

Pacific Journal of Mathematics

ON THE RANGE SETS OF H^p FUNCTIONS

LEON BROWN AND LOWELL J. HANSEN

ON THE RANGE SETS OF H^p FUNCTIONS

LEON BROWN AND LOWELL HANSEN

The object of this paper is to show that, even though every function in H^p ($0 < p < \infty$) has a nontangential limit almost everywhere, "most" functions in H^p have surprisingly "wild" behavior at the boundary. The proof uses category arguments and a method of constructing non-normal functions in H^p .

Preliminaries. We denote the open unit disc by Δ , and the boundary of Δ by T . If f is a complex valued function on Δ , then the *cluster set* of f at the point $P \in T$, denoted by $C(f, P)$, is the set of points w for which there exists a sequence $\{z_n\} \subset \Delta$ with $z_n \rightarrow P$ and $f(z_n) \rightarrow w$. One easily sees that

$$C(f, P) = \bigcap_k \overline{f(D_k)},$$

where $D_k = \{z \in \Delta: |z - P| < 1/k\}$. The *range set* of f at P , $R(f, P)$, is the set of points w such that there exists a sequence $\{z_n\} \subset \Delta$ with $z_n \rightarrow P$ and $f(z_n) = w$. Thus,

$$R(f, P) = \bigcap_k f(D_k).$$

f is said to possess the *angular limit* (nontangential limit) α at $P \in T$ if f converges to α when restricted to each Stolz angle

$$\{z \in \Delta: |\arg(P - z) - \arg P| < \delta\}, \quad 0 < \delta < \pi/2.$$

We say that α is an *asymptotic value* of f at $P \in T$ if there exists a Jordan curve $\varphi: \{0 \leq t < 1\} \rightarrow \Delta$ such that $\lim_{t \rightarrow 1^-} \varphi(t) = P$ and $\lim_{t \rightarrow 1^-} f[\varphi(t)] = \alpha$.

A function defined on D is *normal* if the collection $\{f \circ S: S \in \Gamma\}$ is a normal family of functions, where Γ is the collection of conformal maps of Δ onto itself. Any holomorphic function which omits two complex values is a normal function. It follows from a theorem of Lehto and Virtanen that if a function f is meromorphic on Δ and has two different asymptotic values at $z = 1$, then f is not normal:

THEOREM 1. [9, Theorem 2, p. 53] *Let f be meromorphic and normal in the simply-connected region G , and let f have an asymptotic value α at a boundary point P along a Jordan curve lying in the closure of G . Then f possesses the angular limit α at the point P .*

The reader is referred to [7] for an excellent presentation of the

theory of H^p spaces. We shall make strong use of the following decomposition theorem.

THEOREM 2. *Every function $f \neq 0$ of class H^p ($0 < p \leq \infty$) has a unique factorization of the form*

$$f = BSF,$$

where B is a Blaschke product, S is a singular inner function, and F is an outer function for the class H^p . That is,

$$(1) \quad B(z) = z^n \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}$$

where $\{a_n\}$ are the zeros of f ,

$$(2) \quad S(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(z) \right\}$$

where μ is a singular measure, and

$$(3) \quad F(z) = e^{i\gamma} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) dt \right\}$$

where $\psi(t) = |f(e^{it})|$, $\psi(t) \geq 0$, $\log \psi \in L^1$ and $\psi \in L^p$. Conversely, every such product BSF belongs to H^p .

Main result.

THEOREM 3. *For each function f in H^p ($0 < p < \infty$) outside a set of first category, the range set $R(f, e^{i\theta})$ at each boundary point omits at most one complex number. Thus the cluster set $C(f, e^{i\theta})$ at each boundary point is the full Riemann sphere.*

Proof. The first step in our proof is to show that for "most" functions in H^p , $C \setminus R(f, 1)$ contains at most one point. Let

$$D_k = \left\{ z \in \mathcal{A}: |z - 1| < \frac{1}{k} \right\}.$$

Then we define $A_p(n, k) = A(n, k) = \{f \in H^p | \exists w_1, w_2 \notin f(D_k) \text{ with } |w_i| \leq n, i = 1, 2, \text{ and } |w_1 - w_2| \geq 1/n\}$. Any function in the complement of $\bigcup_{n,k} A(n, k)$ has the required property. Thus it is sufficient to show that $A(n, k)$ is nowhere dense in H^p . We prove this by showing that $A(n, k)$ is closed and has a dense complement.

(a) $A(n, k)$ is closed. If $f_j \in A(n, k)$ and $f_j \rightarrow f$ in H^p , we claim that $f \in A(n, k)$. We use the fact that H^p convergence implies uni-

form convergence on compact subsets of Δ [7, Lemma, p. 36]. If f is a constant function, f clearly is in $A(n, k)$. We assume that f is an open map. Let $w_j^1, w_j^2 \in \{z \mid |z| \leq n\} \setminus f_j(D_k)$ and $|w_j^1 - w_j^2| \geq 1/n$. By choosing an appropriate subsequence we may assume that $w_j^1 \rightarrow w^1$ and $w_j^2 \rightarrow w^2$. It is clear that $|w^1 - w^2| \geq 1/n$ and $|w^i| \leq n$, $i = 1, 2$. If $w^1 \in f(D_k)$, choose $z^1 \in D_k$ such that $f(z^1) = w^1$. Then $f_j - w_j^1$ converges to $f - w^1$ uniformly on compact subsets of Δ . Let $N(z^1)$ be a neighborhood of z^1 such that $\overline{N(z^1)} \subset D_k$ and $w^1 \notin f(\partial N(z^1))$. An application of Rouché's theorem to the functions $f_j - w_j^1$ and $f - w^1$ implies that

$$w_j^1 \in f_j(N(z^1))$$

for j sufficiently large. This contradicts the fact that $w_j^1 \notin f_j(D_k)$. A similar argument shows that $w^2 \notin f(D_k)$. Thus we have shown that $A(n, k)$ is closed in H^p .

(b) The complement of $A(n, k)$ is dense. Let $f \in H^p$. By Theorem 1, we may write $f = BSF$. We shall approximate each of these factors as follows.

(i) If B is a finite product, put $B_n = B$. If B is an infinite product, put

$$B_n(z) = z^n \prod_{k=1}^n \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k}z}.$$

Note that B_n converges uniformly on compact subsets to B , and on T , B_n is continuous and of modulus 1.

(ii) If $S \equiv 1$, we let $S_n \equiv 1$. Otherwise

$$S(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right\}$$

where μ is a singular positive measure. The set of measures with finite support is dense in the space of measures endowed with the weak* topology induced by the space of continuous functions. Thus, since H^p is separable, there exists a sequence of measures μ_n with finite support which converge to μ in the weak* topology. Let

$$S_n(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu_n(t) \right\}.$$

We note that $\|S_n\|_\infty = 1$ and so $\{S_n\}$ forms a normal family. This fact, together with the pointwise convergence of S_n to S , implies that S_n converges uniformly on compact subsets of Δ to S . Furthermore, on T , S_n is continuous and of modulus 1 except on the support of μ_n .

$$(iii) \quad F(z) = e^{iz} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) dt \right\}$$

where $\psi(t) \geq 0$, $\log \psi \in L^1$, $\psi \in L^p$, and $\psi(t) = |F(e^{it})| = |f(e^{it})|$ a.e. Let

$$\psi_n(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{n} \\ |f(e^{it})|, & \frac{1}{n} < t < 2\pi - \frac{1}{n} \\ (2\pi - t)^{-1/2p}, & 2\pi - \frac{1}{n} \leq t < 2\pi \end{cases}$$

and

$$F_n(z) = e^{iz} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi_n(t) dt \right\}.$$

ψ_n has been defined so that $\psi_n \geq 0$, $\log \psi_n \in L^1$, $\psi_n \in L^p$, $\log \psi_n \rightarrow \log \psi$ in L^1 , and $\psi_n \rightarrow \psi$ in L^p . The L^1 convergence of $\log \psi_n$ to $\log \psi$ implies that F_n converges uniformly on compact subsets to F . In addition, since

$$\|F\| = \left(\frac{1}{2\pi} \int \psi(t)^p dt \right)^{1/p}$$

and $\psi_n \rightarrow \psi$ in L^p , we conclude that $\|F_n\| \rightarrow \|F\|$.

We assume that $p > 1$ and claim that $f_n = B_n S_n F_n$ converges to f in H^p . We recall that B_n , S_n , and F_n converge uniformly on compact subsets of Δ to B , S , and F respectively, and

$$\|B_n B_n F_n\| = \|F_n\| \longrightarrow \|F\| = \|f\|.$$

Hence f_n converges weakly to f in H^p and, since H^p ($1 < p < \infty$) is a uniformly convex Banach space, $f_n \rightarrow f$ in H^p .

Let $n \geq 1$ be fixed. Then since the support of μ_n is finite, there exists an arc $\{e^{i\theta}: |\theta| < \delta\}$ on which, except possibly at $\theta = 0$, $|S_n(e^{i\theta})| = 1$. Therefore, since $|B_n(e^{i\theta})| \equiv 1$ and $|F_n(e^{i\theta})| = \psi_n(\theta)$,

$$\lim_{\theta \rightarrow 0^+} f_n(e^{i\theta}) = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 2\pi^-} f_n(e^{i\theta}) = \infty.$$

Let ϕ be a conformal mapping from $\bar{\Delta}$ onto \bar{D}_k such that $\phi(1) = 1$. Thus $f_n \circ \phi$ has 0 and ∞ as asymptotic values at $z = 1$. It follows from Theorem 1 that $f_n \circ \phi$ is not normal and therefore $f_n \circ \phi(\Delta) = f_n(D_k)$ omits at most one value of the complex plane. Thus $f_n \notin A(n, k)$.

Suppose that $0 < p \leq 1$. Then $H^2 \setminus A_2(n, k) \subset H^p \setminus A_p(n, k)$. We have already shown that $H^2 \setminus A_2(n, k)$ is dense in H^2 , and so must be dense in the polynomials (in the H^2 norm). Since $\| \cdot \|_p \leq \| \cdot \|_2$,

$H^2 \setminus A_2(n, k)$ is also dense in the polynomials in the topology of H^p . Therefore $H^2 \setminus A_2(n, k)$ is dense in H^p and so $H^p \setminus A_p(n, k)$ is dense in H^p . This completes the proof that $A(n, k)$ is nowhere dense.

To complete the proof, let $\{e^{i\theta_n}\}$ be a countable dense subset of T . We have just proved that for each fixed n , the set $W(n)$ of functions in H^p for which $C \setminus R(f, e^{i\theta_n})$ has at most one point is a residual set. Thus $\bigcap W(n)$ is a residual set. We claim that if $f \in \bigcap W(n)$, then $C \setminus R(f, e^{i\theta})$ is at most a singleton for every θ . If not, let $f \in \bigcap W(n)$ and $w_1, w_2 \in C \setminus R(f, e^{i\theta})$. Thus there exists $0 < r < 1$ such that $w_1, w_2 \notin f[\{z \in \Delta: |z - e^{i\theta}| < r\}]$. If $|e^{i\theta} - e^{i\theta_n}| < r$, then $w_1, w_2 \in C \setminus R(f, e^{i\theta_n})$, which is a contradiction. This completes the proof of the theorem.

We remark that $\{\theta \mid w \notin R(f, e^{i\theta})\}$ is an open subset of T . Hence if $C \setminus R(f, e^{i\theta})$ is at most a singleton for each θ , then $\bigcup C \setminus R(f, e^{i\theta})$ is at most countable.

We conclude with some historical remarks. A point ζ on the unit circle, T , is called an *ambiguous point* of f if f has two different asymptotic values at ζ . F. Bagemihl [1] proved that the set of ambiguous points of f is at most countable, even if f is an arbitrary function on Δ . F. Bagemihl and W. Seidel [2] proved that if E is any countable subset of T , then there exists a function, holomorphic and of bounded characteristic on Δ , for which every element of E is an ambiguous point. G. T. Cargo [5] has constructed such a function which is in H^p for all $p < \infty$. As we have shown in our theorem, the range set of any holomorphic function at an ambiguous point omits at most one complex value. Thus if we choose a dense countable subset of T , Cargo's construction yields a function with the "wild" behavior of Theorem 3.

Professor W. Seidel has kindly pointed out to us that the Picard-type behavior of a holomorphic function in a neighborhood of an ambiguous point is an old result of E. Lindelöf [10, p. 13].

REFERENCES

1. F. Bagemihl, *Curvilinear cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. U. S., **41** (1955), 379-382.
2. F. Bagemihl and W. Seidel, *Functions of bounded characteristic with prescribed ambiguous points*, Michigan Math. J., **3** (1955-56), 77-81.
3. L. Brown and P. M. Gauthier, *Picard-type theorems for functions defined on a Stein space*, to appear.
4. L. Brown and L. Hansen, *A nonnormal outer function in H^p* , Proc. Amer. Math. Soc., **34** (1972), 175-176.
5. G. T. Cargo, *Almost-bounded holomorphic functions with prescribed ambiguous*

points, *Canad. J. Math.*, **16** (1964), 231-240.

6. E. F. Collingwood and A. J. Lohwater, *The Theory of Cluster Sets*, Cambridge, 1966.

7. P. Duren, *Theory of H^p Spaces*, New York, 1970.

8. S. Kierst and E. Szpilrajn, *Sur certaines singularités des fonctions analytiques uniformes*, *Fund. Math.*, **21** (1933), 276-294.

9. O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions*, *Acta Math.*, **97** (1957), 47-65.

10. E. Lindelöf, *Sur un principe général de l'analyse et ses applications à la théorie de la représentation conforme*, *Acta Soc. Sci. Fenn.*, **46** 4 (1915), 1-35.

Received April 16, 1961 and in revised form December 22, 1971. Supported in part by an NSF Grant GP-20150.

WAYNE STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Tage Bai Andersen, <i>On Banach space valued extensions from split faces</i>	1
David Marion Arnold, <i>A duality for quotient divisible abelian groups of finite rank</i>	11
Donald Pollard Ballou, <i>Shock sets for first order nonlinear hyperbolic equations</i>	17
Leon Brown and Lowell J. Hansen, <i>On the range sets of H^p functions</i>	27
Alexander Munro Davie and Arne Stray, <i>Interpolation sets for analytic functions</i>	33
M. G. Deshpande, <i>Structure of right subdirectly irreducible rings. II</i>	39
Barry J. Gardner, <i>Some closure properties for torsion classes of abelian groups</i>	45
Paul Daniel Hill, <i>Primary groups whose subgroups of smaller cardinality are direct sums of cyclic groups</i>	63
Richard Allan Holzsager, <i>When certain natural maps are equivalences</i>	69
Donald William Kahn, <i>A note on H-equivalences</i>	77
Joong Ho Kim, <i>R-automorphisms of $R[t][[X]]$</i>	81
Shin'ichi Kinoshita, <i>On elementary ideals of polyhedra in the 3-sphere</i>	89
Andrew T. Kitchen, <i>Watts cohomology and separability</i>	99
Vadim Komkov, <i>A technique for the detection of oscillation of second order ordinary differential equations</i>	105
Charles Philip Lanski and Susan Montgomery, <i>Lie structure of prime rings of characteristic 2</i>	117
Andrew Lenard, <i>Some remarks on large Toeplitz determinants</i>	137
Kathleen B. Levitz, <i>A characterization of general Z.P.I.-rings. II</i>	147
Donald A. Lutz, <i>On the reduction of rank of linear differential systems</i>	153
David G. Mead, <i>Determinantal ideals, identities, and the Wronskian</i>	165
Arunava Mukherjea, <i>A remark on Tonelli's theorem on integration in product spaces</i>	177
Hyo Chul Myung, <i>A generalization of the prime radical in nonassociative rings</i>	187
John Piepenbrink, <i>Rellich densities and an application to unconditionally nonoscillatory elliptic equations</i>	195
Michael J. Powers, <i>Lefschetz fixed point theorems for a new class of multi-valued maps</i>	211
Aribindi Satyanarayan Rao, <i>On the absolute matrix summability of a Fourier series</i>	221
T. S. Ravisankar, <i>On Malcev algebras</i>	227
William Henry Ruckle, <i>Topologies on sequences spaces</i>	235
Robert C. Shock, <i>Polynomial rings over finite dimensional rings</i>	251
Richard Tangeman, <i>Strong heredity in radical classes</i>	259
B. R. Wenner, <i>Finite-dimensional properties of infinite-dimensional spaces</i>	267