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A NOTE ON H -EQUIVALENCES

DONALD WILLIAM KAHN

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If X is a space, with base point, the set of homotopy classes of based self-equivalent maps, from X to itself, forms a group, which has been studied by many authors. In this note, we study a related group, in the case where X is an H -space. The main result is that all such groups are finitely-presented. The methods combine results from algebraic topology with combinatorial group theory.

If X is an H -space with multiplication $\mu: X \times X \rightarrow X$, a self-map $f: X \rightarrow X$ is called an H -map if

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & X \\ \downarrow f \times f & & \downarrow f \\ X \times X & \xrightarrow{\mu} & X \end{array}$$

is homotopy commutative. Such maps were first studied in [6], and later in [1]. Arkowitz and Curjel [1] showed that if X is a connected complex, which is an H -space, X has finite-dimensional, commutative, rational Pontrjagin algebra, and the total homotopy groups of X are finitely-generated, then the group of homotopy classes of self-maps, which are H -maps, is finitely-generated. We denote this group by $A(X)$, and remark that it is known to be frequently a complicated, non-Abelian group. Observe that this theorem of [1] suffices to handle the case when X is a finite, connected complex, which is an H -space. The purpose of this note is to show how this result can be strengthened. We shall prove

THEOREM. *If X satisfies the assumptions of the theorem of Arkowitz and Curjel, then $A(X)$ is finitely-presented (see [3] for a definition).*

The class of finitely-presented groups is countable, while it is known that there are uncountably many groups with 2 generators. (This result about uncountability, due to B. H. Neumann, may be found in [3]). Hence, our theorem narrows down the possibilities for $A(X)$ appreciably.

To prove this Theorem, we need several propositions.

PROPOSITION 1. *Let $N \subset G$ be a normal subgroup of the group G .*

Set $K = G/N$. If K and N are finitely presented, so is G .

Proof. See p. 130 in [2]. I believe that this is the first place where this proposition, which is not difficult, has appeared in the literature.

REMARK. On the contrary, if G and K are finitely-presented, N need not even be finitely-generated.

PROPOSITION 2. Let $H \subset G$ be a subgroup of finite index. If G is finitely-presented, so is H .

Proof. See p. 93 of [4].

As a converse of Proposition 2, we have the following proposition which we shall deduce briefly from Proposition 1.

PROPOSITION 3. If $H \subset G$ is a finitely-presented subgroup of finite index, then G is finitely-presented.

Proof. Let H_0 be the intersection of all conjugates of H in G . H_0 is a normal subgroup of finite-index, as there are only finitely-many conjugates. By Proposition 2, H_0 is finitely-presented. G/H_0 is finite, and hence, finitely-presented. The result follows immediately from Proposition 1.

PROPOSITION 4. If G_1, \dots, G_k are finitely-presented, so is the group $\prod_{i=1}^k G_i$.

Proof. For lack of a reference, we indicate the proof. As generators, we select the elements

$$\begin{aligned} &(x_1, 1, \dots, 1), (x_2, 1, \dots, 1), \dots, (x_k, 1, \dots, 1) \\ &(1, y_1, 1, \dots, 1), \dots, (1, y_l, 1, \dots, 1) \\ &\dots \end{aligned}$$

where the x_i generate G_1 , the y_j generate G_2 , etc. A defining set of relations is then given by the relations among the x_i , the relations among the y_j , etc. plus the commutativity relations

$$(x_i, 1, \dots, 1) \cdot (1, y_j, 1, \dots, 1) = (1, y_j, 1, \dots, 1) \cdot (x_i, 1, \dots, 1) \quad \text{etc.}$$

We now prove our Theorem.

(a) Let k be the maximal dimension for which $H_i(X, \mathbb{Q}) \neq 0$. Let $F \subset \pi'_*(X) = \sum_{i=1}^k \bigoplus \pi_i(X)$ be the (graded) free subgroup. We shall denote, by $\text{Aut}_*(G)$, the group of graded automorphism of the

graded group G , reserving the symbol Aut for the usual group of automorphisms. According to [5.], if F_0 is a finitely-generated, free, Abelian group, $\text{Aut}(F_0)$ is finitely-presented. It is clear that $\text{Aut}_1(F)$ is a direct product of such groups, and hence by Proposition 4, it is finitely-presented. Because $\text{Aut}_1(F) \subset \text{Aut}_1(\pi'_*(X))$ is clearly a subgroup of finite index, we conclude from Proposition 3 that the group $\text{Aut}_1(\pi'_*(X))$ is finitely-presented.

(b) It is shown in [1] that the natural map

$$P: A(X) \rightarrow \text{Aut}_1(\pi'_*(x))$$

has finite kernel, and that the image of p (see p. 146 of [1]) is a subgroup of finite index. It is here that the assumptions on X are used.

By (a) above, and Proposition 2, we see that $\text{Im}(p)$ is finitely-presented. $\ker(p)$ being trivially finitely-presented, our theorem follows immediately from Proposition 1.

In conclusion, we would like to make some remarks about the full group of homotopy equivalences, $G(x)$, for such a space X . Clearly, we have a similar homomorphism p_1 and $\text{Im}(p_1)$ is of finite-index. However, $\ker p_1$ is no longer finite. For consider the space

$$X = K(Z, 2n) \times K(Z, 4n) \quad n > 0$$

with the usual H -space structure. A self-map is determined up to homotopy by 2-cohomology classes, the classes $f^*(i_{2n})$ and $f^*(i_{4n})$, these being the images of the fundamental classes. We set, for any integer k ,

$$\begin{aligned} f_k^*(i_{2n}) &= i_{2n} \cdot \\ f_k^*(i_{4n}) &= i_{4n} + k(i_{2n} \cup i_{2n}) \cdot \end{aligned}$$

It is easy to check that such a map f_k induces the identity automorphism on homotopy groups, but that all the different f_k represent distinct homotopy classes. Hence, the kernel of p_1 is infinite. An easy cohomology calculation shows that when $k \neq 0$, f_k is not an H -map. One also see quickly that $A(X)$ does not have finite index in $G(X)$ in this case.

Nevertheless, one can prove that $G(X)$ is finitely-presented, by considering the kernel of p_1 . This will be studied in the forthcoming thesis of Mr. Daniel Sunday.

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