INTERPOLATION SETS FOR ANALYTIC FUNCTIONS

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Let \( U \) be a bounded open subset of the complex plane \( \mathbb{C} \).
Criteria are obtained for a subset \( E \) of \( \overline{U} \) to be an interpolation set for the algebra of all bounded analytic functions on \( U \) extending continuously to \( E \).

In the case where \( U \) is the open unit disc \( \Delta \), this problem was treated by Detraz [3]. She showed that if \( E \) is a subset of the unit circle \( T \) then every bounded continuous function on \( E \) is the restriction of a bounded analytic function on \( \Delta \), extending continuously to \( E \cup (T \cup \mathbb{E}) \), if and only if \( E \) has measure zero. We extend this result to any \( U \) with connected complement, replacing linear measure on \( T \) by harmonic measure (Theorem 1). For the general case the same method yields a criterion in terms of representing measures for \( A(U) \) (Theorem 2). Finally in Theorem 3 we use a localization argument to sharpen Theorem 1 and also treat the case where \( E \) contains points of \( U \) as well as \( \partial U \).

NOTATION. If \( S \) is a plane set then \( \overline{S} \) denotes its closure and \( \partial S \) its boundary. \( A(U) \) denotes the algebra of all continuous functions on \( \overline{U} \), analytic on \( U \); \( H^\infty(U) \) denotes the algebra of all bounded analytic functions on \( U \); \( H^\infty_E(U) \) denotes the algebra of all bounded continuous functions on \( U \cup E \) which are analytic on \( U \). If \( y \in \overline{U} \), a representing measure for \( y \) with respect to \( A(U) \) is a positive borel measure \( \mu \) on \( \overline{U} \) such that \( f(y) = \int f \, d\mu \) for all \( f \in A(U) \). We denote by \( \|f\| \) the supremum of the function \( f \) over its domain of definition. \( \Delta(z, \delta) \) denotes the disc with center \( z \) and radius \( \delta \).

We say that a set \( S \subseteq U \cup E \) is an interpolation set for \( H^\infty_E(U) \) if for any bounded continuous \( f \) on \( S \) we can find \( g \in H^\infty_E(U) \) with \( g|S = f \). We say \( S \) is a peak interpolation set for \( H^\infty_E(U) \) if for any bounded continuous \( f \) on \( S \), and open set \( V \supseteq S \), and any \( \varepsilon > 0 \), we can find \( g \in H^\infty_E(U) \) with \( g|S = f \), \( \|g\| \leq \|f\| \), and \( |g| < \varepsilon \) on \( U \setminus V \).

**Theorem 1.** Suppose \( C \setminus U \) is connected. Let \( F \) be a subset of \( \partial U \) with zero harmonic measure for each point of \( U \) (with respect to \( U \)). Then \( F \) is a peak interpolation set for \( H^\infty_E(U) \).

The proof follows from the following lemma.

**Lemma.** With \( U \) and \( F \) as in the theorem, let \( X \) be a compact
subset of $\bar{U}$, $W$ a neighborhood of $X$, and $\varepsilon > 0$. Then we can find $f \in H^\omega$ with $\|f\| \leq 2, |1 - f| < \varepsilon$ on $F \cap X$, and $|f| < \varepsilon$ on $U \setminus W$.

Proof. We can find a positive harmonic function $\sigma$ on $U$ such that $\sigma(\zeta) \to \infty$ as $\zeta \to z, \zeta \in U$, for each $z \in F$. Let $\tau$ be a harmonic conjugate to $\sigma$ on $U$, and let $\theta = \sigma + i\tau$, an analytic function on $U$. Put $h = \theta/(\theta + 1)$. Since $\theta$ has positive real part, $h \in H^\omega(U)$ with $\|h\| \leq 1$. Moreover $h(\zeta) \to 1$ as $\zeta \to z, \zeta \in U$, for each $z \in F$; hence we can regard $h$ as an element of $H^\omega_F(U)$, with $h = 1$ on $F$.

Now let $\varphi$ be a continuously differentiable function which is 1 on a neighborhood of $X$ and zero outside $W$, with $\|\varphi\| = 1$. Then the function

$$g_n(\zeta) = \varphi(\zeta) h^\omega(\zeta) + \frac{1}{\pi} \int_{\partial U} \frac{h^\omega(z)}{z - \zeta} \frac{\partial \varphi}{\partial z} dm(z)$$

is in $H^\omega_F(U)$. (See [4], p. 210.) Moreover

$$\|g_n - \varphi h^\omega\| \leq \frac{1}{\pi} \left\| \frac{\partial \varphi}{\partial z} \right\| \|h^\omega\|_{L^2(U)} \sup_{\zeta} \left| \frac{1}{\pi} \frac{1}{|z - \zeta|^{1/2}} \right|.$$ 

The last term is bounded by a constant depending only on $U$, and $\|h^\omega\|_{L^2} \to 0$ as $n \to \infty$ since $|h| < 1$ in $U$. Choose $n$ so that $\|g_n - \varphi h^\omega\| < \varepsilon$ and put $f = g_n$. Then $f$ satisfies the requirements of the lemma.

Theorem 1 follows from the lemma in exactly the same way as Theorem 1 follows from Lemma 2 in [2]. (For an alternative approach see the proof of Theorem 4.3 of [3]).

We observe that if $A(U)$ is pointwise boundedly dense in $H^\omega(U)$ then using Theorem 2.1 of [5] we can modify the function $f$ in the lemma so that it is in $H^\omega_F(U)$. Then we can prove that $F$ is a peak interpolation set for $H^\omega_F(U)$. In the general situation (where $C \setminus U$ need not be connected) the same method yields the following result. If $y \in U$ we denote by $M_y$ the set of all (positive) representing measures for $y$ with respect to $A(U)$ on $\bar{U}$. We assume $U$ is connected.

**Theorem 2.** Let $y \in U$ and $F \subseteq \partial U$. Suppose there is a decreasing sequence $\{V_n\}$ of open sets containing $F$, such that $\mu(V_n) \to 0$ uniformly for $\mu \in M_y$.

Then $F$ is a peak interpolation set for $H^\omega(U)$.

**Proof.** We may suppose $\mu(V_n) < 2^{-n}$ for each $\mu \in M_y$. For each $n$ let $\{g_{n,k}\}$ be an increasing sequence of nonnegative continuous functions converging to the characteristic function of $V_n$. Then $\int g_{n,k} d\mu < 2^{-n}$.
for \( \mu \in M_y \) and so by Theorem II 2.1 of [4], we can find \( h_{nk} \in A(U) \) with \( \text{Re}\ h_{nk} \geq g_{nk} \) on \( \bar{U} \), \( \text{Re}\ h_{nk}(y) < 2^{-n} \), and we can also suppose \( \text{Im}\ h_{nk}(y) = 0 \). Passing to a subsequence we have \( h_{nk} \rightharpoonup h_* \) as \( k \to \infty \), pointwise in \( U \), where \( h_* \) is analytic in \( U \) with \( \text{Re}\ h_* \geq 1 \) on \( V_n \cap U \) and \( |h_n(y)| \leq 2^{-n} \), \( \text{Re}\ h_n \geq 0 \) on \( U \), \( \text{Im}\ h_n = 0 \). By Harnack's inequalities the series \( \sum_{n=1}^{\infty} h_n \) converges pointwise on \( U \) to an analytic function \( h \) such that \( \text{Re}\ h \geq 0 \) on \( U \) and \( \text{Re}\ h(\zeta) \to \infty \) as \( \zeta \to z, \zeta \in U \), for each \( \zeta \in F \).

The rest of the proof follows Theorem 1.

Again we observe that if \( A(U) \) is pointwise boundedly dense in \( H^\alpha(U) \) then the interpolation can be achieved by functions in \( H^\alpha_{F \setminus \{0\}}(U) \). Moreover under the same assumption the converse to Theorem 2 holds, for if \( f \) is as in the definition of peak interpolation set, with \( V \) chosen so that \( y \not\in V \), and \( g = 1 \), then we can choose a neighborhood \( W \) of \( F \) so that \( |1 - f| < \epsilon \) on \( U \cap W \); by Theorem 5.1 of [1] we can approximate \( f \) to within \( \epsilon \) on compact subsets of \( W \) by a sequence \( \{f_n\} \) in \( A(U) \) with \( \|f_n\| \leq 1 \), so that \( \mu(W) \) is small for all \( \mu \in M_y \).

The question naturally arises: suppose \( \mu(F) = 0 \) for all \( \mu \in M_y \). Must there exist open sets \( V_n \supseteq F \) such that \( \mu(V_n) \to 0 \) uniformly for \( \mu \in M_y \)? This is easily verified if \( F \) is \( \sigma \)-compact (in this case the conclusion of Theorem 2 can be deduced from the fact that each compact subset of \( F \) is a peak interpolation set for \( A(U) \)). We have no information of the general case.

**Lemma 2.** Let \( F \) be a subset of \( \partial U \) such that for each \( z \in F \) there exists \( \delta > 0 \) such that \( F_z = F \cap \{w: |w - z| \leq \delta/2\} \) is a peak interpolation set for \( H^\alpha_{F \setminus \Delta(z, \delta)}(U \cap \Delta(z, \delta)) \), then \( F \) is a peak interpolation set for \( H^\alpha(U) \).

**Proof.** First we show that \( F_z \) is a peak interpolation set for \( H^\alpha_F(U) \). Let \( g \) be a bounded continuous function on \( F_z \), let \( \epsilon > 0 \), and let \( V \) be an open neighborhood of \( F_z \). Choose \( f \in H^\alpha_{F \setminus \Delta(z, \delta)}(U \cap \Delta(z, \delta)) \) such that \( f = g \) on \( F_z \), \( \|f\| = \|g\| \), and \( |f| < \epsilon \) outside \( V \cap \{w: |w - z| < 3\delta/4\} \).

Choose a continuously differentiable function \( \varphi \) such that \( \varphi = 1 \) in a neighborhood of \( \{w: |w - z| \leq 3\delta/4\} \) and \( \text{supp}\ \varphi \subseteq \{w: |w - z| < \delta\} \). Define

\[
 f_\varphi(w) = f(w)\varphi(w) + \frac{1}{\pi} \int_{U \cap \Delta(z, \delta)} \frac{f(f)}{\zeta - w} \frac{f(\zeta)}{\zeta - w} dm(\zeta)
\]
where $f(w)$ is defined to be zero outside $(F \cup U) \cap \Delta(z, \delta)$. Then $f_i \in H^\infty_{\Delta S}(U)$ and given $t > 0$ we can choose $\varepsilon > 0$ so that $\|f_i - f\| < t$. Moreover $\|f_i\| \leq A\|f\|$, where $A$ is an absolute constant. (See [4], p. 210.) Then we have $|f_i - f| < \varepsilon$ on $F_i$ and $|f_i| < \varepsilon$ on $U \setminus V$. It now follows by a standard argument (see e.g. [2], Theorem 1), that $F_i$ is a peak interpolation set for $H^\infty_{\Delta S}(U)$.

Now let $V$ be an open set containing $F$. Shrinking $V$ if necessary we may suppose that $V$ is contained in the union of the discs $\Delta(z, \delta)$, $z \in F$, constructed above. This implies that for any compact set $K \subseteq V$, we have $K \cap F \subseteq \bigcup_{i=1}^n F_i$, for some $z_i, \ldots, z_n \in F$, which easily implies that $K \cap F$ is a peak interpolation set for $H^\infty_{\Delta S}(U)$. The lemma now follows by the argument used to deduce Theorem 1 from Lemma 3 in [2].

We say that $U$ is locally simple connected at a point $z \in \partial U$ if there exists $\delta > 0$ such that $C \setminus (U \cap \Delta(z, \delta))$ is connected. For example, if the diameters of the components of $C \setminus U$ are bounded away from zero then $U$ is locally simply connected at each point of $\partial U$. (Note that $U \cap \Delta(z, \delta)$ is not required to be connected; we only require that each component be simply connected.)

**Theorem 3.** Let $S$ be a subset of $\overline{U}$ such that $U$ is locally simply connected at each point of $S \cap \partial U$. Then $S$ is an interpolation set for $H^\infty_{\Delta S}(U)$ if and only if:

(i) $U \cap S$ is an interpolating sequence for $H^\infty(U)$,

(ii) $S \cap \partial U$ has zero harmonic measure for each point of $U$, with respect to $U$.

**Proof.** Assume first that $S$ is an interpolation set for $H^\infty_{\Delta S}(U)$. A simple normal family argument shows that (i) holds.

Now let $y \in U$ and choose $f \in H^\infty_{\Delta S}(U)$ such that $\|f\| \leq 1$, $f = 0$ on $S \cap \partial U$, and $f(y) \neq 0$. Then $- \log |f|$ is a positive superharmonic function on $U$, tending to $\infty$ at each point of $S \cap \partial U$, and finite at $y$. Thus $S \cap \partial U$ has zero harmonic measure for $y$ with respect to $U$ which proves (ii).

Now assume (i) and (ii) hold, and let $f$ be a bounded continuous function on $S$ with $\|f\| \leq 1$. By Lemma 2 and Theorem 1, $\partial U \cap S$ is a peak interpolation set for $H^\infty_{\Delta S}(U)$ so we can find $h \in H^\infty_{\Delta S}(U)$ with $\|h\| \leq 1$ and $h = f$ on $\partial U \cap S$. Let $g_i = f - h$ on $S$, then $g_i = 0$ on $\partial U \cap S$ so that for any $\varepsilon > 0$ we can find $F \in H^\infty_{\Delta S}(U)$ so that $F = 0$ on $\partial U \cap S$, $|1 - F| < \varepsilon$ on $S \cap M$, $|F| \leq 2$. Then $|F_i - g_i| \leq 3\varepsilon$ on $S$. Choose $G \in H^\infty(U)$ so that $\|G\| \leq M |g_i| \leq 2M$ and $G = g_i$ on $S \cap M$, where $M$ is the interpolation constant of $S \cap M$; then $FG \in H^\infty_{\Delta S}(U)$ and satisfies $|FG - g_i| \leq 3\varepsilon$ on $S$. Let $\tilde{f} = FG + h \in H^\infty_{\Delta S}(U)$, then $\|\tilde{f} - f\| \leq 3\varepsilon$ on $S$ and $\|\tilde{f}\| \leq 4M + 1$, so the theorem.
follows by choosing $\varepsilon$ with $3\varepsilon < 1$.

References


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