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The object of this paper is to determine the structure and properties of right subdirectly irreducible rings which are either local or self-injective. The rings in the latter class form a special case of the so-called right PF rings. By employing the notion of Feller's X-rings, it is proved that right PF X-rings are finite direct sums of full matrix rings over self-injective right subdirectly irreducible rings. Thus, whether or not right PF X-rings are left PF depends on the answer to the same question for the more elementary case of self-injective right subdirectly irreducible rings.

For a discussion of artinian and noetherian RSI rings, see [2].

1. Notation and preliminaries. All rings considered have an identity and all modules are unitary. A module M_R is R-subdirectly irreducible if the intersection of all nonzero submodules of M is nonzero, which will then be called the heart of M_R. A ring R is RSI (right subdirectly irreducible) if R_R is R-subdirectly irreducible. The heart H of a RSI ring R is a two sided ideal. These and some of the following definitions and observations are given in [2] and we rewrite them for completeness. We will always use the following notation is connection with a RSI ring R. $H =$ heart, $N = H^l = \{x \in R: xH = 0\}$, $D = \mathrm{Hom}_R(H_R, H_R)$, $\hat{R} =$ injective hull of R_R, $K = \mathrm{Hom}_R(\hat{R}, \hat{R})$ and $L = \{f \in K: \ker f \neq 0\}$. In addition, for a local ring R, J will always denote the unique maximal right ideal. A ring R will be termed self-injective if R_R is injective.

We state the following theorem showing the relationship between R, N, H, D, K, L which has been proved in [2, p. 319].

THEOREM 1.1. *If R is RSI, then R/N is isomorphic to a subring of the division ring D and $D \cong K/L$.*

In connection with $QF-1$ algebras, faithful indecomposable modules play an important role. In the following proposition we prove that a RSI ring has a unique faithful indecomposable injective module. In this respect, it may be remarked that an artinian semi-simple ring which is not simple is an example of a ring for which faithful indecomposable injectives don't exist; while over the ring of integers, for each prime p, by using [12, p. 145, Th. 7] or otherwise, one can verify that Z_{p^∞} is a faithful indecomposable injective module, and obviously they are all nonisomorphic.

PROPOSITION 1.2. *A RSI ring R has, up to isomorphism, a unique faithful indecomposable injective module.*

Proof. $\hat{R}_R$ is certainly faithful and injective. It is also indecomposable because every nonzero submodule contains H. Let M_R be any other such module. If $h \in H$ is a nonzero element, then $Mh \neq 0$ because M is faithful. Thus, for some $m \in M$, $mh \neq 0$. The mapping $x \to mx$ is then an isomorphism on H to mH which can be extended to an isomorphism of $\hat{R}$ into M. If N be the image of $\hat{R}$ under this isomorphism, N is injective and hence a direct summand of M. By indecomposability of M, we have $N = M$ and therefore $\hat{R} \cong M$.

2. Local *RSI* rings. We recall that a ring R is a left S-ring in the sense of F. Kasch [5, p. 455] if each proper right ideal has a nonzero left annihilator. It is known that [11, p. 412, Th. 2.9] a (right) self-injective ring is local iff it is right uniform. In the following an analogue of this is considered for *RSI* rings.

PROPOSITION 2.1. *A self-injective ring R is RSI iff it is a local left S-ring.*

Proof. If R is self-injective and *RSI*, then it is right uniform and thus local by the above. If $h \in H$, $h \neq 0$; then h^r is a maximal right ideal and so must be the unique maximal right ideal J. Thus $J^l \neq 0$. If A is any proper right ideal, then $A \subseteqq J$ implies that $0 \neq J^l \subseteqq A^l$ and hence R is a left S-ring. Conversely, R is self-injective and local implies that it is right uniform. Since R is a left S-ring, by [6, p. 237, 2.1] R contains a copy of the simple R-module R/J. Clearly, a right uniform ring containing a minimal right ideal must be *RSI*.

The above proof shows that a *RSI* local ring is necessarily a left S-ring. We now prove the following theorem[1] which will imply that a *LSI* (left subdirectly irreducible) left S-ring is local.

THEOREM 2.2. *Let R be a ring. For the three statements*

(i) *R is local,*

(ii) *there exists a bimodule* ${}_TM_R$ *such that* A^l *in* ${}_TM$ *is nonzero for any proper right ideal A of R, and* ${}_TM$ *is subdirectly irreducible,*

[1] The author is obliged to the referee for this version of the theorem and other helpful suggestions.

(iii) ${}_K\hat{R}$ *is subdirectly irreducible and* R *is a left S-ring; we have* (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). *If, further* R *is RSI, then we also have* (i) $\Rightarrow$ (iii).

Proof. (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). If M_0 is the heart of ${}_TM$, then $M_0 \subseteqq A^l$ for any proper right ideal A of R. Thus $A \subseteqq A^{lr} \subseteqq M_0^r$ which proves that R is local with $J = M_0^r$.

Now we assume that R is RSI and prove (i) $\Rightarrow$ (iii). That R is then a left S-ring is already noted above. Now, let h and a be any two nonzero elements respectively from H and $\hat{R}$. Since R is local and $h^r = J$, we have $a^r \subseteqq h^r$. Thus $ax \to hx$ is a homomorphism of aR into H which can be extended to some element f of K. Then $h = f(a) \in Ka$ which shows that each nonzero submodule of ${}_K\hat{R}$ contains h and so ${}_K\hat{R}$ is subdirectly irreducible. In fact, it can be easily seen that ${}_K\hat{R}$ and R_R have the same heart H.

Since a LSI, left S-ring R satisfies condition (ii) of the above theorem, we have in particular,

COROLLARY 2.3. *For a RSI and LSI ring, the following are equivalent.*

(i) R *is local*

(ii) R *is a left (right) S-ring.*

3. Self-injective RSI rings.

In a local RSI ring R, the left annihilator N of H, and the right annihilator J of H (which will be the unique maximal right ideal) need not coincide, though obviously we must have $N \subseteqq J$. We show by an example[2] that this inclusion can be proper and then prove that for self-injective rings, $N = J$.

EXAMPLE 3.1. Let $F = k(x_1, \cdots, x_n, \cdots)$ be the field of rational functions in $x_1, x_2, \cdots$ over the field k of real numbers and L the subring of fractions with denominators prime to x_1. Let α and β denote an epimorphism and a monomorphism respectively on L to F given by $f(x_1, \cdots, x_n)^\alpha = f(0, x_1, \cdots, x_{n-1})$ and $\beta =$ inclusion. Let R be the ring defined by $(R, +) = L \oplus F$ and $(a, b)(c, d) = (ac, bc^\alpha + a^\beta d)$. If h denotes the element $(0, 1)$ of R, then every element of R can be written as $a + hb$ and $h^2 = 0$. It can be verified that R is RSI with heart $H = hR$ and that $a + hb$ is a unit in R iff a is a unit in L. Since L is a local ring, so is R. For this ring R, $N = H^l = \{(0, b): b \in F\}$ and $J = \{(a, b): a^\alpha = 0\}$. Thus $N \subsetneqq J$.

[2] Professor P. M. Cohn has kindly communicated this example to the author.

PROPOSITION 3.2. *If R is self-injective and RSI, $R/N \cong D$.*

Proof. It was shown in [2] that $f: R/N \to D$ defined by $f(a + N) = f_a$, where $f_a: H \to H$ is the left multiplication by a, is a monomorphism. If R is self-injective and $d: H \to H$ is any element of D, then there exists an element $a \in R$ such that $d = f_a = f(a + N)$ and $R/N \cong D$.

COROLLARY 3.3. *If R is self-injective and RSI, $N = J$.*

Proof. Self-injective and RSI implies local. Since R/N is a division ring, N must be a maximal right ideal and hence $N = J$.

We now give a characterization of RSI rings in the class of self-injective rings which is analogous to McCoy's Theorem [9, p. 382, Th. 1] for the commutative subdirectly irreducible rings.

THEOREM 3.4. *Let R be self-injective. Then R is RSI iff there exists a nonzero principal right ideal $X = xR$ and an ideal Y in R such that*

(i) $Y^l = X$, *so that X is a two sided ideal,*

(ii) $X^l = Y$,

(iii) *R/Y is a division ring, and*

(iv) *If a is an element of Y not in X, there exists an element b of Y not in X, such that $ab = x$.*

Proof. ($\Rightarrow$). If R is RSI, we choose X to be $H = hR$ and $Y = N$. Then (ii) holds by definition of N and (iii) is a consequence of 3.2 above. Since R is in this case local with $N =$ unique maximal right ideal, $aN = 0$ if and only if aR is a minimal right ideal. Thus $N^l = H$ which proves (i). Now let $a \in N$ such that $a \notin N$. Since $H \subseteqq aR$ we have $h = ab$ for some $b \in R$. (ii) implies that $b \notin H$. Also if $b \notin N$ then b is a unit which implies $a = hb^{-1} \in H$ contradicting the hypothesis on a.

($\Leftarrow$). By assuming (i), (ii), (iii), and (iv) we will show that every nonzero right ideal of R contains the fixed element x of R. Accordingly, let a be a nonzero element in a right ideal A of R. If $a \notin Y$, by (iii) we have $1 - ay \in Y$ for a suitable $y \in R$. Then by (ii), $(1 - ay)x = 0$ which implies $x = ayx \in aR \subseteqq A$. If $a \in Y$ and $a \notin X$ then by (iv) we have $x \in aR \subseteqq A$. Lastly, if $a \in X$, then $a = xc$ for some $c \in R$. c cannot be in Y because then we would have $a = xc \in XY = 0$. Thus, again by (iii) $1 - cd = u \in Y$ for some $d \in R$. Then $ad = xcd = x(1 - u) = x$ which proves that $x \in A$. Thus R is RSI with heart $X = xR$.

4. Right *PF* X-rings. Utumi [14, p. 56] defined a ring R to be right PF if every faithful right R-module is completely faithful.

These rings afford a nice generalization of QF rings and have been discussed by Azumaya [1], Osofsky [10], Kato [6, 7], Utumi [13, 14] and others. A ring R is right PF if and only if [1, p. 701, Th. 6] it is a finite direct sum of indecomposable injective right ideals each of which contains a unique minimal right ideal. It is not known whether a right PF ring is also left PF. Clearly, a self-injective RSI ring is right PF. Either by using the theory of right PF rings developed by the authors mentioned above, or directly as a consequence of our Theorem 2.2 it can be seen that a self-injective RSI ring is left subdirectly irreducible. (In this case, R is local, $\hat{R} = R$ and $K \cong R$). We state this as a

PROPOSITION 4.1. *A self-injective RSI ring is LSI.*

The following definition is due to Feller [3, p. 20, 2.2].

DEFINITION 4.2. *A ring R is called an X-ring if for every pair e, f of primitive idempotents such that $eR \not\cong fR$; $a \in eR$ and $a^r \cap fR \neq 0$ implies $afR = 0$.*

We state the following useful lemma whose proof is straightforward.

LEMMA 4.3. *If A and B are rings and $R = A \oplus B$ is the ring theoretic direct sum, then R is a self-injective ring iff each of A and B are self-injective rings.*

We are now in a position to prove the following

THEOREM 4.4. *A ring R is a right PF X-ring if and only if R is isomorphic to a finite direct sum of full matrix rings over self-injective RSI rings. Further, this decomposition is unique.*

Proof. If S is a self-injective RSI ring with heart H, then the $n \times n$ matrix ring S_n is self-injective [13, p. 172, Th. 8.3] and is the direct sum of indecomposable right ideals $e_{ii}S_n$, $i = 1, 2, \cdots, n$ each of which contains a unique minimal right ideal $e_{ii}H_n$. Consequently S_n is a right PF ring which is trivially an X-ring. Now, by using 4.3 we can see that any finite direct sum of such rings is again right PF. In order to show that it is also an X-ring, it is enough to remark that eR and fR are nonisomorphic only if they belong to different matrix rings, in which case $eRfR = 0$.

Conversely, Let R be a right PF X-ring. Then

4.5. $$R = e_1R \oplus \cdots \oplus e_nR$$

where $e_1, \cdots, e_n$ are primitive and let us assume that $e_1R, \cdots, e_kR$ $(k \leqq n)$ denote a complete set of nonisomorphic right ideals among the n summands in 4.5. Since each e_iR is indecomposable and injective, it is right uniform. By the same argument as in [3, p. 20, Th. 2.3] we conclude that $R = A_1 \oplus \cdots \oplus A_k$ where each A_i is the sum of all summands in 4.5 which are isomorphic to e_iR and $A_1, \cdots, A_k$ are all two sided ideals. Further, each A_i is isomorphic to a full matrix ring over e_iRe_i. That this decomposition is unique follows from [4, p. 42, Th. 1]. Also, by 4.3, each of the rings $A_1, \cdots, A_k$ is self-injective and hence by [13, p. 172, Th. 8.3] so are the rings e_iRe_i. Finally, if H_i is the unique minimal right ideal of R contained in e_iR, it can be verified that e_iRe_i is *RSI* with heart $e_iH_ie_i$. This prove the Theorem.

As a consequence of this theorem, a right *PF X*-ring will be left *PF* (if and) only if the self-injective *RSI* rings over which matrix rings appear in the above decomposition are also left self-injective. The author does not know if such is always the case.

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