

# Pacific Journal of Mathematics

**STRUCTURE OF RIGHT SUBDIRECTLY IRREDUCIBLE  
RINGS. II**

M. G. DESHPANDE

## STRUCTURE OF RIGHT SUBDIRECTLY IRREDUCIBLE RINGS II

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The object of this paper is to determine the structure and properties of right subdirectly irreducible rings which are either local or self-injective. The rings in the latter class form a special case of the so-called right *PF* rings. By employing the notion of Feller's *X*-rings, it is proved that right *PF X*-rings are finite direct sums of full matrix rings over self-injective right subdirectly irreducible rings. Thus, whether or not right *PF X*-rings are left *PF* depends on the answer to the same question for the more elementary case of self-injective right subdirectly irreducible rings.

For a discussion of artinian and noetherian *RSI* rings, see [2].

1. Notation and preliminaries. All rings considered have an identity and all modules are unitary. A module  $M_R$  is *R*-subdirectly irreducible if the intersection of all nonzero submodules of  $M$  is nonzero, which will then be called the heart of  $M_R$ . A ring *R* is *RSI* (right subdirectly irreducible) if  $R_R$  is *R*-subdirectly irreducible. The heart  $H$  of a *RSI* ring *R* is a two sided ideal. These and some of the following definitions and observations are given in [2] and we rewrite them for completeness. We will always use the following notation in connection with a *RSI* ring *R*.  $H = \text{heart}$ ,  $N = H^l = \{x \in R: xH = 0\}$ ,  $D = \text{Hom}_R(H_R, H_R)$ ,  $\hat{R} = \text{injective hull of } R_R$ ,  $K = \text{Hom}_R(\hat{R}, \hat{R})$  and  $L = \{f \in K: \ker f \neq 0\}$ . In addition, for a local ring *R*, *J* will always denote the unique maximal right ideal. A ring *R* will be termed self-injective if  $R_R$  is injective.

We state the following theorem showing the relationship between  $R, N, H, D, K, L$  which has been proved in [2, p. 319].

**THEOREM 1.1.** *If  $R$  is *RSI*, then  $R/N$  is isomorphic to a subring of the division ring  $D$  and  $D \cong K/L$ .*

In connection with  $QF - 1$  algebras, faithful indecomposable modules play an important role. In the following proposition we prove that a *RSI* ring has a unique faithful indecomposable injective module. In this respect, it may be remarked that an artinian semi-simple ring which is not simple is an example of a ring for which faithful indecomposable injectives don't exist; while over the ring of integers, for each prime  $p$ , by using [12, p. 145, Th. 7] or otherwise,

one can verify that  $Z_{p^\infty}$  is a faithful indecomposable injective module, and obviously they are all nonisomorphic.

**PROPOSITION 1.2.** *A RSI ring  $R$  has, up to isomorphism, a unique faithful indecomposable injective module.*

*Proof.*  $\hat{R}_R$  is certainly faithful and injective. It is also indecomposable because every nonzero submodule contains  $H$ . Let  $M_R$  be any other such module. If  $h \in H$  is a nonzero element, then  $Mh \neq 0$  because  $M$  is faithful. Thus, for some  $m \in M$ ,  $mh \neq 0$ . The mapping  $x \rightarrow mx$  is then an isomorphism on  $H$  to  $mH$  which can be extended to an isomorphism of  $\hat{R}$  into  $M$ . If  $N$  be the image of  $\hat{R}$  under this isomorphism,  $N$  is injective and hence a direct summand of  $M$ . By indecomposability of  $M$ , we have  $N = M$  and therefore  $\hat{R} \cong M$ .

2. *Local RSI rings.* We recall that a ring  $R$  is a left  $S$ -ring in the sense of F. Kasch [5, p. 455] if each proper right ideal has a nonzero left annihilator. It is known that [11, p. 412, Th. 2.9] a (right) self-injective ring is local iff it is right uniform. In the following an analogue of this is considered for *RSI* rings.

**PROPOSITION 2.1.** *A self-injective ring  $R$  is RSI iff it is a local left  $S$ -ring.*

*Proof.* If  $R$  is self-injective and *RSI*, then it is right uniform and thus local by the above. If  $h \in H$ ,  $h \neq 0$ ; then  $h^r$  is a maximal right ideal and so must be the unique maximal right ideal  $J$ . Thus  $J^i \neq 0$ . If  $A$  is any proper right ideal, then  $A \subseteq J$  implies that  $0 \neq J^i \subseteq A^i$  and hence  $R$  is a left  $S$ -ring. Conversely,  $R$  is self-injective and local implies that it is right uniform. Since  $R$  is a left  $S$ -ring, by [6, p. 237, 2.1]  $R$  contains a copy of the simple  $R$ -module  $R/J$ . Clearly, a right uniform ring containing a minimal right ideal must be *RSI*.

The above proof shows that a *RSI* local ring is necessarily a left  $S$ -ring. We now prove the following theorem<sup>1</sup> which will imply that a *LSI* (left subdirectly irreducible) left  $S$ -ring is local.

**THEOREM 2.2.** *Let  $R$  be a ring. For the three statements*

- (i)  *$R$  is local,*
- (ii) *there exists a bimodule  ${}_T M_R$  such that  $A^i$  in  ${}_T M$  is nonzero for any proper right ideal  $A$  of  $R$ , and  ${}_T M$  is subdirectly irreducible,*

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<sup>1</sup> The author is obliged to the referee for this version of the theorem and other helpful suggestions.

(iii)  ${}_K\hat{R}$  is subdirectly irreducible and  $R$  is a left  $S$ -ring; we have (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i). If, further  $R$  is  $RSI$ , then we also have (i)  $\Rightarrow$  (iii).

*Proof.* (iii)  $\Rightarrow$  (ii) is obvious.

(ii)  $\Rightarrow$  (i). If  $M_0$  is the heart of  ${}_T M$ , then  $M_0 \subseteq A^l$  for any proper right ideal  $A$  of  $R$ . Thus  $A \subseteq A^{lr} \subseteq M_0^r$  which proves that  $R$  is local with  $J = M_0^r$ .

Now we assume that  $R$  is  $RSI$  and prove (i)  $\Rightarrow$  (iii). That  $R$  is then a left  $S$ -ring is already noted above. Now, let  $h$  and  $a$  be any two nonzero elements respectively from  $H$  and  $\hat{R}$ . Since  $R$  is local and  $h^r = J$ , we have  $a^r \subseteq h^r$ . Thus  $ax \rightarrow hx$  is a homomorphism of  $aR$  into  $H$  which can be extended to some element  $f$  of  $K$ . Then  $h = f(a) \in Ka$  which shows that each nonzero submodule of  ${}_K\hat{R}$  contains  $h$  and so  ${}_K\hat{R}$  is subdirectly irreducible. In fact, it can be easily seen that  ${}_K\hat{R}$  and  $R_R$  have the same heart  $H$ .

Since a  $LSI$ , left  $S$ -ring  $R$  satisfies condition (ii) of the above theorem, we have in particular,

**COROLLARY 2.3.** *For a  $RSI$  and  $LSI$  ring, the following are equivalent.*

- (i)  $R$  is local
- (ii)  $R$  is a left (right)  $S$ -ring.

**3. Self-injective  $RSI$  rings.** In a local  $RSI$  ring  $R$ , the left annihilator  $N$  of  $H$ , and the right annihilator  $J$  of  $H$  (which will be the unique maximal right ideal) need not coincide, though obviously we must have  $N \subseteq J$ . We show by an example<sup>2</sup> that this inclusion can be proper and then prove that for self-injective rings,  $N = J$ .

**EXAMPLE 3.1.** Let  $F = k(x_1, \dots, x_n, \dots)$  be the field of rational functions in  $x_1, x_2, \dots$  over the field  $k$  of real numbers and  $L$  the subring of fractions with denominators prime to  $x_1$ . Let  $\alpha$  and  $\beta$  denote an epimorphism and a monomorphism respectively on  $L$  to  $F$  given by  $f(x_1, \dots, x_n)^\alpha = f(0, x_1, \dots, x_{n-1})$  and  $\beta = \text{inclusion}$ . Let  $R$  be the ring defined by  $(R, +) = L \oplus F$  and  $(a, b)(c, d) = (ac, bc^\alpha + a^\beta d)$ . If  $h$  denotes the element  $(0, 1)$  of  $R$ , then every element of  $R$  can be written as  $a + hb$  and  $h^2 = 0$ . It can be verified that  $R$  is  $RSI$  with heart  $H = hR$  and that  $a + hb$  is a unit in  $R$  iff  $a$  is a unit in  $L$ . Since  $L$  is a local ring, so is  $R$ . For this ring  $R$ ,  $N = H^l = \{(0, b) : b \in F\}$  and  $J = \{(a, b) : a^\alpha = 0\}$ . Thus  $N \subsetneq J$ .

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<sup>2</sup> Professor P. M. Cohn has kindly communicated this example to the author.

PROPOSITION 3.2. *If  $R$  is self-injective and  $RSI$ ,  $R/N \cong D$ .*

*Proof.* It was shown in [2] that  $f: R/N \rightarrow D$  defined by  $f(a + N) = f_a$ , where  $f_a: H \rightarrow H$  is the left multiplication by  $a$ , is a monomorphism. If  $R$  is self-injective and  $d: H \rightarrow H$  is any element of  $D$ , then there exists an element  $a \in R$  such that  $d = f_a = f(a + N)$  and  $R/N \cong D$ .

COROLLARY 3.3. *If  $R$  is self-injective and  $RSI$ ,  $N = J$ .*

*Proof.* Self-injective and  $RSI$  implies local. Since  $R/N$  is a division ring,  $N$  must be a maximal right ideal and hence  $N = J$ .

We now give a characterization of  $RSI$  rings in the class of self-injective rings which is analogous to McCoy's Theorem [9, p. 382, Th. 1] for the commutative subdirectly irreducible rings.

THEOREM 3.4. *Let  $R$  be self-injective. Then  $R$  is  $RSI$  iff there exists a nonzero principal right ideal  $X = xR$  and an ideal  $Y$  in  $R$  such that*

- (i)  $Y^l = X$ , so that  $X$  is a two sided ideal,
- (ii)  $X^l = Y$ ,
- (iii)  $R/Y$  is a division ring, and
- (iv) *If  $a$  is an element of  $Y$  not in  $X$ , there exists an element  $b$  of  $Y$  not in  $X$ , such that  $ab = x$ .*

*Proof.* ( $\Rightarrow$ ). If  $R$  is  $RSI$ , we choose  $X$  to be  $H = hR$  and  $Y = N$ . Then (ii) holds by definition of  $N$  and (iii) is a consequence of 3.2 above. Since  $R$  is in this case local with  $N =$  unique maximal right ideal,  $aN = 0$  if and only if  $aR$  is a minimal right ideal. Thus  $N^l = H$  which proves (i). Now let  $a \in N$  such that  $a \notin N$ . Since  $H \subseteq aR$  we have  $h = ab$  for some  $b \in R$ . (ii) implies that  $b \notin H$ . Also if  $b \in N$  then  $b$  is a unit which implies  $a = hb^{-1} \in H$  contradicting the hypothesis on  $a$ .

( $\Leftarrow$ ). By assuming (i), (ii), (iii), and (iv) we will show that every nonzero right ideal of  $R$  contains the fixed element  $x$  of  $R$ . Accordingly, let  $a$  be a nonzero element in a right ideal  $A$  of  $R$ . If  $a \notin Y$ , by (iii) we have  $1 - ay \in Y$  for a suitable  $y \in R$ . Then by (ii),  $(1 - ay)x = 0$  which implies  $x = ayx \in aR \subseteq A$ . If  $a \in Y$  and  $a \notin X$  then by (iv) we have  $x \in aR \subseteq A$ . Lastly, if  $a \in X$ , then  $a = xc$  for some  $c \in R$ .  $c$  cannot be in  $Y$  because then we would have  $a = xc \in XY = 0$ . Thus, again by (iii)  $1 - cd = u \in Y$  for some  $d \in R$ . Then  $ad = xcd = x(1 - u) = x$  which proves that  $x \in A$ . Thus  $R$  is  $RSI$  with heart  $X = xR$ .

4. Right  $PF$   $X$ -rings. Utumi [14, p. 56] defined a ring  $R$  to be right  $PF$  if every faithful right  $R$ -module is completely faithful.

These rings afford a nice generalization of  $QF$  rings and have been discussed by Azumaya [1], Osofsky [10], Kato [6, 7], Utumi [13, 14] and others. A ring  $R$  is right  $PF$  if and only if [1, p. 701, Th. 6] it is a finite direct sum of indecomposable injective right ideals each of which contains a unique minimal right ideal. It is not known whether a right  $PF$  ring is also left  $PF$ . Clearly, a self-injective  $RSI$  ring is right  $PF$ . Either by using the theory of right  $PF$  rings developed by the authors mentioned above, or directly as a consequence of our Theorem 2.2 it can be seen that a self-injective  $RSI$  ring is left subdirectly irreducible. (In this case,  $R$  is local,  $\hat{R} = R$  and  $K \cong R$ ). We state this as a

PROPOSITION 4.1. *A self-injective RSI ring is LSI.*

The following definition is due to Feller [3, p. 20, 2.2].

DEFINITION 4.2. *A ring  $R$  is called an  $X$ -ring if for every pair  $e, f$  of primitive idempotents such that  $eR \not\cong fR$ ;  $a \in eR$  and  $a' \in fR$  implies  $a'fR = 0$ .*

We state the following useful lemma whose proof is straightforward.

LEMMA 4.3. *If  $A$  and  $B$  are rings and  $R = A \oplus B$  is the ring theoretic direct sum, then  $R$  is a self-injective ring iff each of  $A$  and  $B$  are self-injective rings.*

We are now in a position to prove the following

THEOREM 4.4. *A ring  $R$  is a right  $PF$   $X$ -ring if and only if  $R$  is isomorphic to a finite direct sum of full matrix rings over self-injective  $RSI$  rings. Further, this decomposition is unique.*

*Proof.* If  $S$  is a self-injective  $RSI$  ring with heart  $H$ , then the  $n \times n$  matrix ring  $S_n$  is self-injective [13, p. 172, Th. 8.3] and is the direct sum of indecomposable right ideals  $e_{ii}S_n, i = 1, 2, \dots, n$  each of which contains a unique minimal right ideal  $e_{ii}H_n$ . Consequently  $S_n$  is a right  $PF$  ring which is trivially an  $X$ -ring. Now, by using 4.3 we can see that any finite direct sum of such rings is again right  $PF$ . In order to show that it is also an  $X$ -ring, it is enough to remark that  $eR$  and  $fR$  are nonisomorphic only if they belong to different matrix rings, in which case  $eRfR = 0$ .

Conversely, Let  $R$  be a right  $PF$   $X$ -ring. Then

4.5. 
$$R = e_1R \oplus \dots \oplus e_nR$$

where  $e_1, \dots, e_n$  are primitive and let us assume that  $e_1R, \dots, e_kR$  ( $k \leq n$ ) denote a complete set of nonisomorphic right ideals among the  $n$  summands in 4.5. Since each  $e_iR$  is indecomposable and injective, it is right uniform. By the same argument as in [3, p. 20, Th. 2.3] we conclude that  $R = A_1 \oplus \dots \oplus A_k$  where each  $A_i$  is the sum of all summands in 4.5 which are isomorphic to  $e_iR$  and  $A_1, \dots, A_k$  are all two sided ideals. Further, each  $A_i$  is isomorphic to a full matrix ring over  $e_iRe_i$ . That this decomposition is unique follows from [4, p. 42, Th. 1]. Also, by 4.3, each of the rings  $A_1, \dots, A_k$  is self-injective and hence by [13, p. 172, Th. 8.3] so are the rings  $e_iRe_i$ . Finally, if  $H_i$  is the unique minimal right ideal of  $R$  contained in  $e_iR$ , it can be verified that  $e_iRe_i$  is *RSI* with heart  $e_iH_ie_i$ . This prove the Theorem.

As a consequence of this theorem, a right *PF* *X*-ring will be left *PF* (if and) only if the self-injective *RSI* rings over which matrix rings appear in the above decomposition are also left self-injective. The author does not know if such is always the case.

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Received January 12, 1971 and in revised form June 16, 1971. A portion of this paper is included in the author's doctoral dissertation under the supervision of Professor Edmund H. Feller, submitted to University of Wisconsin-Milwaukee, in June, 1969.

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