Pacific Journal of Mathematics

ON BANACH SPACE VALUED EXTENSIONS FROM SPLIT FACES

TAGE BAI ANDERSEN

Vol. 42, No. 1 January 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

ON BANACH SPACE VALUED EXTENSIONS FROM SPLIT FACES

TAGE BAI ANDERSEN

The object of this note is the following theorem: Suppose a is a continuous affine map from a closed split face F of a compact convex set K with values in a Banach space B enjoying the approximation property. Suppose also that p is a strictly positive lower semi-continuous concave function on K such that $\|a(k)\| \leqq p(k)$ for all k in F. Then a admits a continuous affine extension $\tilde{a}$ to K into B such that $\|\tilde{a}(k)\| \leqq p(k)$ for all k in K.

We shall use the methods of tensor products of compact convex sets as developed by Semadeni [12], Lazar [9], Namioka and Phelps [10] and Behrends and Wittstock [6] to reduce the problem to the case $B = \boldsymbol{R}$, and in this case the result follows from the work of Alfsen and Hirsberg [3] and the present author [4].

We shall be concerned with compact convex sets K_1 and K_2 in locally convex spaces E_1 and E_2 respectively. By $A(K_i)$ we shall denote the continuous real affine functions on K_i for $i = 1, 2$. We let $BA(K_1 \times K_2)$ be the Banach space of continuous biaffine functions on $K_1 \times K_2$. We observe that $1 \in BA(K_1 \times K_2)$ and that $BA(K_1 \times K_2)$ separates points of $K_1 \times K_2$. As usual we define the projective tensor product of K_1 and K_2, $K_1 \otimes K_2$, to be the state space of $BA(K_1 \times K_2)$ equipped with the w^*-topology. Then $K_1 \otimes K_2$ is a compact convex set, and we have a homeomorphic embedding $\omega_{K_1 \times K_2}$ (called ω, when no confusion can arise) from $K_1 \times K_2$ into $K_1 \otimes K_2$ defined by the following rule: For all a in $BA(K_1 \times K_2)$ and all (x_1, x_2) in $K_1 \times K_2$

$$\omega(x_1, x_2)(a) = a(x_1, x_2) .$$

We notice that ω is a biaffine map. It was proved in [10; Prop. 1.3, Th. 2.3] and [6; Satz 1.1.3] that $\partial_e(K_1 \otimes K_2) = \omega(\partial_e K_1 \times \partial_e K_2)$, where in general we denote the extreme points of a convex set K by $\partial_e K$.

For a in $A(K_1)$ and b in $A(K_2)$ we define the continuous biaffine function $a \otimes b$ by

$$a \otimes b(x_1, x_2) = a(x_1)b(x_2), \text{ all } (x_1, x_2) \in K_1 \times K_2 .$$

We let $A(K_1) \otimes A(K_2)$ be the real vector space

$$A(K_1) \otimes A(K_2) = \{\sum_{i=1}^{n} a_i \otimes b_i \mid a_i \in A(K_1), b_i \in A(K_2)\}$$

which is a copy of the algebraic tensor product of $A(K_1)$ and $A(K_2)$. We denote by $A(K_1) \otimes_\varepsilon A(K_2)$ the uniform closure of $A(K_1) \otimes A(K_2)$ in $BA(K_1 \times K_2)$.

We recall that a Banach space B is said to have the approximation property if for each compact convex subset C of B and each $\varepsilon > 0$ there is a continuous linear map $T: B \to B$ such that $T(B)$ is finite dimensional and such that $\|Tx - x\| < \varepsilon$ for all $x \in C$. It is proved in [10; Lem. 2.5] that if $A(K_1)$ (or $A(K_2)$) has the approximation property then $BA(K_1 \times K_2) = A(K_1) \otimes_\varepsilon A(K_2)$.

Following Lazar [9] we define T_1 and T_2 as the natural embeddings of $A(K_1)$ and $A(K_2)$ into $BA(K_1 \times K_2)$, i.e.

$$T_1 a = a \otimes 1, \text{ all } a \in A(K_1)$$
$$T_2 b = 1 \otimes b, \text{ all } b \in A(K_2) .$$

Let P_i be the adjoint map of T_i for $i = 1, 2$. Then P_i is an affine and continuous map of $K_1 \otimes K_2$ onto K_i ($=$ state space of $A(K_i)$), and

$$P_i \omega(k_1, k_2) = k_i, \; i = 1, 2 .$$

The first part of the following proposition was proved by Lazar in the case where K_1 and K_2 are simplexes, but the proof holds in general. The last part was proved by Lazar in the simplex case by means of the Stone-Weierstrass Theorem for simplexes.

PROPOSITION 1. *Let F_1 and F_2 be closed faces of compact convex sets K_1 and K_2 resp. Let $F = P_1^{-1}(F_1) \cap P_2^{-1}(F_2)$*

(i) *Then F is a closed face in $K_1 \otimes K_2$ and $F = \overline{\text{co}}(\omega(F_1 \times F_2))$*

(ii) *If $A(F_1)$ or $A(F_2)$ has the approximation property then $F_1 \otimes F_2$ is affinely homeomorphic to F.*

Proof. Since P_i is continuous and affine it is immediate that $P_i^{-1}(F_i)$ is a closed face of $K_1 \otimes K_2$, and hence F is a closed face.

Now let $p = \omega(k_1, k_2) \in \omega(F_1 \times F_2)$. Then $P_i p = k_i \in F_i$, and hence $p \in P_1^{-1}(F_1) \cap P_2^{-1}(F_2) = F$. By the Krein Milman Theorem: $\overline{\text{co}}(\omega(F_1 \times F_2)) \subseteqq F$.

Conversely, let $p \in \partial_e F$. Since F is a closed face we get

$$p \in \partial_e F = F \cap \partial_e(K_1 \otimes K_2) = F \cap \omega(\partial_e K_1 \times \partial_e K_2) .$$

Hence $p = \omega(x_1, x_2)$, $x_i \in \partial_e K_i$. Then $P_i p = x_i$ belongs to F_i by the definition of F. Hence $p \in \omega(F_1 \times F_2)$, and again by the Krein Milman Theorem $F \subseteqq \overline{\text{co}}(\omega(F_1 \times F_2))$, and (i) is proved.

Now we shall prove (ii) under the assumption that $A(F_1)$ has the approximation property. We shall define a continuous affine map

$T: F_1 \otimes F_2 \to K_1 \otimes K_2$ by

$$(T\varphi)(b) = \varphi(b|_{F_1 \times F_2}),\ \varphi \in F_1 \otimes F_2,\ b \in BA(K_1 \times K_2)\ .$$

Then $T(F_1 \otimes F_2)$ is compact and convex in $K_1 \otimes K_2$. If $\varphi \in \partial_e(F_1 \otimes F_2)$ then $\varphi = \omega_{F_1 \times F_2}(x_1, x_2)$, where $x_i \in \partial_e F_i$, $i = 1, 2$. But then

$$(T\varphi)(b) = b(x_1, x_2) = \omega_{K_1 \times K_2}(x_1, x_2)(b),\ \text{all}\ b \in BA(K_1 \times K_2)\ .$$

Hence $T\varphi = \omega_{K_1 \times K_2}(x_1, x_2) \in \overline{\text{co}}(\omega_{K_1 \times K_2}(F_1 \times F_2)) = F$. By the Krein Milman Theorem we conclude that $T(F_1 \otimes F_2) \subseteq F$.

Conversely, if $\psi \in \partial_e F$ then as F is a closed face, we get by Milman's theorem

$$\psi \in \omega_{K_1 \times K_2}(F_1 \times F_2) \cap \omega_{K_1 \times K_2}(\partial_e K_1 \times \partial_e K_2) = \omega_{K_1 \times K_2}(\partial_e F_1 \times \partial_e F_2)\ .$$

If $\psi = \omega_{K_1 \times K_2}(x_1, x_2)$, $x_i \in \partial_e F_i$, then $\omega_{F_1 \times F_2}(x_1, x_2) \in \partial_e(F_1 \otimes F_2)$, and as above $\psi = T(\omega_{F_1 \times F_2}(x_1, x_2))$. By the Krein Milman Theorem we get $F \subseteq T(F_1 \otimes F_2)$, and so T is surjective.

We proceed to show that T is injective. This is the case if $BA(K_1 \times K_2)|_{F_1 \times F_2}$ is dense in $BA(F_1 \times F_2)$. We show that $A(K_1) \otimes A(K_2)|_{F_1 \times F_2}$ is dense in $BA(F_1 \times F_2)$. Hence let $c \in BA(F_1 \times F_2)$ and $\varepsilon > 0$. Since $A(F_1)$ has the approximation property, we have that $A(F_1) \otimes_\varepsilon A(F_2) = BA(F_1 \otimes F_2)$, so there exist $a_1, \cdots, a_n \in A(F_1)$, $b_1, \cdots, b_n \in A(F_2)$ such that

$$\left\| c - \sum_{i=1}^n a_i \otimes b_i \right\|_{F_1 \times F_2} < \frac{\varepsilon}{2}\ .$$

Now $A(K_i)|_{F_i}$ is dense in $A(F_i)$, so we can choose $a'_i \in A(K_1)$, $b'_i \in A(K_2)$, $i = 1, \cdots n$, such that

$$\left\| \sum_{i=1}^n a_i \otimes b_i - \sum_{i=1}^n a'_i \otimes b'_1 \right\|_{F_1 \times F_2} < \frac{\varepsilon}{2}\ .$$

Then $\|c - \sum_{i=1}^n a'_i \otimes b'_i\|_{F_1 \times F_2} < \varepsilon$, and the claim follows.

The next step is to prove that $\overline{\text{co}}(\omega(F_1 \times F_2))$ is a closed split face of $K_1 \otimes K_2$ provided F_i is a closed split face of K_i for $i = 1, 2$, and f.ex. $A(F_1)$ has the approximation property.

We shall remind the reader of the following definitions and facts: If F is a closed face of a compact convex K, then the complementary σ-face F' is the union of all faces disjoint from F. It is always true that $K = \text{co}(F \cup F')$. F is called a split face if F' is a face and each point in $K \backslash (F \cup F')$ can be decomposed uniquely as convex combination of a point in F and a point in F'. It follows from a slight modification of the proof of [2; Th. 3.5] that a closed face is a split face if and only if each nonnegative u.s.c. affine function of F admits an u.s.c.

affine extension to K, which is equal to 0 on F'. This characterization is sometimes inconvenient because of the "nonsymmetric" properties of the affine functions involved. Using the above characterization we shall give a new one involving the space $A_s(K)$ which is the smallest uniformly closed subspace of the bounded functions on K containing the bounded u.s.c. affine functions. This space has been used f.ex. by Krause [8] and Behrends and Wittstock [6] in simplex theory and by Combes [7] in C^*-algebra theory. We shall state some of the known properties of $A_s(K)$.

LEMMA 2.
(i) *If* $a \in A_s(K)$ *and* $a \geqq 0$ *on* $\partial_e K$ *then* $a \geqq 0$ *on* K.
(ii) *If* $a \in A_s(K)$ *then* $\|a\|_K = \|a\|_{\partial_e K}$.
(iii) *If* $a \in A_s(K)$ *then* a *satisfies the barycentric calculus.*

Sketch of proof. If s and t are u.s.c. affine functions on K and $s \leqq t$ on $\partial_e K$ it follows by [5; Lem. 1] that $s \leqq t$ on K. Hence (i) follows by a limit argument. Now (ii) follows by (i), since on $\partial_e K$: $-\|a\|_{\partial_e K} \leqq a \leqq \|a\|_{\partial_e K}$. Hence the same inequality holds on K, and so $\|a\|_K \leqq \|a\|_{\partial_e K}$. The converse inequality is trivial. Finally (iii) follows from Lebesgue's theorem on dominated convergence, since the barycentric calculus holds for (differences of) u.s.c bounded affine functions, cf. [1; Cor. I. 1.4].

PROPOSITION 3. *Let* F *be a closed face of a compact convex set* K. *Then* F *is a split face if and only if each* $a \in A_s(F)$ (*or* $A_s(F)^+$, $A(F)$, $A(F)^+$, $A(F; K)$, $A(F; K)^+$) *has an extension* $\tilde{a} \in A_s(K)$ *such that* $\tilde{a} = 0$ *on* F'. *If such an extension exists then it is unique.*

Proof. The uniqueness statement follows from Lemma 2 (ii), since $\partial_e K \subseteqq F \cup F'$.

Assume F is a split face and let $a \in A_s(F)$. If a is u.s.c. affine and nonnegative a has as noted above an u.s.c. affine extension $\tilde{a}$ with $\tilde{a} = 0$ on F'. Hence the result follows if a is the difference of two nonnegative u.s.c. affine functions on K. In general there are b_n, c_n u.s.c. affine and nonnegative, $a_n = b_n - c_n$, such that $\|a_n - a\|_F \underset{n\to\infty}{\longrightarrow} 0$. We use Lemma 2 (ii) and the fact that $\partial_e K \subseteqq F \cup F'$ to conclude that

$$\|\tilde{a}_n - \tilde{a}_m\| = \|\tilde{a}_n - \tilde{a}_m\|_{\partial_e K} = \|a_n - a_m\|_{\partial_e F} = \|a_n - a_m\|_F .$$

Hence $\{\tilde{a}_n\}_1^\infty$ is Cauchy in $A_s(K)$. Then $\tilde{a} = \lim \tilde{a}_n \in A_s(K)$ will be an extension of a with $\tilde{a} = 0$ on F'.

Conversely, assume that each $a \in A(F; K)^+$ has an extension $\tilde{a} \in A_s(K)$ such that $\tilde{a} = 0$ on F'. Let $x \in K \backslash (F \cup F')$, $x = \lambda y + (1 - \lambda)z$, where $y \in F, z \in F'$ and $0 < \lambda < 1$. Then $\lambda = \tilde{1}(x)$, and since λ is uniquely determined, $\hat{\chi}_F$ is affine, and hence $F' = \hat{\chi}_F^{-1}(0)$ is a face, cf. [2; Prop. 1.1, Cor. 1.2]. Now the uniqueness of F, F' components is easy, since $A(F; K)^+$ separates points of F.

The following lemma can be derived from [6; Formula (1), p. 263, Satz 2.1.3]. For the readers convenience we shall give a proof.

LEMMA 4. *Let K_1 and K_2 be compact convex sets and $a \in A_s(K_1)$, $b \in A_s(K_2)$. Then there is a function $c \in A_s(K_1 \otimes K_2)$, denoted by $a \otimes b$, such that*

$$c(\omega(x_1, x_2)) = a(x_1)b(x_2), \text{ all } (x_1, x_2) \in K_1 \times K_2.$$

Proof. First we shall consider the case where a and b are nonnegative u.s.c. and affine. Then there exist nets $\{a_\alpha\} \subseteqq A(K_1)^+$, $\{b_\beta\} \subseteqq A(K_2)^+$ such that $a_\alpha \searrow a$, $b_\beta \searrow b$, pointwise. Then $\{a_\alpha \otimes b_\beta\}$ is a decreasing net in $BA(K_1 \times K_2)^+$, and therefore there is an u.s.c. affine function c on $K_1 \otimes K_2$ such that

$$c(\varphi) = \inf_{\alpha,\beta} \varphi(a_\alpha \otimes b_\beta), \text{ all } \varphi \in K_1 \otimes K_2 .$$

Especially, for all $(x_1, x_2) \in K_1 \times K_2$

$$c(\omega(x_1, x_2)) = \inf a_\alpha(x_1)b_\beta(x_2) = a(x_1)b(x_2) .$$

If

$$a = a_1 - a_2, b = b_1 - b_2 \tag{*}$$

where a_i is u.s.c. nonnegative and affine on K_1, b_i is u.s.c. nonnegative and affine on K_2, then $(x_1, x_2) \to a(x_1)b(x_2)$ is linear combination of four terms of the kind considered in the first part of the proof, and we can choose c as the corresponding linear combination of elements from $A_s(K_1 \otimes K_2)$.

If $a \in A_s(K_1)$, $b \in A_s(K_2)$ are arbitrary then we can find a'_n, b'_n of the type (*), such that $\|b - b'_n\|_{K_2} < 1/n$, $\|a - a'_n\|_{K_1} < 1/n$ and $c_n \in A_s(K \otimes K_2)$ such that

$$c_n(\omega(x_1, x_2)) = a'_n(x_1)b'_n(x_2), \text{ all } (x_1, x_2) \in K_1 \times K_2 . \tag{**}$$

Then for all $(x_1, x_2) \in \partial_e K_2$

$$|a(x_1)b(x_2) - c_n(\omega(x_1, x_2))| < \frac{1}{n^2} + \frac{1}{n}(\|a\|_{K_1} + \|b\|_{K_2}) .$$

From this it follows that $\{c_n|_{\partial_e(K_1 \otimes K_2)}\}$ is Cauchy, and hence $\{c_n\}$ is Cauchy on $K_1 \otimes K_2$ by Lemma 2 (ii). Let $c = \lim c_n \in A_s(K_1 \otimes K_2)$. Then it is obvious from (**) that c satisfies the requirement.

THEOREM 5. *Let K_1 and K_2 be compact convex sets, and F_1 and F_2 closed faces of K_1 and K_2 respectively. Let F be the face $\overline{\text{co}}(\omega(F_1 \times F_2))$ in $K_1 \otimes K_2$. Then the following holds*

(i) *If F is a split face of $K_1 \otimes K_2$ then F_1 and F_2 are split faces of K_1 and K_2.*

(ii) *If either $A(F_1)$ or $A(F_2)$ has the the approximation property, and F_1 and F_2 are split faces of K_1 and K_2, then F is a split face of $K_1 \otimes K_2$.*

Proof. To prove (i) we assume that F is a split face. As noted before $\partial_e F = \omega(\partial_e F_1 \times \partial_e F_2)$. Let $a \in A(K_1)$ such that $a \geqq 0$ on F_1, i.e. $a|_{F_1} \in A(F_1; K_1)^+$. By Proposition 3 it will suffice to show that $(a \cdot \chi_{F_1})^\wedge$ is affine K_1. We know that $((a \otimes 1) \cdot \chi_F)^\wedge$ is u.s.c. and affine on $K_1 \otimes K_2$, since $a \otimes 1$ is nonnegative on $\omega(F_1 \times F_2)$ and hence on F. Now we fix $x_2 \in \partial_e F_2$. Then the function $g(x_2)\colon x \to ((a \otimes 1) \cdot \chi_F)^\wedge(\omega(x, x_2))$ is u.s.c. and affine on K_1. On F_1 $g(x_2)$ agrees with a, and since $\omega(\partial_e F_1' \times \partial_e F_2) \subseteqq F'$, we have that $g(x_2) = 0$ on $\partial_e F_1'$

Since $g(x_2)$ and $(a \cdot \chi_{F_1})^\wedge$ agree on $\partial_e K_1$, and $g(x_2)$ is u.s.c. affine, while $(a \cdot \chi_{F_1})^\wedge$ is u.s.c. concave it follows from Bauers principle [5; Lem. 1] that $g(x_2) \leqq (a \cdot \chi_{F_1})^\wedge$. Moreover $g(x_2) \geqq a \cdot \chi_{F_1}$, and since $(a \cdot \chi_{F_1})^\wedge$ is the smallest u.s.c. concave majorant of $a \cdot \chi_{F_1}$, we have $g(x_2) \geqq (a \cdot \chi_{F_1})^\wedge$, and (i) follows.

To prove (ii) we shall assume that F_1 and F_2 are split faces, and that $A(F_1)$ has the approximation property. By Proposition 3 we have to show that if $a \in A(F)^+$ then a admits an extension $\tilde{a} \in A_s(K_1 \otimes K_2)$ such that $\tilde{a} = 0$ on F'. Now $a \circ (\omega_{K_1 \times K_2}|_{F_1 \times F_2})$ belongs to $BA(F_1 \times F_2) = A(F_1) \otimes_\varepsilon A(F_2)$. If $\varepsilon > 0$ is arbitrary we can choose $a_1, \cdots, a_n \in A(F_1)$ and $b_1, \cdots, b_n \in A(F_2)$ such that

$$\left\| a \circ \omega_{K_1 \times K_2} - \sum_{i=1}^{n} a_i \otimes b_i \right\|_{F_1 \times F_2} < \varepsilon \,.$$

By Proposition 3 we can choose $\tilde{a}_i \in A_s(K_1)$, $\tilde{b}_i \in A_s(K_2)$ such that $\tilde{a}_i = a_i$ on F_1 and $\tilde{a}_i = 0$ on F_1', while $\tilde{b}_i = b_i$ on F_2 and $\tilde{b}_i = 0$ on F_2'.

By Lemma 4 $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i \in A_s(K_1 \otimes K_2)$ and on $\omega(F_1 \times F_2)$ it equals $\sum_{i=1}^n a_i \otimes b_i$, while $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i = 0$ on $\partial_e(K_1 \otimes K_2) \backslash \partial_e F$.

As $A_s(K_1 \otimes K_2)$ is complete in $\| \quad \|_{\partial_e(K_1 \otimes K_2)}$ and the norm of $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i$ is obtained at $\omega(F_1 \times F_2)$, this argument leads to the existence of $\tilde{a} \in A_s(K_1 \otimes K_2)$ such that $\tilde{a} = a$ on $\omega(F \times F_2)$, and $\tilde{a} = 0$ on $\partial_e F' = \partial_e(K_1 \otimes K_2) \backslash F$. It remains to show that $\tilde{a} = a$ on F and $\tilde{a} = 0$ on F'.

Now let $x \in F$ and represent x by a probability measure μ on $\omega(F_1 \times F_2)$. Since $\tilde{a}$ satisfies the barycentric calculus we get

$$\tilde{a}(x) = \int_{K_1 \otimes K_2} \tilde{a} d\mu = \int_{w(F_1 \times F_2)} \tilde{a} d\mu = \int_F a d\mu = a(x)$$

and so $\tilde{a} = a$ on F.

To show that $\tilde{a} = 0$ on F' we let $b \in A(K_1 \otimes K_2)$ with $b > 0$ on $K_1 \otimes K_2$ and $b > a$ on F. Then $b \geqq \tilde{a}$ on $\partial_e(K_1 \otimes K_2)$, and by Lemma 2 (i), $b \geqq \tilde{a}$ on $K_1 \otimes K_2$. For $\rho \in K_1 \otimes K_2$ we have

$$(a \cdot \chi_F)^\wedge(\rho) = \inf \{b(\rho) \mid b \in A(K_1 \otimes K_2),\ b > a \cdot \chi_F\} \geqq \tilde{a}(\rho) \geqq 0 .$$

Since $(a \cdot \chi_F)^\wedge = 0$ on F', we get $\tilde{a} = 0$ on F', and the proof is complete.

REMARK. It is easy to see from Lemma 4 that the embedding of the product of two parallel faces F_1 and F_2 in the sense of [11] gives rise to a parallel face F without the assumption of the presence of the approximation property in $A(F_1)$. In fact, $\hat{\chi}_F = \hat{\chi}_{F_1} \otimes \hat{\chi}_{F_2}$ is affine.

THEOREM 6. *Let F be a closed split face of a compact convex set K. Let B be a real Banach space having the approximation property. Let p be a concave l.s.c. strictly positive real function on K. Let $a: F \to B$ be an affine continuous map such that*

$$\|a(k)\| \leqq p(k), \text{ all } k \in F .$$

Then a has an extension to a continuous affine map $\tilde{a}: K \to B$ such that

$$\|\tilde{a}(k)\| \leqq p(k), \text{ all } k \in K .$$

Proof. Let C be the unit ball of B^* with w^*-topology. $B \times \boldsymbol{R}$ is normed by $\|(x, r)\| = \|x\| + |r|$. It was observed in [10] that $(x, r) \to (\cdot)(x) + r$ is an isometric isomorphism of $B \times \boldsymbol{R}$ onto $A(C)$. Hence if B has the approximation property then $A(C)$ has.

We define a biaffine continuous function b on $F \times C$ by

$$b(x, x^*) = x^*(a(x)), \text{ all } x \in F,\ x^* \in C .$$

By Proposition 1 (ii) there is an affine homeomorphism between $F \otimes C$ and $\overline{\mathrm{co}}(\omega_{K \times C}(F \times C))$ defined by

$$T(\rho)(d) = \rho(d|_{F \times C}) \text{ for } d \in BA(K \times C) .$$

Since b is naturally a continuous affine function on $F \otimes C$ there is a continuous affine function b_1 on $\overline{\mathrm{co}}(\omega_{K \times C}(F \times C))$ such that

$$b_1(T\ \omega_{F \times C}(x, x^*)) = x^*(a(x)), \text{ all } (x, x^*) \in F \times C .$$

Moreover $\rho \to p(P_1(\rho))$ is concave, strictly positive and l.s.c. on $K \otimes C$. For $\rho \in \partial_e(\mathrm{co}(\omega_{K \times C}(F \times C))) = \omega_{K \times C}(\partial_e F \times \partial_e C)$ we have $\rho = \omega_{K \times C}(x, x^*)$ with $(x, x^*) \in \partial_e F \times \partial_e C$ and hence

$$|b_1(\rho)| = |x^*(a(x))| \leqq \|a(x)\| \leqq p(x) = p(P_1(\rho)) .$$

Since $\rho \to |b_1(\rho)|$ is convex and continuous and $\rho \to p(P_1(\rho))$ is concave and l.s.c., it follows from Bauers principle [5; Lem. 1] that $|b_1| \leqq p\circ P_1$ on $\overline{\mathrm{co}}(\omega_{K\times C}(F \times C))$.

Now it follows from Theorem 5 that $\overline{\mathrm{co}}(\omega_{K\times C}(F \times C))$ is a split face of $K \otimes C$. By [1; Th. II. 6. 12] and [3; Th. 2.2 and Th. 4.5] it follows that there is a function $c \in A(K \otimes C)$ such that c extends b_1 and

$$|c(\rho)| \leqq p(P_1(\rho)),\ \text{all}\ \rho \in K \otimes C\ .$$

(Actually, it follows from [1; Cor. I. 5.2] that a concave l.s.c. function on a compact convex set is $A(K)$-superharmonic in the sense of [3]. Moreover it should be remarked that the theorems 2.2 and 4.5 of [3] are stated for complex spaces, but the proofs hold almost unchanged for the real case.)

Now we can define a continuous affine map $c_1: K \to A(C)$ by

$$c_1(k)(\cdot) = c(\omega(k, \cdot))\ .$$

Then for $k \in K$

$$||c_1(k)|| = \sup_{x^* \in C} ||c(\omega(k, x^*))|| \leqq \sup p(P_1(k, x^*))) = p(k)\ .$$

By composing the isometry S between $A(C)$ and $B \times \boldsymbol{R}$ with the canonical projection Q from $B \times \boldsymbol{R}$ to B, which has norm 1. we get an affine continuous map $\tilde{a}(= Q\circ S\circ c_1)$ of K into B such that

$$||\tilde{a}(k)|| = ||(Q\circ S\circ c_1)(k)|| \leqq ||c_1(k)|| \leqq p(k)$$

for all $k \in K$. Moreover, for $k \in F$, $x^* \in C$

$$\begin{aligned} x^*(\tilde{a}(k)) &= x^*((Q\circ S\circ c_1)(k)) = c_1(k)(x^*) \\ &= c(\omega(k, x^*)) = b_1(\omega(k, x^*)) = x^*(a(k))\ . \end{aligned}$$

Hence for $k \in F$: $\tilde{a}(k) = a(k)$.

COROLLARY. *Let F be a closed split face of a compact convex set K. Let B be a real Banach space having the approximation property. Let $a: F \to B$ be a continuous affine map. Then a admits an extension to a continuous affine function $\tilde{a}: K \to B$ such that* $\max_{k \in F} ||a(k)|| = \max_{k \in K} ||\tilde{a}(k)||$.

REMARK. Conclusions similar to those of Theorem 6 and the Corollary hold with no assumptions on B, if instead we know that $A(F)$ has the approximation property. This is f.ex. the case, if K is a simplex.

REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Mathematik, Springer Verlag, Germany, 1971.
2. E. M. Alfsen and T. B. Andersen, *Split faces of compact convex sets*, Proc. London Math. Soc., **21** (1970), 415-442.
3. E. M. Alfsen and B. Hirsberg, *On dominated extensions in linear subspaces of $C_C(X)$*, Pacific J. of Math. **36** (1971), 567-584.
4. T. B. Andersen, *On dominated extension of continuous affine functions on split faces*, (to appear in Math. Scand.)
5. H. Bauer, *Kennzeichnung kompakter Simplexe mit abgeschlossener Extremalpunkt menge*, Archiv der Mathematik, **14** (1963), 415-421.
6. E. Behrends and G. Wittstock, *Tensorprodukte kompakter konvexer Mengen*, Inventiones Math., **10** (1970), 251-266.
7. F. Combes, *Quelques propriétés des C*-algèbres*, Bull. Sci. Math., **94** (1970), 165-192.
8. U. Krause, *Der Satz von Choquet als ein abstrakter Spektralsatz und vice versa*, Math. Ann., **184** (1970), 275-296.
9. A Lazar, *Affine products of simplexes*, Math. Scand., **22** (1968), 165-175.
10. I. Namioka and R. R. Phelps, *Tensor products of compact convex sets*, Pacific J. Math., **31** (1969), 469-480.
11. M. Rogalski, *Topologies faciales dans les convexes compacts, calcul fonctionnel et decomposition spectrale dans le centre d'un espace $A(X)$*, Seminaire Choquet, 1969-70, No. 4.
12. Z. Semadeni, *Categorical methods in convexity*, Proc. Colloq. on Convexity, Copenhagen 1965, 281-307.

Received March 30, 1971. This work was in part supported by the Danish Science Council, J. No. 511-630. The author wants to thank E. Alfsen for helpful comments.

UNIVERSITY OF OSLO
AND
UNIVERSITY OF ÅRHUS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 42, No. 1 January, 1972