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Let K be a polygonal simple closed curve (a knot) in a 3-sphere S^3. For each nonnegative integer d the dth elementary ideal E_d of K in the integral group-ring over an infinite cyclic group is defined by R. H. Fox. The ideal E_d of K is a topological invariant of the position of K in S^3. This method has been applied to various more general settings, for instance, links in S^3, S^{n-2} in S^n $(n > 2)$ and etc. In this paper the dth elementary ideals $E_d(l)$ are associated to each $(n-2)$-dimensional integral cycle l on a polyhedron L in an n-sphere $S^n(n > 2)$ that does not separate S^n. The collection of $E_d(l)$ for all possible l on L forms a toplogical invariant of the position of L in S^n.

In §2 we prove theorems of the dth elementary ideal $E_d(l)$ associated with an $(n-2)$-dimensional integral l on a polyhedron L in S^n that does not separate $S^n(n > 2)$. In §3 we will consider to the case of polyhedra in S^3. After studying an example of a θ-curve in S^3 in §4, we reconsider knots and links from this point of view in §5. In §6 we will give a remark on a Torres' formula for a link in S^3 ([7]) from this point of view.

Our discussion will be based on Fox's free differential calculus ([1], [2], [3]), though other methods, especially the covering space technique, would also be helpful. We need some minor adjustment of free differential calculus that will be given in §1.

The method of the paper is essentially different from that of [4], though a close relation between them will be observed.

1. From free differential calculus. Let G be a group with a presentation $(x_1, \cdots, x_m; r_1, \cdots, r_n)(n \leqq \infty)$. Let H be a multiplicative abelian group and ψ a homomorphism of G into H. Let ϕ be the canonical homomorphism of the free group $F(x_1, \cdots, x_m)$ onto G. These homomorphism ϕ and ψ are naturally extended to ring homomorphisms of the integral group-ring JF onto JG and of JG into JH, respectively. Using Fox's free differential calculus ([1], [2], [3]), we have an $n \times m$ matrix $A(G, \psi) = (r_{ij})$, where

$$r_{ij} = \left(\frac{\partial r_i}{\partial x_j}\right)^{\psi\phi} \in JH\,.$$

Generally, let (r_{ij}) be an $n \times m$ matrix $(n \leqq \infty)$ over a commutative ring R with unity. Then for each nonnegative integer d the dth elementary ideal E_d of the matrix (r_{i_j}) is the ideal of R generated by all minors of order $m-d$, if $m-d>0$ and $n-(m-d)\geqq 0$. If $m-d>0$, but $n-(m-d)<0$, let $E_d=(0)$. If $m-d\leqq 0$, let $E_d=(1)$.

It was proved in [3] that the dth elementary ideal E_d of the matrix $A(G,\alpha)$ is an invariant of the group G, where α is the abelianizer of G. Since there is a unique homomorphism σ of the abelianization of G into H with $\psi=\sigma\alpha$, the dth elementary ideal $E_d(G,\psi)$ of the matrix $A(G,\psi)$ is an invariant of G with respect to ψ.

Now let us consider the following diagram:

$$G_1 \xrightarrow{\phi} G_2 \xrightarrow{\sigma} H\,,$$

where G_1 and G_2 are groups, H a multiplicative abelian group, and ψ is a homomorphism of G_1 onto G_2 and σ of G_2 into H. Then we have the following theorem.

THEOREM 1. $E_d(G_1,\sigma\psi)\subset E_d(G_2,\sigma)$.

Proof. Let $(x_1,\cdots,x_m\colon r_1,\cdots,r_n)(n\leqq\infty)$ be a presentation of G_1. Then $x_1^\psi,\cdots,x_m^\psi$ are generators of G_2, since ψ is a homomorphism of G_1 onto G_2, and $r_1^\psi,\cdots,r_n^\psi$ are some of the relators of G_2. From this it follows that one of $A(G_1,\sigma\psi)$ is a submatrix of one of $A(G_2,\sigma)$, where the both matrices have the same number of columns. The theorem is now clear.

A trivializer of a group G is a homomorphism of G onto the trivial group that consists of only one element. Any trivializer will be denoted by the same notation o in this paper. Further, the group-ring $Jo(G)$ will be identified with J.

For the convenience of the readers we refer to the following two theorems from free differential calculus ([2]).

THEOREM A (Fox). *Let G be a group and o a trivializer of G. Then a matrix $A(G,o)$ is elementarily equivalent* (see [1], [3]) *to the following matrix*:

$$\begin{bmatrix} \tau_1 & & & & & 0 \\ & \ddots & & & & \\ & & \tau_n & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix}$$

where $\tau_1,\cdots,\tau_n$ are torsion numbers and μ the Betti number of the *abelianization of* G. (We assume that the abelianization of G is finitely generated.)

THEOREM B (Fox). *Let* H *be a multiplicative free abelian group of rank* μ ($\mu \geqq 1$) *with generators* $t_1, \cdots, t_\mu$ *and let* i *be the identity of* H. *Then we have*

$$\begin{cases} E_0(H, i) = (0) , \\ E_d(H, i) = (t_1 - 1, \cdots, t_\mu - 1)^{\mu - d}, \text{ if } 1 \leqq d < \mu , \\ E_d(H, i) = (1), \text{ if } d \geqq \mu . \end{cases}$$

2. **On polyhedra in S^n.** Let L be a polyhedron in an n-sphere S^n ($n \geqq 3$) that does not separate S^n, and let G_L be the fundamental group of $S^n - L$. Let l be an $(n-2)$-dimensional cycle with integral coefficients on L. There is a homomorphism ψ of the group G_L into the multiplicative infinite cyclic group H generated by t such that for each $g \in G_L$,

$$g^\psi = t^{\operatorname{link}(g, l)} ,$$

where link (g, l) is the linking number between g and l in S^n. Since the dth elementary ideal $E_d(G_L, \psi)$ is an invariant of the group G_L with respect ψ, it is a topological invariant of $S^n - L$ with respect to l on L, and from this it follows that $E_d(G_L, \psi)$ is a topological invariant of the position of l on L in S^n. We will denote it by $E_d(l)$. If two $(n-2)$-cycles l and l' are homologous on L, then the corresponding dth elementary ideals $E_d(l)$ and $E_d(l')$ are the same. The collection of $E_d(l)$ for all possible $(n-2)$-cycles l on L forms a topological invariant of the position of L in S^n.

THEOREM 2. Let L *be a polyhedron in an* n*-sphere* S^n ($n \geqq 3$) *that does not separate* S^n. *Let* $p_{n-2}(L)$ *be the* $(n-2)$*-dimensional Betti number of* L *and* $\tau_1, \tau_2, \cdots, \tau_r$ *are* $(n-3)$*-dimensional torsion numbers with* $\tau_i | \tau_{i+1}$ ($i = 1, 2, \cdots, r-1$). *Then for each* $(n-2)$*-dimensional cycle* l *on* L *the dth elementary ideal* $E_d(l)$ *of* l *on* L *in* S^n *satisfies the following conditions*:

$$\begin{cases} (E_d(l))^\circ = (0), \text{ if } d < p_{n-2}(L) , \\ (E_d(l))^\circ = \tau_1\tau_2 \cdots \tau_m, \text{ where } m = r - (d - p_{n-2}(L)) , \\ \qquad\qquad \text{if } p_{n-2}(L) \leqq d < p_{n-2}(L) + r , \\ (E_d(l))^\circ = (1), \text{ if } d \geqq p_{n-2}(L) + r . \end{cases}$$

Proof. By the Alexander duality theorem we have

$$p_1(S^n - L) = p_{n-2}(L)$$

and

$$T_1(S^n - L) \approx T_{n-3}(L) ,$$

where $T_i(K)$ is the i-dimensional torsion group of a complex K. Since the abelianizations of $\pi(S^n - L)$ is the 1-dimensional homology group of $S^n - L$, the theorem is clear by Theorem A.

Especially if l_0 is the $(n-2)$-cycle on L such that the coefficients of l_0 on every $(n-2)$-simplexes of L are 0, then for every $g \in G_L$ we have link $(g, l_0) = 0$. Hence the homomorphism ψ of G_L into H with respect to l_0 is a trivializer. Therefore, we have the following theorem.

THEOREM 3. *Let L be a polyhedron in $S^n (n \geqq 3)$ that does not separate S^n and l_0 the $(n-2)$-cycle on L with coefficients* 0 *on every $(n-2)$-simplexes on L. Then we have*

$$E_d(l_0) = (E_d(l))^\circ ,$$

where l is an $(n-2)$-cycle on L.

Now let μ be the number of components of L (i.e. $\mu = p_0(L)$) and $L_i (i = 1, 2, \cdots, \mu)$ the ith component of L. Then an $(n-2)$-cycle l on L can be expressed as $\sum_{i=1}^{\mu} l_i$, where $l_i (i = 1, 2, \cdots, \mu)$ is an $(n-2)$-cycle on L defined by

$$\begin{cases} l_i | L_i = l | L_i , \\ l_i | L_j = 0 \qquad (i \neq j) . \end{cases}$$

Let H_0 be a multiplicative free abelian group of rank μ, i.e. $H_0 = \prod_{i=1}^{\mu} H_i$, where $H_i = (t_i : \ ) (i = 1, 2, \cdots, \mu)$. Define a homomorphism ψ_0 of G_L into H_0 by

$$g^{\psi_0} = \prod_{i=1}^{\mu} t_i^{\,\mathrm{link}(g, l_i)} .$$

Then, as before, we have the dth elementary ideal $E_d(G_L, \psi_0)$ in JH_0, that is a topological invariant of the position of l on L in S^n. $E_d(G_L, \psi_0)$ will be denoted by $E_d[l]$.

Let σ be a homomorphism of H_0 onto $H = (t: \ )$ defined by $t_i^\sigma = t$. Since $g^\psi = t^{\mathrm{link}(g,l)}$, we have $\sigma\psi_0 = \psi$. Since σ is an homomorphism of H_0 onto H, we have the following theorem.

THEOREM 4. *We have*

$$\begin{cases} (E_d[l])^\sigma = E_d(l) \text{ and} \\ (E_d[l])^\circ = (E_d(l))^\circ . \end{cases}$$

Now assume that $p_{n-2}(L) \geqq 1$. We have a sequence of homomorphism

$$\begin{tikzcd} G_L \arrow[r, "\alpha"] \arrow[rrr, bend right, "\psi_0"'] & H_L \arrow[r, "\alpha'"] & H' \arrow[r, "\sigma'"] & H_0 \arrow[from=2-2, to=2-4, bend left, "\sigma_0"] \end{tikzcd} ,$$

where α is the abelianizer of G_L, $H_L \cong H'$ (a free abelian group of rank $p_{n-2}(L)$) $\times$ (the torsion subgroup of H_L) and α' is the projection of H_L onto H', $\psi_0 = \sigma_0\alpha$, and $\sigma_0 = \sigma'\alpha'$. Then, by Theorems 1 and B, we have

$$E_d(G_L, \alpha'\alpha) \subset E_d(H', i)$$
$$= \begin{cases} (0), \text{ if } d = 0, \text{ and} \\ (1 - s_1, \cdots, 1 - s_{p_{n-2}(L)})^{p_{n-2}(L)-d}, \text{ if } 1 \leqq d < p_{n-2}(L) , \end{cases}$$

in JH', where we assume that H' is generated by $s_1, \cdots, s_{p_{n-2}(L)}$. In [2] it is proved that

$$(1 - s_1^\sigma, \cdots, 1 - s_{p_{n-2}(L)}^\sigma) \subset (1 - t_1, \cdots, 1 - t_\mu) .$$

Hence we have the following theorem.

THEOREM 5. *Assume that* $p_{n-2}(L) \geqq 1$. *Then we have*

$$\begin{cases} E_0[l] = (0), \text{ and} \\ E_d[l] \subset (1 - t_1, \cdots, 1 - t_\mu)^{p_{n-2}(L)-d}, \text{ if } 1 \leqq d < p_{n-2}(L) . \end{cases}$$

THEOREM 6. *Assume that* $p_{n-2}(L) \geqq 1$. *Then we have*

$$\begin{cases} E_0(l) = (0), \text{ and} \\ E_d(l) \subset (1 - t)^{p_{n-2}(L)-d}, \text{ if } 1 \leqq d < p_{n-2}(L) . \end{cases}$$

REMARK. The first formula in Theorem 2 follows from Theorem 6.

3. On polyhedra in S^3.

THEOREM 7[1]. *Let M^3 be a 3-dimensional manifold and L a polyhedron in M^3 that does not separate M^3. Then $\pi(M^3 - L)$ has a presentation with deficiency $p_1(M^3 - L) - p_2(M^3 - L)$.*

Proof. Let K be a connected 2-dimensional polyhedron. Then $\pi(K)$ has a presentation with $\alpha_1 - (\alpha_0 - 1)$ generators and α_2 relators, where $\alpha_i (i = 0, 1, 2)$ is the number of i-dimensional simplexes. Since $M^3 - L$ has a connected 2-dimensional polyhedron as its deformation retract, say K, there is a presentation of $\pi(M^3 - L)$ with deficiency

[1] This theorem is due to the referee.

$$\begin{aligned}\alpha_1 - (\alpha_0 - 1) - \alpha_2 &= 1 - p_0(K) + p_1(K) - p_2(K) \\ &= p_1(M^3 - L) - p_2(M^3 - L) .\end{aligned}$$

COROLLARY. *Let L be a polyhedron in S^3 that does not separate S^3. Then $\pi(S^3 - L)$ has a presentation with deficiency* $1 - p_0(L) + p_1(L)$.

Proof. By the Alexander duality theorem we have

$$\begin{aligned}p_1(S^3 - L) - p_2(S^3 - L) &= p_1(L) - (p_0(L) - 1) \\ &= 1 - p_0(L) + p_1(L) .\end{aligned}$$

THEOREM 8. *Let L be a polyhedron in S^3 that does not separate S^3. Then for each* 1-*cycle l on L we have*

$$E_d(l) = (0), \text{ if } d < 1 - p_0(L) + p_1(L) .$$

REMARK. A greater number than $1 - p_0(L) + p_1(L)$ can be obtained, if one (or more) component of L is contractible in itself.

The following Theorem 9 and Theorem 10 are corollaries of Theorem 2 and Theorems 5 and 6, respectively.

THEOREM 9. *Let L be a polyhedron in* 3-*sphere S^3 that does not separate S^3. Then for each* 1-*cycle l on L we have*

$$\begin{cases}(E_d(l))^\circ = (0), \text{ if } d < p_1(L), \text{ and} \\ (E_d(l))^\circ = (1), \text{ if } d \geqq p_1(L) .\end{cases}$$

THEOREM 10. *Let L be a polyhedron in* 3-*sphere S^3 that does not separate S^3. Assume that $p_1(L) > 0$ and let $\mu = p_0(L)$. Then we have*

$$\begin{cases}E_0[l] = 0, \text{ and} \\ E_d[l] \subset (1 - t_1, \cdots, t - t_\mu)^{p_1(L)-d}, \text{ if } 1 \leqq d < p_1(L) .\end{cases}$$

Especially we have

$$\begin{cases}E_0(l) = 0, \text{ and} \\ E_d(l) \subset (1 - t)^{p_1(L)-d}, \text{ if } 1 \leqq d < p_1(L) .\end{cases}$$

4. EXAMPLES. Let L_1 be a θ-curve, which is trivially imbedded in S^3. Then the fundamental group G_{L_1} of $S^3 - L_1$ is a free group of rank 2. From this it follows that for each 1-cycle l on L_1 we have

$$\begin{cases}E_d(l) = (0), \text{ if } d < 2 , \\ E_d(l) = (1), \text{ if } d \geqq 2 .\end{cases}$$

Now let L_2 be a θ-curve in S^3, which is shown in Fig. 1.

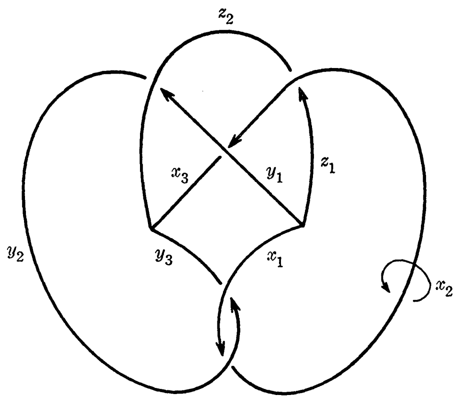

FIGURE 1

Then a presentation of the fundamental group G_{L_2} of $S^3 - L_2$ is as follows:

$$(x_1, y_1, y_2: y_2x_1y_1y_2^{-1}x_1y_2y_1^{-1}y_2^{-1}x_1^{-1}y_2y_1^{-1}x_1^{-1}y_2^{-1}x_1y_2x_1^{-1} = 1) .$$

Now let l be a 1-cycle on L_2 and suppose that link $(x_1, l) = c_1$ and link $(y_1, l) =$ link $(y_2, l) = c_2$. Then we have

$$A(G_{L_2}, \psi) \approx (t^{c_1+c_2} + t^{c_2} + 1, 2, 0) .$$

Hence, we have

$$\begin{cases} E_d(l) = (0), \text{ if } d < 2 , \\ E_2(l) = (t^{c_1+c_2} + t^{c_2} + 1, 2) , \\ E_d(l) = (1), \text{ if } d > 2 . \end{cases}$$

This means that the position of L_1 and that of L_2 in S^3 are topologically inequivalent. Note that any one of three simple closed curves on L_2 is a trivial knot in S^3. (The example L_2 was also discussed in [4], but there was a mistake in calculation, that was pointed out by R. H. Fox to the author of the paper. Of course the underlying theory in that paper is also different to that of this paper as noted in the introduction.) We may also note that generally $E_2(l)$ of l on L_2 in S^3 is not a principal ideal, that generally $E_2(l)$ does not satisfy the symmetricity property, that appears for knots and links in S^3. There are several of this kind of example in [6], too.

In another paper the author of the paper will prove that for any integral polynomial $f(t)$ with $f(t) = \pm 1$ there exists a θ-curve L in S^3 and a 1-cycle l on L such that $E_2(l) = (f(t))$.

5. **On knots and links.** Let K be an oriented polyhedral $(n -$

2)-sphere in S^n ($n \geqq 3$) and k an $(n-2)$-cycle on K such that $k = ck'$, where k' is the fundamental cycle of K. Consider the fundamental group G_K of $S^n - K$ and its abelianization

$$\alpha: G_K \to H_K = (t': \ ) .$$

We choose the generator t' in such a way that for each $g \in G_K$

$$g^\alpha = (t')^{\operatorname{link}(g,k')} .$$

On the other hand, we have

$$g^\psi = t^{\operatorname{link}(g,k)} = t^{c(\operatorname{link}(g,k'))} .$$

Now define a homomorphism σ of H_K into H by $(t')^\sigma = t^c$. Then we have $\psi = \sigma\alpha$. From this it follows that

$$E_d(k)(t) = E_d(k')(t^c)$$

for each $d \geqq 0$. Further, since H_K is infinite cyclic, we have $E_0(k) = (0)$ and $(E_d(k))^\circ = (1)$ for $d \geqq 1$. If $n = 3$, we have

$$E_1(k)(t) = (\Delta_K(t^c)) ,$$

where $\Delta_K(t)$ is the Alexander polynomial of the oriented simple closed curve K in S^3.

Let L be an oriented polyhedral $(n-2)$-link with μ components in S^n ($n \geqq 3$), i.e. an ordered collection of μ number of mutually disjoint oriented polyhedral $(n-2)$-spheres in S^n. We assume that $\mu \geqq 2$. Let L_i be the ith component of L for each $i (i = 1, \cdots, \mu)$. Let l be an $(n-2)$-cycle on L and let l'_i be as follows:

$$\begin{cases} l'_i \mid L_i = \text{the fundamental cycle on } L_i, \text{ and} \\ l'_i \mid L_j = 0, \text{ if } i \neq j . \end{cases}$$

Hence, we have $l = \sum_{i=1}^\mu c_i l'_i$. Let $l_i = c_i l'_i$ and $l' = \sum_{i=1}^\mu l'_i$. Consider the fundamental group G_L of $S^n - L$ and its abelianization

$$\alpha: G_L \to H_L ,$$

which is a free abelian group of rank μ. We choose the generator $t'_i (i = 1, \cdots, \mu)$ in such a way that for each $g \in G_L$

$$g^\alpha = \prod_{i=1}^\mu (t'_i)^{\operatorname{link}(g,l'_i)} .$$

On the other hand, we have

$$g^{\psi_0} = \prod_{i=1}^\mu t_i^{\operatorname{link}(g,l_i)} = \prod_{i=1}^\mu (t_i)^{c_i \operatorname{link}(g,l)'_i)} .$$

Define a homomorphism σ of H_L into H_0 by $(t'_i)^\sigma = t_i^{c_i}$ for each i ($i = 1, \cdots, \mu$). Then we have $\psi_0 = \sigma\alpha$. From this it follows that

$$E_d[l](t_1, \cdots, t_\mu) = E_d[l'](t_1^{c_1}, \cdots, t_\mu^{c_\mu}) .$$

Further we have $E_0(l) = 0$, $(E_d(l))^\circ = (0)$ for $d < \mu$, and $(E_d(l))^\circ = (1)$ for $d \geqq \mu$. If $n = 3$, we have

$$E_1[l](t_1, \cdots, t_\mu) = (1 - t_1^{c_1}, \cdots, 1 - t_\mu^{c_\mu})\Delta_L(t_1^{c_1}, \cdots, t_\mu^{c_\mu}) ,$$

where $\Delta_L(t_1, \cdots, t_\mu)$ is the Alexander polynomial of the link L in S^3 and, hence

$$E_1(l)(t) = (1 - t^c)\Delta_L(t^{c_1}, \cdots, t^{c_\mu}) ,$$

where $c = \text{g.c.d.}\,(c_1, \cdots, c_\mu)$.

REMARK. Further, it is proved by Shinohara and Sumners [5] that if $n \geqq 4$, $E_d(l) = (0)$ for $d < \mu$.

6. On a Torres' formula for a link. The following consideration may be interesting: Suppose that L is an oriented link with multiplicity $\mu(> 1)$ in S^3 and L_i the ith component of L. $(i = 1, 2, \cdots, \mu)$. Let l be a 1-cycle on L and express l as $\sum_{i=1}^{\mu} l_i$ as before. Denote $\sum_{i=1}^{\mu-1} l_i$ by l^*. Now let $L^\# = L - L_\mu$ and let $l^\#$ be the 1-cycle on $L^\#$ such that $l^\# = l\,|\,L^\#$. Hence we have $l^\# = l^*\,|\,L^\#$. Let H_0 be a multiplicative free abelian group of rank μ, i.e. $H_0 = \prod_{i=0}^{\mu} H_i$, where $H_i = (t_i\colon\ )(i = 1, 2, \cdots, \mu)$. Let $H_0^\# = \prod_{i=1}^{\mu-1} H_i$.

Now let ψ_0 be the homomorphism of G_L into H_0 such that for each $g \in G_L$

$$g^{\psi_0} = \prod_{i=1}^{\mu} t_i^{\operatorname{link}(g, l_i)} ,$$

ψ_0^* the homomorphism of G_L into H_0 such that for each $g \in G_L$

$$g^{\psi_0^*} = \prod_{i=1}^{\mu-1} t_i^{\operatorname{link}(g, l_i)}$$

and $\psi_0^\#$ the homomorphism of $G_L^\#$ into $H_0^\#$ such that for each $g \in G_L^\#$

$$g^{\psi_0^\#} = \prod_{i=1}^{\mu-1} t^{\operatorname{link}(g, l_i)} .$$

Let c_i be the coefficient of l on $L_i (i = 1, \cdots, \mu)$. Then we have the following theorem.

THEOREM 11. *We have*

$$E_1[l^*, \psi_0^*] = (t_1^{c_1 l_{1\mu}} \cdots t_{\mu-1}^{c_{\mu-1} l_{\mu-1\,\mu}} - 1)E_1[l, \psi_0^\#] ,$$

where $l_{ij} = \operatorname{link}\,(L_i, L_j)(i, j = 1, 2, \cdots, \mu)$.

Proof. Let

$$E_1[l, \psi_0] = (t_1^{c_1} - 1, \cdots, t_\mu^{c_\mu} - 1)\Delta_L(t_1^{c_1}, \cdots, t_\mu^{c_\mu}) .$$

Then we have

$$E_1[l^*, \psi_0^*] = (t_1^{c_1} - 1, \cdots, t_{\mu-1}^{c_{\mu-1}} - 1)\Delta_L(t_1^{c_1}, \cdots, t_{\mu-1}^{c_{\mu-1}}, 1) .$$

On the other hand, we have

$$E_1[l^\#, \psi^\#] = (t_1^{c_1} - 1, \cdots, t_{\mu-1}^{c_{\mu-1}} - 1)\Delta_{L^\#}(t_1^{c_1} \cdots, t_{\mu-1}^{c_{\mu-1}}) .$$

A Torres' formula of the Alexander polynomial of links ([7]) is as follows:

If $\mu = 2$, then

$$\Delta_L(t_1, 1) = \Delta_{L^\#}(t_1)(t_1^{l_{12}} - 1)/(t_1 - 1) .$$

If $\mu > 2$, then

$$\Delta_L(t_1, \cdots, t_{\mu-1}, 1) = (t_1^{l_{1\mu}} \cdots t_{\mu-1}^{l_{\mu-1\mu}} - 1)\Delta_{L^\#}(t_1, \cdots, t_{\mu-1}) .$$

Hence if $\mu > 2$, the statement of the theorem follows immediately. If $\mu = 2$ and $c_1 \neq 0$, we have

$$\begin{aligned} E_1[l^*, \psi_0^*] &= (t_1^{c_1} - 1)\Delta_L(t_1^{c_1}, 1) \\ &= (t_1^{c_1 l_{12}} - 1)\Delta_{L^\#}(t_1^{c_1}) = (t_1^{c_1 l_{12}} - 1)E_1[l^\#, \psi_0^\#] . \end{aligned}$$

The statement of the theorem is trivial, if $c_1 = 0$ and $\mu = 2$. Thus the proof of the theorem is complete.

REMARK. Theorem 11 can be proved directly and the Torres' theorem can be obtained as a corollary to this theorem.

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