ON THE RANGE SETS OF $H^p$ FUNCTIONS

Leon Brown and Lowell J. Hansen
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The object of this paper is to show that, even though every function in $H^p(0 < p < \infty)$ has a nontangential limit almost everywhere, "most" functions in $H^p$ have surprisingly "wild" behavior at the boundary. The proof uses category arguments and a method of constructing non-normal functions in $H^p$.

Preliminaries. We denote the open unit disc by $A$, and the boundary of $A$ by $T$. If $f$ is a complex valued function on $A$, then the cluster set of $f$ at the point $P \in T$, denoted by $C(f, P)$, is the set of points $w$ for which there exists a sequence $\{z_n\} \subset A$ with $z_n \to P$ and $f(z_n) \to w$. One easily sees that

$$C(f, P) = \bigcap_k f(D_k),$$

where $D_k = \{z \in A: |z - P| < 1/k\}$. The range set of $f$ at $P$, $R(f, P)$, is the set of points $w$ such that there exists a sequence $\{z_n\} \subset A$ with $z_n \to P$ and $f(z_n) = w$. Thus,

$$R(f, P) = \bigcap_k f(D_k).$$

$f$ is said to possess the angular limit (nontangential limit) $\alpha$ at $P \in T$ if $f$ converges to $\alpha$ when restricted to each Stolz angle

$$\{z \in A: |\arg (P - z) - \arg P| < \delta\}, \quad 0 < \delta < \pi/2.$$

We say that $\alpha$ is an asymptotic value of $f$ at $P \in T$ if there exists a Jordan curve $\varphi$: $\{0 \leq t < 1\} \to A$ such that $\lim_{t \to 1^-} \varphi(t) = P$ and $\lim_{t \to 1^-} f[\varphi(t)] = \alpha$.

A function defined on $D$ is normal if the collection $\{f \circ S: S \in \Gamma\}$ is a normal family of functions, where $\Gamma$ is the collection of conformal maps of $A$ onto itself. Any holomorphic function which omits two complex values is a normal function. It follows from a theorem of Lehto and Virtanen that if a function $f$ is meromorphic on $A$ and has two different asymptotic values at $z = 1$, then $f$ is not normal:

**Theorem 1.** [9, Theorem 2, p. 53] Let $f$ be meromorphic and normal in the simply-connected region $G$, and let $f$ have an asymptotic value $\alpha$ at a boundary point $P$ along a Jordan curve lying in the closure of $G$. Then $f$ possesses the angular limit $\alpha$ at the point $P$.

The reader is referred to [7] for an excellent presentation of the
theory of $H^p$ spaces. We shall make strong use of the following decomposition theorem.

THEOREM 2. Every function $f \neq 0$ of class $H^p$ ($0 < p \leq \infty$) has a unique factorization of the form

$$f = BSF,$$

where $B$ is a Blaschke product, $S$ is a singular inner function, and $F$ is an outer function for the class $H^p$. That is,

$$(1) \quad B(z) = z^n \prod_{a_n} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_nz}$$

where $\{a_n\}$ are the zeros of $f$;

$$(2) \quad S(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(z) \right\}$$

where $\mu$ is a singular measure, and

$$(3) \quad F(z) = e^{i\theta} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) \, dt \right\}$$

where $\psi(t) = |f(e^{it})|$, $\psi(t) \geq 0$, $\log \psi \in L^1$ and $\psi \in L^p$. Conversely, every such product $BSF$ belongs to $H^p$.

Main result.

THEOREM 3. For each function $f$ in $H^p$ ($0 < p < \infty$) outside a set of first category, the range set $R(f, e^{i\theta})$ at each boundary point omits at most one complex number. Thus the cluster set $C(f, e^{i\theta})$ at each boundary point is the full Riemann sphere.

Proof. The first step in our proof is to show that for "most" functions in $H^p$, $C \setminus R(f, 1)$ contains at most one point. Let

$$D_k = \{z \in \Delta: |z - 1| < \frac{1}{k}\}.$$ 

Then we define $A_p(n, k) = A(n, k) = \{f \in H^p \mid \exists w_i, w_i \in f(D_k) \text{ with } |w_i| \leq n, i = 1, 2, \text{ and } |w_1 - w_2| \geq 1/n\}$. Any function in the complement of $\bigcup_{n,k} A(n, k)$ has the required property. Thus it is sufficient to show that $A(n, k)$ is nowhere dense in $H^p$. We prove this by showing that $A(n, k)$ is closed and has a dense complement.

(a) $A(n, k)$ is closed. If $f_j \in A(n, k)$ and $f_j \rightarrow f$ in $H^p$, we claim that $f \in A(n, k)$. We use the fact that $H^p$ convergence implies uni-
form convergence on compact subsets of $\Delta$ [7, Lemma, p. 36]. If $f$ is a constant function, $f$ clearly is in $A(n, k)$. We assume that $f$ is an open map. Let $w^i_j, w^i_j \in \{z \mid |z| \leq n\} \setminus f_j(D_k)$ and $|w^i_j - w^i_j| \geq 1/n$. By choosing an appropriate subsequence we may assume that $w^i_j \to w^i$ and $w^i_j \to w^i$. It is clear that $|w^i - w^i_j| \geq 1/n$ and $|w^i| \leq n$, $i = 1, 2$.

If $w^i \in f(D_k)$, choose $z^i \in D_k$ such that $f(z^i) = w^i$. Then $f_j - w^i_j$ converges to $f - w^i$ uniformly on compact subsets of $\Delta$. Let $N(z^i)$ be a neighborhood of $z^i$ such that $\overline{N(z^i)} \subset D_k$ and $w^i \notin f(\partial N(z^i))$. An application of Rouche’s theorem to the functions $f_j - w^i_j$ and $f - w^i$ implies that

$$w^i_j \in f_j(N(z^i))$$

for $j$ sufficiently large. This contradicts the fact that $w^i_j \notin f_j(D_k)$. A similar argument shows that $w^i \notin f(D_k)$. Thus we have shown that $A(n, k)$ is closed in $H^p$.

(b) The complement of $A(n, k)$ is dense. Let $f \in H^p$. By Theorem 1, we may write $f = BSF$. We shall approximate each of these factors as follows.

(i) If $B$ is a finite product, put $B_n = B$. If $B$ is an infinite product, put

$$B_n(z) = z^n \prod_{k=1}^{n} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - a_k z}.$$

Note that $B_n$ converges uniformly on compact subsets to $B$, and on $T$, $B_n$ is continuous and of modulus 1.

(ii) If $S = 1$, we let $S_n = 1$. Otherwise

$$S(z) = \exp \left\{ - \int_{0}^{z} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right\},$$

where $\mu$ is a singular positive measure. The set of measures with finite support is dense in the space of measures endowed with the weak* topology induced by the space of continuous functions. Thus, since $H^p$ is separable, there exists a sequence of measures $\mu_n$ with finite support which converge to $\mu$ in the weak* topology. Let

$$S_n(z) = \exp \left\{ - \int_{0}^{z} \frac{e^{it} + z}{e^{it} - z} d\mu_n(t) \right\}.$$

We note that $\|S_n\|_\infty = 1$ and so $\{S_n\}$ forms a normal family. This fact, together with the pointwise convergence of $S_n$ to $S$, implies that $S_n$ converges uniformly on compact subsets of $\Delta$ to $S$. Furthermore, on $T$, $S_n$ is continuous and of modulus 1 except on the support of $\mu_n$. 


(iii) \[ F(z) = e^{it} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) \, dt \right\} \]

where \( \psi(t) \geq 0, \log \psi \in L^1, \psi \in L^p, \) and \( \psi(t) = |F(e^{it})| = |f(e^{it})| \) a.e.

Let

\[ \psi_n(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{n} \\ |f(e^{it})|, & \frac{1}{n} < t < 2\pi - \frac{1}{n} \\ (2\pi - t)^{-1/2}, & 2\pi - \frac{1}{n} \leq t < 2\pi \end{cases} \]

and

\[ F_n(z) = e^{it} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi_n(t) \, dt \right\}. \]

\( \psi_n \) has been defined so that \( \psi_n \geq 0, \log \psi_n \in L^1, \psi_n \in L^p, \log \psi_n \rightarrow \log \psi \) in \( L^1 \), and \( \psi_n \rightarrow \psi \) in \( L^p \). The \( L^1 \) convergence of \( \log \psi_n \) to \( \log \psi \) implies that \( F_n \) converges uniformly on compact subsets to \( F \). In addition, since

\[ \| F \| = \left( \frac{1}{2\pi} \int \psi(t)^p \, dt \right)^{1/p} \]

and \( \psi_n \rightarrow \psi \) in \( L^p \), we conclude that \( \| F_n \| \rightarrow \| F \| \).

We assume that \( p > 1 \) and claim that \( f_n = B_n S_n F_n \) converges to \( f \) in \( H^p \). We recall that \( B_n, S_n, \) and \( F_n \) converge uniformly on compact subsets of \( \Delta \) to \( B, S, \) and \( F \) respectively, and

\[ \| B_n B_n F_n \| = \| F_n \| \longrightarrow \| F \| = \| f \| . \]

Hence \( f_n \) converges weakly to \( f \) in \( H^p \) and, since \( H^p(1 < p < \infty) \) is a uniformly convex Banach space, \( f_n \rightarrow f \) in \( H^p \).

Let \( n \geq 1 \) be fixed. Then since the support of \( \mu_n \) is finite, there exists an arc \( \{e^{i\theta}: |\theta| < \delta\} \) on which, except possibly at \( \theta = 0, |S_n(e^{i\theta})| = 1 \). Therefore, since \( |B_n(e^{i\theta})| = 1 \) and \( |F_n(e^{i\theta})| = \psi_n(\theta) \),

\[ \lim_{\theta \rightarrow 0^+} f_n(e^{i\theta}) = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 2\pi^-} f_n(e^{i\theta}) = \infty \]

Let \( \phi \) be a conformal mapping from \( \overline{\Delta} \) onto \( \overline{D_k} \) such that \( \phi(1) = 1 \). Thus \( f_n \circ \phi \) has 0 and \( \infty \) as asymptotic values at \( z = 1 \). It follows from Theorem 1 that \( f_n \circ \phi \) is not normal and therefore \( f_n \circ \phi(\Delta) = f_n(D_k) \) omits at most one value of the complex plane. Thus \( f_n \notin A(n, k) \).

Suppose that \( 0 < p \leq 1 \). Then \( H^1 \setminus A_2(n, k) \subset H^p \setminus A_p(n, k) \). We have already shown that \( H^1 \setminus A_2(n, k) \) is dense in \( H^2 \), and so must be dense in the polynomials (in the \( H^2 \) norm). Since \( \| \| \|_p \leq \| \|_2 \),
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$H^p \backslash A_\sigma(n, k)$ is also dense in the polynomials in the topology of $H^p$. Therefore $H^p \backslash A_\sigma(n, k)$ is dense in $H^p$ and so $H^p \backslash A_\tau(n, k)$ is dense in $H^p$. This completes the proof that $A(n, k)$ is nowhere dense.

To complete the proof, let $\{e^{i\theta}w\}$ be a countable dense subset of $T$. We have just proved that for each fixed $n$, the set $W(n)$ of functions in $H^p$ for which $C\backslash R(f, e^{i\theta}w)$ has at most one point is a residual set. Thus $\bigcap W(n)$ is a residual set. We claim that if $f \in \bigcap W(n)$, then $C\backslash R(f, e^{i\theta})$ is at most a singleton for every $\theta$. If not, let $f \in \bigcap W(n)$ and $w_1, w_2 \in C\backslash R(f, e^{i\theta})$. Thus there exists $0 < r < 1$ such that $w_1, w_2 \in f[\{z \in \Delta : |z - e^{i\theta}| < r\}]$. If $|e^{i\theta} - e^{i\theta}w_1| < r$, then $w_1, w_2 \in C\backslash R(f, e^{i\theta}w)$, which is a contradiction. This completes the proof of the theorem.

We remark that $\{\theta \mid w \in R(f, e^{i\theta})\}$ is an open subset of $T$. Hence if $C\backslash R(f, e^{i\theta})$ is at most a singleton for each $\theta$, then $\bigcup C\backslash R(f, e^{i\theta})$ is at most countable.

We conclude with some historical remarks. A point $\zeta$ on the unit circle, $T$, is called an ambiguous point of $f$ if $f$ has two different asymptotic values at $\zeta$. F. Bagemihl [1] proved that the set of ambiguous points of $f$ is at most countable, even if $f$ is an arbitrary function on $\Delta$. F. Bagemihl and W. Seidel [2] proved that if $E$ is any countable subset of $T$, then there exists a function, holomorphic and of bounded characteristic on $\Delta$, for which every element of $E$ is an ambiguous point. G. T. Cargo [5] has constructed such a function which is in $H^p$ for all $p < \infty$. As we have shown in our theorem, the range set of any holomorphic function at an ambiguous point omits at most one complex value. Thus if we choose a dense countable subset of $T$, Cargo's construction yields a function with the "wild" behavior of Theorem 3.

Professor W. Seidel has kindly pointed out to us that the Picard-type behavior of a holomorphic function in a neighborhood of an ambiguous point is an old result of E. Lindelöf [10, p. 13].

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