

# Pacific Journal of Mathematics

**A NOTE ON  $H$ -EQUIVALENCES**

DONALD WILLIAM KAHN

## A NOTE ON $H$ -EQUIVALENCES

DONALD W. KAHN

If  $X$  is a space, with base point, the set of homotopy classes of based self-equivalent maps, from  $X$  to itself, forms a group, which has been studied by many authors. In this note, we study a related group, in the case where  $X$  is an  $H$ -space. The main result is that all such groups are finitely-presented. The methods combine results from algebraic topology with combinatorial group theory.

If  $X$  is an  $H$ -space with multiplication  $\mu: X \times X \rightarrow X$ , a self-map  $f: X \rightarrow X$  is called an  $H$ -map if

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \downarrow f \times f & & \downarrow f \\ X \times X & \xrightarrow{\mu} & X \end{array}$$

is homotopy commutative. Such maps were first studied in [6], and later in [1]. Arkowitz and Curjel [1] showed that if  $X$  is a connected complex, which is an  $H$ -space,  $X$  has finite-dimensional, commutative, rational Pontrjagin algebra, and the total homotopy groups of  $X$  are finitely-generated, then the group of homotopy classes of self-maps, which are  $H$ -maps, is finitely-generated. We denote this group by  $A(X)$ , and remark that it is known to be frequently a complicated, non-Abelian group. Observe that this theorem of [1] suffices to handle the case when  $X$  is a finite, connected complex, which is an  $H$ -space. The purpose of this note is to show how this result can be strengthened. We shall prove

**THEOREM.** *If  $X$  satisfies the assumptions of the theorem of Arkowitz and Curjel, then  $A(X)$  is finitely-presented (see [3] for a definition).*

The class of finitely-presented groups is countable, while it is known that there are uncountably many groups with 2 generators. (This result about uncountability, due to B. H. Neumann, may be found in [3]). Hence, our theorem narrows down the possibilities for  $A(X)$  appreciably.

To prove this Theorem, we need several propositions.

**PROPOSITION 1.** *Let  $N \subset G$  be a normal subgroup of the group  $G$ .*

Set  $K = G/N$ . If  $K$  and  $N$  are finitely presented, so is  $G$ .

*Proof.* See p. 130 in [2]. I believe that this is the first place where this proposition, which is not difficult, has appeared in the literature.

REMARK. On the contrary, if  $G$  and  $K$  are finitely-presented,  $N$  need not even be finitely-generated.

PROPOSITION 2. Let  $H \subset G$  be a subgroup of finite index. If  $G$  is finitely-presented, so is  $H$ .

*Proof.* See p. 93 of [4].

As a converse of Proposition 2, we have the following proposition which we shall deduce briefly from Proposition 1.

PROPOSITION 3. If  $H \subset G$  is a finitely-presented subgroup of finite index, then  $G$  is finitely-presented.

*Proof.* Let  $H_0$  be the intersection of all conjugates of  $H$  in  $G$ .  $H_0$  is a normal subgroup of finite-index, as there are only finitely-many conjugates. By Proposition 2,  $H_0$  is finitely-presented.  $G/H_0$  is finite, and hence, finitely-presented. The result follows immediately from Proposition 1.

PROPOSITION 4. If  $G_1, \dots, G_k$  are finitely-presented, so is the group  $\prod_{i=1}^k G_i$ .

*Proof.* For lack of a reference, we indicate the proof. As generators, we select the elements

$$\begin{aligned} &(x_1, 1, \dots, 1), (x_2, 1, \dots, 1), \dots, (x_k, 1, \dots, 1) \\ &(1, y_1, 1, \dots, 1), \dots, (1, y_i, 1, \dots, 1) \\ &\dots \end{aligned}$$

where the  $x_i$  generate  $G_1$ , the  $y_j$  generate  $G_2$ , etc. A defining set of relations is then given by the relations among the  $x_i$ , the relations among the  $y_j$ , etc. plus the commutativity relations

$$(x_i, 1, \dots, 1) \cdot (1, y_j, 1, \dots, 1) = (1, y_j, 1, \dots, 1) \cdot (x_i, 1, \dots, 1) \quad \text{etc.}$$

We now prove our Theorem.

(a) Let  $k$  be the maximal dimension for which  $H_i(X, \mathbb{Q}) \neq 0$ . Let  $F \subset \pi'_*(X) = \sum_{i=1}^k \pi_i(X)$  be the (graded) free subgroup. We shall denote, by  $\text{Aut}_1(G)$ , the group of graded automorphism of the

graded group  $G$ , reserving the symbol  $\text{Aut}$  for the usual group of automorphisms. According to [5.], if  $F_0$  is a finitely-generated, free, Abelian group,  $\text{Aut}(F_0)$  is finitely-presented. It is clear that  $\text{Aut}_1(F)$  is a direct product of such groups, and hence by Proposition 4, it is finitely-presented. Because  $\text{Aut}_1(F) \subset \text{Aut}_1(\pi'_*(X))$  is clearly a subgroup of finite index, we conclude from Proposition 3 that the group  $\text{Aut}_1(\pi'_*(X))$  is finitely-presented.

(b) It is shown in [1] that the natural map

$$P: A(X) \rightarrow \text{Aut}_1(\pi'_*(x))$$

has finite kernel, and that the image of  $p$  (see p. 146 of [1]) is a subgroup of finite index. It is here that the assumptions on  $X$  are used.

By (a) above, and Proposition 2, we see that  $\text{Im}(p)$  is finitely-presented.  $\ker(p)$  being trivially finitely-presented, our theorem follows immediately from Proposition 1.

In conclusion, we would like to make some remarks about the full group of homotopy equivalences,  $G(x)$ , for such a space  $X$ . Clearly, we have a similar homomorphism  $p_1$  and  $\text{Im}(p_1)$  is of finite-index. However,  $\ker p_1$  is no longer finite. For consider the space

$$X = K(Z, 2n) \times K(Z, 4n) \quad n > 0$$

with the usual  $H$ -space structure. A self-map is determined up to homotopy by 2-cohomology classes, the classes  $f^*(i_{2n})$  and  $f^*(i_{4n})$ , these being the images of the fundamental classes. We set, for any integer  $k$ ,

$$\begin{aligned} f_k^*(i_{2n}) &= i_{2n} \cdot \\ f_k^*(i_{4n}) &= i_{4n} + k(i_{2n} \cup i_{2n}) \cdot \end{aligned}$$

It is easy to check that such a map  $f_k$  induces the identity automorphism on homotopy groups, but that all the different  $f_k$  represent distinct homotopy classes. Hence, the kernel of  $p_1$  is infinite. An easy cohomology calculation shows that when  $k \neq 0$ ,  $f_k$  is not an  $H$ -map. One also see quickly that  $A(X)$  does not have finite index in  $G(X)$  in this case.

Nevertheless, one can prove that  $G(X)$  is finitely-presented, by considering the kernel of  $p_1$ . This will be studied in the forthcoming thesis of Mr. Daniel Sunday.

#### REFERENCES

1. M. Arkowitz and C. Curjel, *On maps of  $H$ -spaces* Topology **6** No. 2 (1967).
2. H. Behr, *Über die endliche Definierbarkeit* J. Reine u. Angew. Mathematik, **211** (1962).

3. A. Kurosh, *Theory of Groups Vol. 2*, Chelsea Co., New York.
4. Magnus, Karass, and Solitar, *Combinatorial Group Theory*, J. Wiley, New York.
5. W. Magnus, *Über  $n$ -dim Gittertransformationen*, Acta Math., **64** (1934).
6. H. Samelson, *Groups and spaces of loops*, Comm. Math. Helv., **28** (1954).

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