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**INTERPOLATION SETS FOR ANALYTIC FUNCTIONS**

ALEXANDER MUNRO DAVIE AND ARNE STRAY

## INTERPOLATION SETS FOR ANALYTIC FUNCTIONS

A. M. DAVIE AND A. STRAY

**Let  $U$  be a bounded open subset of the complex plane  $C$ . Criteria are obtained for a subset  $E$  of  $\bar{U}$  to be an interpolation set for the algebra of all bounded analytic functions on  $U$  extending continuously to  $E$ .**

In the case where  $U$  is the open unit disc  $\Delta$ , this problem was treated by Détraz [3]. She showed that if  $E$  is a subset of the unit circle  $T$  then every bounded continuous function on  $E$  is the restriction of a bounded analytic function on  $\Delta$ , extending continuously to  $E \cup (T/\bar{E})$ , if and only if  $E$  has measure zero. We extend this result to any  $U$  with connected complement, replacing linear measure on  $T$  by harmonic measure (Theorem 1). For the general case the same method yields a criterion in terms of representing measures for  $A(U)$  (Theorem 2). Finally in Theorem 3 we use a localization argument to sharpen Theorem 1 and also treat the case where  $E$  contains points of  $U$  as well as  $\partial U$ .

NOTATION. If  $S$  is a plane set then  $\bar{S}$  denotes its closure and  $\partial S$  its boundary.  $A(U)$  denotes the algebra of all continuous functions on  $\bar{U}$ , analytic on  $U$ ;  $H^\infty(U)$  denotes the algebra of all bounded analytic functions on  $U$ ;  $H_E^\infty(U)$  denotes the algebra of all bounded continuous functions on  $U \cup E$  which are analytic on  $U$ . If  $y \in \bar{U}$ , a *representing measure* for  $y$  with respect to  $A(U)$  is a positive borel measure  $\mu$  on  $\bar{U}$  such that  $f(y) = \int f d\mu$  for all  $f \in A(U)$ . We denote by  $\|f\|$  the supremum of the function  $f$  over its domain of definition.  $\Delta(z, \delta)$  denotes the disc with center  $z$  and radius  $\delta$ .

We say that a set  $S \subseteq U \cup E$  is an *interpolation set* for  $H_E^\infty(U)$  if for any bounded continuous  $f$  on  $S$  we can find  $g \in H_E^\infty(U)$  with  $g|_S = f$ . We say  $S$  is a *peak interpolation set* for  $H_E^\infty(U)$  if for any bounded continuous  $f$  on  $S$ , and open set  $V \supseteq S$ , and any  $\varepsilon > 0$ , we can find  $g \in H_E^\infty(U)$  with  $g|_S = f$ ,  $\|g\| \leq \|f\|$ , and  $|g| < \varepsilon$  on  $U \setminus V$ .

**THEOREM 1.** *Suppose  $C \setminus U$  is connected. Let  $F$  be a subset of  $\partial U$  with zero harmonic measure for each point of  $U$  (with respect to  $U$ ). Then  $F$  is a peak interpolation set for  $H_F^\infty(U)$ .*

The proof follows from the following lemma.

**LEMMA.** *With  $U$  and  $F$  as in the theorem, let  $X$  be a compact*

subset of  $\bar{U}$ ,  $W$  a neighborhood of  $X$ , and  $\varepsilon > 0$ . Then we can find  $f \in H_F^\infty$  with  $\|f\| \leq 2$ ,  $|1 - f| < \varepsilon$  on  $F \cap X$ , and  $|f| < \varepsilon$  on  $U \setminus W$ .

*Proof.* We can find a positive harmonic function  $\sigma$  on  $U$  such that  $\sigma(\zeta) \rightarrow \infty$  as  $\zeta \rightarrow z$ ,  $\zeta \in U$ , for each  $z \in F$ . Let  $\tau$  be a harmonic conjugate to  $\sigma$  on  $U$ , and let  $\theta = \sigma + i\tau$ , an analytic function on  $U$ . Put  $h = \theta/(\theta + 1)$ . Since  $\theta$  has positive real part,  $h \in H^\infty(U)$  with  $\|h\| \leq 1$ . Moreover  $h(\zeta) \rightarrow 1$  as  $\zeta \rightarrow z$ ,  $\zeta \in U$ , for each  $z \in F$ ; hence we can regard  $h$  as an element of  $H_F^\infty(U)$ , with  $h = 1$  on  $F$ .

Now let  $\varphi$  be a continuously differentiable function which is 1 on a neighborhood of  $X$  and zero outside  $W$ , with  $\|\varphi\| = 1$ . Then the function

$$g_n(\zeta) = \varphi(\zeta)h^n(\zeta) + \frac{1}{\pi} \int_U \frac{h^n(z)}{z - \zeta} \frac{\partial \varphi}{\partial \bar{z}} dm(z)$$

is in  $H_F^\infty(U)$ . (See [4], p. 210.) Moreover

$$\|g_n - \varphi h^n\| \leq \frac{1}{\pi} \left\| \frac{\partial \varphi}{\partial \bar{z}} \right\| \|h^n\|_{L^3(U)} \sup_\zeta \left\| \frac{1}{z - \zeta} \right\|_{L^3 \setminus 2(U)}.$$

The last term is bounded by a constant depending only on  $U$ , and  $\|h^n\|_{L^3} \rightarrow 0$  as  $n \rightarrow \infty$  since  $|h| < 1$  in  $U$ . Choose  $n$  so that  $\|g_n - \varphi h^n\| < \varepsilon$  and put  $f = g_n$ . Then  $f$  satisfies the requirements of the lemma.

Theorem 1 follows from the lemma in exactly the same way as Theorem 1 follows from Lemma 2 in [2]. (For an alternative approach see the proof of Theorem 4.3 of [3]).

We observe that if  $A(U)$  is pointwise boundedly dense in  $H^\infty(U)$  then using Theorem 2.1 of [5] we can modify the function  $f$  in the lemma so that it is in  $H_{F \cup (\partial U \setminus \bar{F})}^\infty(U)$ . Then we can prove that  $F$  is a peak interpolation set for  $H_{F \cup (\partial U \setminus \bar{F})}^\infty(U)$ .

In the general situation (where  $C \setminus U$  need not be connected) the same method yields the following result. If  $y \in U$  we denote by  $M_y$  the set of all (positive) representing measures for  $y$  with respect to  $A(U)$  on  $\bar{U}$ . We assume  $U$  is connected.

**THEOREM 2.** *Let  $y \in U$  and  $F \subseteq \partial U$ . Suppose there is a decreasing sequence  $\{V_n\}$  of open sets containing  $F$ , such that  $\mu(V_n) \rightarrow 0$  uniformly for  $\mu \in M_y$ .*

*Then  $F$  is a peak interpolation set for  $H_F^\infty(U)$ .*

*Proof.* We may suppose  $\mu(V_n) < 2^{-n}$  for each  $\mu \in M_y$ . For each  $n$  let  $\{g_{nk}\}$  be an increasing sequence of nonnegative continuous functions converging to the characteristic function of  $V_n$ . Then  $\int g_{nk} d\mu < 2^{-n}$

for  $\mu \in M_y$  and so by Theorem II 2.1 of [4], we can find  $h_{nk} \in A(U)$  with  $\operatorname{Re} h_{nk} \geq g_{nk}$  on  $\bar{U}$ ,  $\operatorname{Re} h_{nk}(y) < 2^{-n}$ , and we can also suppose  $\operatorname{Im} h_{nk}(y) = 0$ . Passing to a subsequence we have  $h_{nk} \rightarrow h_n$  as  $k \rightarrow \infty$ , pointwise in  $U$ , where  $h_n$  is analytic in  $U$  with  $\operatorname{Re} h_n \geq 1$  on  $V_n \cap U$  and  $|h_n(y)| \leq 2^{-n}$ ,  $\operatorname{Re} h_n \geq 0$  on  $U$ ,  $\operatorname{Im} h_n = 0$ . By Harnack's inequalities the series  $\sum_{n=1}^{\infty} h_n$  converges pointwise on  $U$  to an analytic function  $h$  such that  $\operatorname{Re} h \geq 0$  on  $U$  and  $\operatorname{Re} h(\zeta) \rightarrow \infty$  as  $\zeta \rightarrow z$ ,  $\zeta \in U$ , for each  $\zeta \in F$ .

The rest of the proof follows Theorem 1.

Again we observe that if  $A(U)$  is pointwise boundedly dense in  $H^\infty(U)$  then the interpolation can be achieved by functions in  $H_{F \cup (\partial U \setminus \bar{F})}^\infty(U)$ . Moreover under the same assumption the converse to Theorem 2 holds, for if  $f$  is as in the definition of peak interpolation set, with  $V$  chosen so that  $y \notin V$ , and  $g = 1$ , then we can choose a neighborhood  $W$  of  $F$  so that  $|1 - f| < \varepsilon$  on  $U \cap W$ ; by Theorem 5.1 of [1] we can approximate  $f$  to within  $\varepsilon$  on compact subsets of  $W$  by a sequence  $\{f_n\}$  in  $A(U)$  with  $\|f_n\| \leq 1$ , so that  $\mu(W)$  is small for all  $\mu \in M_y$ .

The question naturally arises: suppose  $\mu(F) = 0$  for all  $\mu \in M_y$ . Must there exist open sets  $V_n \supseteq F$  such that  $\mu(V_n) \rightarrow 0$  uniformly for  $\mu \in M_y$ ? This is easily verified if  $F$  is  $\sigma$ -compact (in this case the conclusion of Theorem 2 can be deduced from the fact that each compact subset of  $F$  is a peak interpolation set for  $A(U)$ ). We have no information of the general case.

**LEMMA 2.** *Let  $F$  be a subset of  $\partial U$  such that for each  $z \in F$  there exists  $\delta > 0$  such that  $F_z = F \cap \{w : |w - z| \leq \delta/2\}$  is a peak interpolation set for  $H_{F \cap \Delta(z, \delta)}^\infty(U \cap \Delta(z, \delta))$ , then  $F$  is a peak interpolation set for  $H_F^\infty(U)$ .*

*Proof.* First we show that  $F_z$  is a peak interpolation set for  $H_{F_z}^\infty(U)$ . Let  $g$  be a bounded continuous function on  $F_z$ , let  $\varepsilon > 0$ , and let  $V$  be an open neighborhood of  $F_z$ . Choose  $f \in H_{F \cap \Delta(z, \delta)}^\infty(U \cap \Delta(z, \delta))$  such that  $f = g$  on  $F_z$ ,  $\|f\| = \|g\|$ , and  $|f| < \varepsilon$  outside  $V \cap \{w : |w - z| < 3\delta/4\}$ .

Choose a continuously differentiable function  $\varphi$  such that  $\varphi = 1$  in a neighborhood of  $\{w : |w - z| \leq 3\delta/4\}$  and  $\operatorname{supp} \varphi \subseteq \{w : |w - z| < \delta\}$ . Define

$$f_1(w) = f(w)\varphi(w) + \frac{1}{\pi} \int_{U \cap \Delta(z, \delta)} \frac{f(f)}{\zeta - w} \frac{f(\zeta)}{\zeta - w} dm(\zeta)$$

where  $f(w)$  is defined to be zero outside  $(F \cup U) \cap \Delta(z, \delta)$ . Then  $f_1 \in H_F^\infty(U)$  and given  $t > 0$  we can choose  $\varepsilon > 0$  so that  $\|f_1 - f\| < t$ . Moreover  $\|f_1\| \leq A\|f\|$ , where  $A$  is an absolute constant. (See [4], p. 210.) Then we have  $|f_1 - f| < \varepsilon$  on  $F_z$  and  $|f_1| < \varepsilon$  on  $U \setminus V$ . It now follows by a standard argument (see e.g. [2], Theorem 1), that  $F_z$  is a peak interpolation set for  $H_F^\infty(U)$ .

Now let  $V$  be an open set containing  $F$ . Shrinking  $V$  if necessary we may suppose that  $V$  is contained in the union of the discs  $\Delta(z, \delta)$ ,  $z \in F$ , constructed above. This implies that for any compact set  $K \subseteq V$ , we have  $K \cap F \subseteq \bigcup_{i=1}^n F_{z_i}$  for some  $z_1, \dots, z_n \in F$ , which easily implies that  $K \cap F$  is a peak interpolation set for  $H_F^\infty(U)$ . The lemma now follows by the argument used to deduce Theorem 1 from Lemma 3 in [2].

We say that  $U$  is locally simple connected at a point  $z \in \partial U$  if there exists  $\delta > 0$  such that  $C \setminus (U \cap \Delta(z, \delta))$  is connected. For example, if the diameters of the components of  $C \setminus U$  are bounded away from zero then  $U$  is locally simply connected at each point of  $\partial U$ . (Note that  $U \cap \Delta(z, \delta)$  is not required to be connected; we only require that each component be simply connected.)

**THEOREM 3.** *Let  $S$  be a subset of  $\bar{U}$  such that  $U$  is locally simply connected at each point of  $S \cap \partial U$ . Then  $S$  is an interpolation set for  $H_{S \cap \partial U}^\infty(U)$  if and only if:*

- (i)  $U \cap S$  is an interpolating sequence for  $H^\infty(U)$ ,
- (ii)  $S \cap \partial U$  has zero harmonic measure for each point of  $U$ , with respect to  $U$ .

*Proof.* Assume first that  $S$  is an interpolation set for  $H_{S \cap \partial U}^\infty(U)$ . A simple normal family argument shows that (i) holds.

Now let  $y \in U$  and choose  $f \in H_{S \cap \partial U}^\infty(U)$  such that  $\|f\| \leq 1$ ,  $f = 0$  on  $S \cap \partial U$ , and  $f(y) \neq 0$ . Then  $-\log |f|$  is a positive superharmonic function on  $U$ , tending to  $\infty$  at each point of  $S \cap \partial U$ , and finite at  $y$ . Thus  $S \cap \partial U$  has zero harmonic measure for  $y$  with respect to  $U$  which proves (ii).

Now assume (i) and (ii) hold, and let  $f$  be a bounded continuous function on  $S$  with  $\|f\| \leq 1$ . By Lemma 2 and Theorem 1,  $\partial U \cap S$  is a peak interpolation set for  $H_{\partial U \cap S}^\infty(U)$  so we can find  $h \in H_{\partial U \cap S}^\infty(U)$  with  $\|h\| \leq 1$  and  $h = f$  on  $\partial U \cap S$ . Let  $g_1 = f - h$  on  $S$ , then  $g_1 = 0$  on  $\partial U \cap S$  so that for any  $\varepsilon > 0$  we can find  $F \in H_{\partial U \cap S}^\infty(U)$  so that  $F = 0$  on  $\partial U \cap S$ ,  $|1 - F| < \varepsilon$  on  $\{z \in S: |g_1(z)| > \varepsilon\}$ ,  $\|F\| \leq 2$ . Then  $|Fg_1 - g_1| \leq 3\varepsilon$  on  $S$ . Choose  $G \in H^\infty(U)$  so that  $\|G\| \leq M\|g_1\| \leq 2M$  and  $G = g_1$  on  $S \cap U$ , where  $M$  is the interpolation constant of  $S \cap U$ ; then  $FG \in H_{\partial U \cap S}^\infty$  and satisfies  $|FG - g_1| \leq 3\varepsilon$  on  $S$ . Let  $\tilde{f} = FG + h \in H_{\partial U \cap S}^\infty(U)$ , then  $|\tilde{f} - f| \leq 3\varepsilon$  on  $S$  and  $\|\tilde{f}\| \leq 4M + 1$ , so the theorem

follows by choosing  $\varepsilon$  with  $3\varepsilon < 1$ .

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