine group. Pick any $[g] \in [K, L, h; e]_f$. Let $\iota : [K, L, h; e]_f \to H^0(K, L, f; \mathscr{E})$ be given by $\iota[g] = \Delta(g_0, g; h)$. This function is well-defined, one-to-one, and onto, and induces an affine group structure on $[K, L, h; e]_f$ which is
independent of the choice of $g_0$, by Theorems 4.4, 4.5, 4.6, and 4.7. We leave the details to the reader.

5. $B$-spectrum maps and a spectral sequence for $H^*(K, L, f; \mathcal{E})$. Let $\mathcal{E} = \{e_i\}_{i \geq m}, \{e_i\}$ and $\mathcal{A} = \{a_i\}_{i \geq m}, \{a_i\}$ be $B$-spectra. We define a $B$-spectrum map $f: \mathcal{E} \rightarrow \mathcal{A}$ of degree $d$ to be an indexed collection $\{f_i\}_{i \geq p}$ of pointed $B$-bundle maps, where $p \geq \max(m, n-d)$, such that for any $i \geq p$, $f_i: e_i \rightarrow a_{i+d}$ and the following diagram is commutative:

We can define $f_k: H^k(K, L, f; \mathcal{E}) \rightarrow H^{k+d}(K, L, f; \mathcal{A})$ for any integer $k$ to be the direct limit of the $(f_i)_k$; similarly we can define

for any integer $k$.

Let $\mathcal{D} = \{d_i\}_{i \geq p}, \{d_i\}$ be the fiber of $f$, defined as follows. For any $i \geq p$, $d_i = (D_i, d_i, d_i')$ where

$$d_i(x, \sigma) = e_i x$$

for all $(x, \sigma) \in D_i$ and $d_i'b = (e_i'b, \langle b \rangle)$ for all $b \in B$, where $\langle b \rangle t = a_{i+d}b$ for all $t \in I$. Let $\delta_i: d_i' \rightarrow \Omega d_{i+1}$ be defined as follows: For any $(x, \sigma) \in D_i$ and any $t \in I$, $(\delta_i(x, \sigma))t = ((e_i x)t, \tau)$, where $\tau u = (a_{i+d}(\sigma u))t$ for all $u \in I$. Consider the sequence of $B$-spectra and $B$-spectrum maps (called the fibration sequence of $f$):

(5-1)

where $\mathcal{F} = \{g_i\}_{i \geq p}$ has degree 0 and $\mathcal{A} = \{h_i\}_{i \geq p + d - 1}$ has degree $-d+1$; defined as follows: For any $(x, \sigma) \in D_i$, $h_i(x, \sigma) = x$; and for any $y \in A_i$, $g_i y = ((e_{i-d+1} \circ a_i) y, a_i y)$. The sequence (5-1) is analogous to the fibration sequence for any map of pointed spaces (where $F$ is the fiber of $f$):

As in that case, we may, in a straightforward manner, verify the exactness of the long sequences:
We say that $f: \mathcal{E} \to \mathcal{A}$ is $k$-connected if $\mathcal{D}$ is $k$-connected, and we say that $f$ is $k$-coconnected if $\mathcal{D}$ is $k$-coconnected, i.e., $\pi_r(\mathcal{D}) = 0$ for all $r \geq k$.

Henceforth in this section, let $\mathcal{E} = ([e_i]_{i \geq m}, \{e_i\})$ be a $B$-spectrum. We define a resolution of $\mathcal{E}$ to be a commutative diagram of $B$-spectra, where each map has degree 0:

\[
\cdots \longrightarrow \mathcal{E}_{k+1} \xrightarrow{\varepsilon_{k+1}} \mathcal{E}_k \xrightarrow{\varepsilon_k} \mathcal{E}_{k-1} \longrightarrow \cdots
\]

such that for any integer $r$, there exists an integer $N$ such that $\varepsilon_{k}$ is $r$-connected for all $k \geq N$, and an integer $M$ such that $\mathcal{E}_k$ is $r$-coconnected for all $k \leq M$. We are thus assured that $H^*(K, L, f; \mathcal{E})$ is isomorphic to the inverse limit $\lim_{k \to \infty} H(K, L, f; \mathcal{E}_k)$ under the homomorphisms $(\varepsilon_k)_*$. An important special case of a resolution of $\mathcal{E}$ is a Postnikov resolution: that is where $(\varepsilon_k)_*: \pi_r(\mathcal{E} \to \pi_r(\mathcal{E}_k) \to \pi_r(\mathcal{E})$ is an isomorphism for all $r \leq k$, and where each $\mathcal{E}_k$ is $(k+1)$-coconnected.

In § 6, we shall show that every $B$-spectrum has a Postnikov resolution.

Using a resolution of $\mathcal{E}$, (5-2), we construct a spectral sequence for $H^*(K, L, f; \mathcal{E})$. For any integer $r$, we have a filtration of $H^r(K, L, f; \mathcal{E})$:

\[
0 \subset \cdots \subset G^{r+q, q} \subset G^{r+q-1, q-1} \subset \cdots H^r(K, L, f; \mathcal{E})
\]

where $G^{p,q}$ is the kernel of

\[
(\varepsilon_q)_*: H^{p+q}(K, L, f; \mathcal{E}) \longrightarrow H^{p+q}(K, L, f; \mathcal{E})
\]

(The conditions that $\varepsilon_k$ is highly connected for large $k$ and $\mathcal{E}_k$ is highly coconnected for small $k$ insures that the filtration has only finitely many distinct terms.) For any $k$, consider the fibration sequence of $\varepsilon_k$:
Recall that \( \gamma_k \) and \( ?^k \) have degree 0, and \( \gamma_k \) has degree 1. For any integers \( p \) and \( q \), define 
\[
E_{2}^{p,q} = H^{p-q}(K, L, f; \mathcal{E}_q)
\]
and 
\[
D_{2}^{p,q} = H^{p-q-1}(K, L, f; \mathcal{E}_q).
\]
Let \((\mathcal{E}_q)_{\delta} = i_{\delta} \colon D_{2}^{p,q} \to D_{2}^{p-1,q-1}\), \((\mathcal{E}_q)_{\gamma} = j_{\gamma} \colon D_{2}^{p,q} \to E_{2}^{p+2,q+1}\), and 
\[
(\mathcal{E}_q)_{\kappa} = k_{\kappa} \colon E_{2}^{p,q} \to D_{2}^{p,q}.
\]
Using general spectral sequence arguments, we can verify that
\[
E_{r} = H^{p-q}(K, L, f; \mathcal{E}_q) \quad \text{for all } r \geq 2,
\]
and that \( E_{\infty}^{p,q} = G^{p-q-1}/G^{p,q} \) for all \( p \) and \( q \).

In the special case that \((5-2)\) is a Postnikov resolution, we can construct an \( E_{1} \) term of the spectral sequence as follows. Let \( K^{r} \) be the \( r \)-skeleton of \( K \), for any \( r \): \( K^{r} = \emptyset \) if \( r < 0 \). For any \( p \) and \( q \), let \( D_{2}^{p,q} = H^{p,q}(K^{r} \cup L, f; \mathcal{E}) \) and \( E_{2}^{p,q} = C^{p}(K, L, f^{-1}\pi_{q}(\mathcal{E})) \), the group of cochains with coefficients in the local system \( f^{-1}\pi_{q}(\mathcal{E}) \) over \( K \). Let \( i_{\delta} \colon D_{2}^{p,q} \to D_{1}^{p-1,q-1}\) and \( k_{\kappa} \colon E_{2}^{p,q} \to D_{1}^{p,q} \) be the homomorphisms induced by the appropriate inclusions, and let \( j_{\gamma} \colon D_{2}^{p,q} \to E_{2}^{p+1,q+1} \) be the connecting homomorphism of the pair \((K^{p+1} \cup L, K^p \cup L)\). The differential \( d_{\gamma} \colon C^{p}(K, L, f^{-1}\pi_{q}(\mathcal{E})) \to C^{p+1}(K, L, f^{-1}\pi_{q}(\mathcal{E})) \) is then the usual coboundary on cochains with local coefficients, hence
\[
E_{2}^{p,q} = H^{p-q}(K, L, f^{-1}\pi_{q}(\mathcal{E})).
\]
We leave the rather routine verification that the above \( E_{1}, D_{1}, i_{\delta}, j_{\gamma}, \) and \( k_{\kappa} \) yield the correct \( E_{2}, D_{2}, \) etc., to the reader. (Hint: If \( \mathcal{E} \) is \( k \)-connected, \( H^{p}(K, L, f; \mathcal{E}) = 0 \) for all \( p \geq n-k \), where \( n = \dim(K/L) \)).

We now explore the relation between the single obstruction and the classical obstructions. Let us suppose that \( e = (E, e) \) is a \( k \)-connected \( B \)-bundle, for some \( k \geq 1 \), and that diagram \((5-2)\) is a Postnikov system for \( \mathcal{E} = \mathcal{E}(e) \). For any integer \( r \), let \( \tau_{r} \colon \pi_{*}e \to \tau_{*}(\mathcal{E}) \) be the composition
\[
\pi_{*}e \to \pi_{*}PSe \cong \pi_{*}\Omega Se \cong \pi_{r+1,e_{1}} \to \pi_{*}(\mathcal{E}),
\]
an isomorphism if \( r \leq 2k \). Now suppose that \( f \mid K^{m} \cap L \) has a rel \( h \) lifting, \( g^{m} \), for some integer \( m \). Then
\[
i^{*}\Gamma(f, h) = \Gamma(f \mid K^{m} \cup L; h) = 0
\]
by Theorem 4.2. Consider the commutative diagram of groups and homomorphisms:
Since \( \mathcal{E}_{m-1} \) is \( m \)-coconnected,

\[
\iota^\#: H^i(K, L; f; \mathcal{E}_{m-1}) \longrightarrow H^i(K^m \cup L, L; f; \mathcal{E}_{m-1})
\]
is an isomorphism. Thus \( (\iota_m)_* \Gamma(f; h) = 0 \). Since \( \mathcal{X}_m \) is the fiber of \( \iota_m \), \( (\iota_m)_* \Gamma(f; h) \in (\iota_m)_* H^i(K, L, \mathcal{X}_m) \). The classical obstruction to extending \( g_m \) over \( K^{m+1} \cup L \), \( \gamma(g_m) \in H^{k+1}(K, L; f^{-1} \pi_m e) \) up to some indeterminacy. It is a routine matter of checking definitions to verify that \( (\iota_m)_* (\iota_m)_* \alpha^* (g_m) = (\iota_m)_* \Gamma(f; h) \).

6. Construction of the Postnikov resolution of \( \mathcal{E} \). For every integer, \( n \), we define a functor \( K_n: \mathcal{X}^* \rightarrow \mathcal{X}^* \) as follows. If \( n < 0 \), let \( K_n \) be the identity. Otherwise, if \( e = (E, e, e') \) is a pointed \( B \)-bundle, let \( B^{n+1} \) be a (topological) \( (n+1) \)-ball with boundary \( S^n \) and basepoint \( * \in S^n \). Let \( E^n_B \) be the space of all continuous maps

\[
h: S^n \rightarrow E \text{ such that } h(*) \in e'(B) \text{ and } e \circ h \text{ is constant.}
\]

Let \( E^n_B \rightarrow E \) be the evaluation map, and let \( (K_n)_* E = E \cup (E^n_B \times B^{n+1}) \). We define \( K_n e \) to be the pointed \( B \)-bundle \( ((K_n)_* E, k, k') \), where \( k' = e' \), \( k|E = e \), and \( k(h, b) = (e \circ h)(*) \) for all \( (h, b) \in (E^n_B \times B^{n+1}) \). If \( \alpha: e \rightarrow a \) is any pointed \( B \)-bundle map, we define \( K_n \alpha: K_n e \rightarrow K_n a \) in the obvious way: \( K_n \alpha| = \alpha \), and \( (K_n \alpha)(h, b) = (\alpha \circ h)(b) \) for all \( (h, b) \in E^n_B \times B^{n+1} \). A very simple homotopy argument shows:

**Remark 6.1.**  (i) For all \( k < n \), \( i_k^*: \pi_k e \rightarrow \pi_k(K_n e) \) is an isomorphism, where \( i: e \rightarrow K_n e \) is the inclusion.  (ii) \( \pi_n(K_n e) = 0 \).

We define functors \( K_i: \mathcal{X}^* \rightarrow \mathcal{X}^* \) for all integers \( n \leq r \), inductively, as follows: \( K_n^n = K_n \), and \( K_n^{r+1} = K_{r+1} K_n^r \) for all \( n \leq r \). It is very simple to see that the "union" \( \bigcup_{r=n}^\infty K_r^n \) is also a functor, which we call \( K_n^\infty: \mathcal{X}^* \rightarrow \mathcal{X}^\infty \). We call \( K_n, K_r^n, \) and \( K_n^\infty \) homotopy-killing functors. The following remark is an immediate Corollary of 6.1:
REMARK 6.2. (i) \( \pi_k \rightarrow \pi_k(K\Gamma e) \) is an isomorphism for all \( k < n \), where \( i: e \rightarrow K\gamma e \) is the inclusion. (ii) \( \pi_k(K\gamma e) = 0 \) for all \( k \geq n \).

Thus \( K_\alpha \) is the analogue of the \((n-1)\)th stage in the Postnikov tower of a space. In order to pass to spectra, we must examine the relationship between the homotopy-killing functors and the looping functor. We define a pointed \( B \)-bundle map \( T_n: K\gamma e \rightarrow \Omega K_{n+1} e \) for all integers \( n \) as follows: If \( n \leq -2 \), \( T_n \) is the identity. If \( n = -1 \), \( T_n = \Omega i: \Omega e \rightarrow \Omega K_0 e \), where \( i: e \rightarrow K_0 e \) is the inclusion. Otherwise, let \( T_n: \Omega_{B} E \cup_i ((\Omega_{B} E)^{S^n} \times B^{n+1}) \rightarrow \Omega_{B} (E \cup_i (S_B^{n+1} \times B^{n+1})) \) be the identity on \( \Omega_{B} E \), and for any \((h, b) \in (\Omega_{B} E)^{S^n} \times B^{n+1}\), and any \( t \in I \), let \((T_n(h, b))t = (h, [b, t])\). Note: \( B^{n+2} = \sum B^{n+1} \) and \( (\Omega_{B} E)^{S^n} = B_{B}^{S^n} \). We leave it to the reader to verify that \( (T_n)_*: \pi_k(K\gamma e) \rightarrow \pi_k(\Omega K_{n+1} e) \) is an isomorphism for all \( k \leq n \).

Similarly, we define \( T_n^*: K^n_{\gamma} e \rightarrow K^{n+1}_{\gamma} e \) inductively for all \( n \leq r \) as follows: \( T^n_n = T_n \), and \( T^{n+1}_n = T_{r+1} \circ (K_{r+1} T^n_r) \) for all \( r \geq n \). In an obvious way we can then define \( T^n_n: K^n_{\gamma} e \rightarrow \Omega K_{n+1} e \). We leave the proof of the following to the reader:

REMARK 6.3. The \( B \)-bundle map \( T^n_n: K^n_{\gamma} e \rightarrow \Omega K_{n+1} e \) is a weak homotopy equivalence.

We are now ready to define the Postnikov resolution of \( B \)-spectrum \( E \equiv (\{e_i\}_{i \geq m}, \{\varepsilon_i\}) \). For each integer \( n \), let

\[
E_n = \{(K^{\infty}_{n+i+1} e_i)_{i \geq m}, \{T^{n+i+1}_i \circ (K^{\infty}_{n+i+1} e_i)\}\}.
\]

Let \( \gamma_n: E \rightarrow E_n = \{p_i\}_{i \geq m} \) where \( p_i: e_i \rightarrow K_{n+i+1} e_i \) is the inclusion, and let \( \gamma_n: E_n \rightarrow E_{n-1} = \{q_{n,i}\}_{i \geq m} \), where \( q_{n,i} = K^{\infty}_{n+i+1} j: K^{\infty}_{n+i+1} e_i \rightarrow K^{\infty}_{n-i+1} e_i \), where \( j: e_i \rightarrow K_{n+i} e_i \) is the inclusion. The resolution of \( E \) described above (see diagram (5-2)) is a Postnikov resolution, by Remarks 6.2 and 6.3.

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