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WATTS COHOMOLOGY AND SEPARABILITY

ANDREW T. KITCHEN

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A cohomology theory, $H_K^p A$, for commutative K -algebras, A , is discussed for the case where K is a field. This was originally introduced by C. E. Watts in connection with rings of continuous functions. N. Greenleaf computed $H_K^p A$ in the case where A is an extension field of K . In this paper it is shown that, for any K -algebra A , the separable closure of K in A can be identified with $H_K^0 A$. Furthermore Greenleaf's result is extended to a substantial class of local algebras.

1. Let K be a field and A a commutative K -algebra with unit element 1. In [4] Watts described a cochain complex $C_K A$, based on the additive Amitsur complex $F_K A$ [3]. He showed that in the case where $K = \mathbf{R}$ and $A = C(X)$, the ring of continuous real valued functions on the compact Hausdorff space X , the cohomology of this complex is naturally isomorphic to the real Čech cohomology of X . At the other extreme Greenleaf in [2] proved the following result. If L is an arbitrary extension field of K then $C_K L$ is naturally isomorphic to $F_{L_s} L$, where L_s is the separable closure of K in L . Thus the homology of $C_K L$ is trivial, except in dimension zero where $H^0(C_K L) \cong L_s$.

In this paper we investigate further the part separability plays in this theory. Letting A_s be the separable closure of K in A (see §2) and writing $H_K^p A$ for $H^p(C_K A)$, we prove the following results.

THEOREM 1. *If A is an arbitrary K -algebra then $H_K^0 A = A_s$.*

THEOREM 2. *Let A be a (not necessarily Noetherian) local K -algebra with unique maximal ideal, m . Suppose the image of A_s , under the canonical map of A onto A/m , is separably closed in A/m ; then $C_K A$ is naturally isomorphic to $F_{A_s} A$.*

From Theorem 2 it follows that, for such an algebra, $H_K^p A = 0$ for $p > 0$.

At the end of the paper we mention some interesting classes of local algebras which satisfy the hypothesis of Theorem 2.

2. The complex $F_K A$ is the additive Amitsur complex [3, §4] with a dimension shift of 1: $F_K^p A$ is the $p + 1$ - fold tensor product of A over K , and the coboundary map $d^p: F_K^p A \rightarrow F_K^{p+1} A$ is given by $d^p(f_0 \otimes \cdots \otimes f_p) = \sum_{i=0}^{p+1} (-1)^i f_0 \otimes \cdots \otimes f_{i-1} \otimes 1 \otimes f_i \otimes \cdots \otimes f_p$.

PROPOSITION 1. *The complex $F_K A$ has zero homology, except in dimension zero where $H^0(F_K A) \cong K$.*

Proof. See [3, Lemma 4.1].

Let $\mu_p: F_K^p A \rightarrow A$ by $\mu_p(f_0 \otimes \cdots \otimes f_p) = f_0 \cdots f_p$. The subcomplex $N_K A$ is defined as follows

$$N_K^p A = \{f \in F_K^p A \mid \exists g \in F_K^p A \text{ with } \mu_p g \text{ a unit and } fg = 0\},$$

this is easily seen to be equivalent to Watts' definition. The Watts cohomology is then the homology of the complex $C_K A = F_K A / N_K A$.

An element $f \in A$ is said to be *separable over K* if there exists a polynomial $p \in K[X]$, such that $p(f) = 0$ and $p'(f)$ is a unit in A . The *separable closure*, A_s , of K in A is the set of elements of A which are separable over K , it is a subalgebra of A (see §3, Corollary to Theorem 1).

3. From the definition of $C_K A$ it is clear that we can consider $H_K^0 A$ to be embedded in A .

PROPOSITION 2. *If A is an arbitrary K -algebra then $A_s \subset H_K^0 A$.*

Proof. If $f \in A_s$, let $p = a_n X^n + \cdots + a_0 \in K[X]$ be such that $p(f) = 0$ and $p'(f)$ is a unit. Define

$$h_{k-1} = \sum_{i=1}^k f^{i-1} \otimes f^{k-i} \in F_K^1 A.$$

Then $\mu_1 h_{k-1} = k f^{k-1}$. If $g = a_n h_{n-1} + \cdots + a_1 h_0$, then $(1 \otimes f - f \otimes 1)g = 0$ and $\mu_1 g = p'(f)$. Thus $d^1 f \in N_K^1 A$, so $f \in H_K^0 A$.

LEMMA. *If R is the Jacobson radical of A then $R \cap H_K^0 A = 0$.*

Proof. If $f \in H_K^0 A$, then there exists $g \in F_K^1 A$ such that $\mu_1 g$ is a unit and $(1 \otimes f - f \otimes 1)g = 0$. Suppose f is also in R . It then follows that, for each maximal ideal m , the image of f under the natural map $\varphi: A \rightarrow A/m (= L)$ is zero. Thus $(1 \otimes f)g' = 0$ in $L \otimes_K A$, where g' is the image of g under the map $\varphi \otimes 1: A \otimes_K A \rightarrow L \otimes_K A$. Now g' can be written $g' = \sum \lambda_i \otimes g_i$, where the elements $\lambda_i \in L$ are linearly independent over K . It then follows that $f g_i = 0$ for all i . As $\mu_1 g$ is a unit a simple argument shows that, for some i , $\varphi g_i \neq 0$. So, for each maximal ideal m , there exists g_m such that $g_m \notin m$ and $f g_m = 0$. Therefore $\text{Ann}(f) = A$ and $f = 0$.

Proof of Theorem 1. If $f \in H_K^0 A$ then there exists $g = \sum_{i=1}^n g_i \otimes h_i$

$\in F_K^1 A$ such that $\sum g_i \otimes h_i f = \sum g_i f \otimes h_i$ and μ, g is a unit. In fact

$$\sum g_i \otimes h_i f^k = \sum g_i f^k \otimes h_i$$

for $k = 0, 1, 2, \dots$. We can assume that g_1, \dots, g_n are linearly independent over K , in which case $h_i f^k$ is in the K -module spanned by h_1, \dots, h_n . It follows that there exists a polynomial $q_i \in K[X]$ such that $h_i q_i(f) = 0$. Hence, because μ, g is a unit, $q(f) = q_1(f) \cdots q_n(f) = 0$. Thus f is algebraic over K .

For each maximal ideal m , the image of f under $\varphi: A \rightarrow A/m (= L)$ is in $H_K^0 L$. Hence, by Greenleaf's result [2], there exists an irreducible polynomial $p_m \in K[X]$ such that $p_m(\varphi f) = 0$ and $p'_m(\varphi f) \neq 0$. Now φf satisfies q , so p_m divides q and there are, therefore, only a finite number of distinct p_m . Let p_1, \dots, p_r be those distinct polynomials and let $p = p_1 \cdots p_r$. Clearly $p(f) \in R \cap H_K^0 A$ so $p(f) = 0$. A simple argument shows that $p'(f)$ is a unit. Thus $H_K^0 A \subset A_s$.

REMARK. The proof shows that $H_K^0 A$, and thus A_s , can be described as follows: $f \in H_K^0 A$ if and only if there exist distinct irreducible separable polynomials $p_1, \dots, p_r \in K[X]$ such that $p_1(f) \cdots p_r(f) = 0$.

COROLLARY. *The separable closure, A_s , of K in A is a K -algebra. Furthermore if A is a local algebra then A_s is a field extension of K .*

Proof. By Theorem 1 we can identify A_s with $H_K^0 A$. The first part of the result can then be proved easily once we observe the identity

$$1 \otimes fg - fg \otimes 1 = (1 \otimes f - f \otimes 1)(1 \otimes g) + (f \otimes 1)(1 \otimes g - g \otimes 1).$$

If A is local and f is a nonzero element of $H_K^0 A$, then the minimal polynomial of f , constructed in the proof of Theorem 1, is clearly irreducible over K . Thus the subalgebra $K[f]$ of $H_K^0 A$ is a field, and so $f^{-1} \in H_K^0 A$. Therefore $H_K^0 A$ is a field.

4. The following proposition is proved in [2].

PROPOSITION 3. *If L is a separable (algebraic) extension field of K then $N_K^p L = \ker \mu_p$.*

Using an inductive argument based on Proposition 2, we can in fact remove the restriction that L be a field.

PROPOSITION 4. *If the field L is separable over K and A is an L -algebra, then the natural map $\theta: F_K A \rightarrow F_L A$ induces an isomorphism, $C_K A \cong C_L A$.*

Proof. The induced map is certainly a surjection. On the other hand, by Proposition 3, the sequence

$$0 \longrightarrow N_K^p L \longrightarrow F_K^p L \longrightarrow L \longrightarrow 0$$

is exact. Applying the exact functor $F_K^p A \otimes_B (\)$, where $B = F_K^p L$, we obtain the exact sequence

$$0 \longrightarrow F_K^p A \otimes_B N_K^p L \longrightarrow F_K^p A \longrightarrow F_K^p A \otimes_B L \longrightarrow 0 .$$

However in $F_K^p A \otimes_B L$

$$\alpha_0 \otimes \cdots \otimes \lambda a_i \otimes \cdots \otimes a_p \otimes 1 = \alpha_0 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_p \otimes \lambda .$$

So the map of $F_K^p A \otimes_B L$ onto $F_L^p A$, induced by taking $\alpha_0 \otimes \cdots \otimes a_p \otimes \lambda$ to $\alpha_0 \otimes \cdots \otimes \lambda a_p$, is an isomorphism. The composition of this map with $1 \otimes \mu_p$ is θ_p , and the kernel of θ_p is thus the image of $F_K^p A \otimes_B N_K^p L$ in $F_K^p A$. It follows therefore that $\ker \theta_p \subset N_K^p A$. Suppose $f \in F_K^p A$ with $\theta_p f \in N_L^p A$; then there exists $g \in F_K^p A$ such that $\mu_p g$ is a unit and $f g \in \ker \theta_p$. So there exists $h \in F_K^p A$, such that $\mu_p h$ is a unit and $f g h = 0$. Since $\mu_p h g = (\mu_p h)(\mu_p g)$ is a unit, $f \in N_K^p A$. This completes the proof.

A ring in which every zero divisor is nilpotent we will call a *ZDN* ring.

PROPOSITION 5. *Let A and A' be K -algebras which are *ZDN* rings, and let N be the ideal of nilpotents of A . Suppose K is separably closed in the field of quotients of A/N , then $A \otimes_K A'$ is a *ZDN* ring.*

Proof. If B is a subring of A then it is a *ZDN* ring, with ideal of nilpotents $N \cap B$. The domain $B/(N \cap B)$ embeds in A/N , so K is separably closed in the quotient field of $B/(N \cap B)$. We can therefore restrict ourselves to a finitely generated subalgebra of A , and so assume that A is Noetherian. Let L be the quotient field of A/N , then $(A/N) \otimes_K A' \subset L \otimes_K A'$. So by [2, Proposition 3] $(A/N) \otimes_K A'$ is a *ZDN* ring and hence $N \otimes_K A'$ is primary. However (0) is a primary ideal of A with associated prime N . Thus it follows, putting $E = A$ and $F = B = A \otimes_K A'$ in [1, Chapter IV, §2.6, Theorem 2], that the associated primes of (0) in $A \otimes_K A'$ are also the associated primes of $N \otimes_K A'$. Hence (0) is a primary in $A \otimes_K A'$ also, and so $A \otimes_K A'$ is a *ZDN* ring.

Note that if A is a local ring (A has a unique maximal ideal m)

and n is a positive integer, then A/m^n is a ZDN ring.

PROPOSITION 6. *Let A be a Noetherian local K -algebra; then the natural map of $F_K^p A$ into the projective limit (inverse limit) of the system $\{F_K^p(A/m^n)\}_n$ is an injection.*

Proof. As A is Noetherian, $\bigcap_{n=1}^{\infty} m^n = 0$, and so $A \rightarrow \text{proj lim}_n (A/m^n)$ is an injection. The proof can be completed by induction on p , using the following lemma, the demonstration of which is straightforward.

LEMMA. *If $\{M_i, f_{ji}\}$ and $\{N_i, g_{ji}\}$ are projective systems of K -modules (K a field) indexed over the same directed set, and if M and N are the projective limits of these systems, then the natural map of $M \otimes_K N$ into $\text{proj lim}_i (M_i \otimes_K N_i)$ is an injection.*

PROPOSITION 7. *Let A be a local K -algebra and let K be separably closed in A/m . If z is a zero divisor in $F_K^p A$ then $\mu_p z \in m$, and hence $N_K^p A = 0$.*

Proof. Suppose z is a zero divisor in $F_K^p A$; then there exists $w \neq 0$ such that $zw = 0$. Choose a finitely generated subalgebra, B , of A such that w and z are in $F_K^p B$. The ideal $B \cap m$ is prime in B . So, localizing B at $B \cap m$, we get a local Noetherian subalgebra B' of A , such that $B' \cap m$ is the maximal ideal of B' , and z and w are elements of $F_K^p(B')$. We can therefore assume that A is Noetherian. By Proposition 6, there exists n such that the image of w in $F_K^p(A/m^n)$ is nonzero. Thus z' , the image of z , is a zero divisor in $F_K^p(A/m^n)$. However K is separably closed in A/m and so, by induction from Proposition 5, we see that $F_K^p(A/m^n)$ is a ZDN ring. The image z' is thus nilpotent and the same is true of $\mu_p z' \in A/m^n$. As the image of $\mu_p z$ in A/m^n is $\mu_p z'$, it follows that $\mu_p z \in m$.

Proof of Theorem 2. As A_s is a field we can apply Proposition 4 to get $C_K A \cong C_{A_s} A$. However Proposition 7 shows that $C_{A_s} A = F_{A_s} A$. This completes the proof.

The following corollary to Theorem 2 is immediate on applying Proposition 1.

COROLLARY. *If A satisfies the hypotheses of Theorem 2, then $H_K^p A = 0$ for $p > 0$.*

Clearly any local algebra over a separably closed field (i.e. separably closed in its algebraic closure) satisfies the hypotheses of Theorem 2.

If A is a complete Noetherian local K -algebra, there exists [5, Chapter VIII, §12, Theorem 27] a subfield L of A which is mapped onto A/m by the natural map. Under these circumstances A_s is mapped isomorphically onto the separable closure of K in A/m . Thus it follows that, for such an algebra also, the hypotheses of Theorem 2 are satisfied.

Our ultimate goal is to prove the conclusion of Theorem 2 for all local K -algebras; then, loosely speaking, to study this cohomology theory for an arbitrary K -algebra by using sheaf theoretic methods to patch the algebra together from its localizations (at prime or maximal ideals). Partial results in this direction have been obtained.

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Tage Bai Andersen, <i>On Banach space valued extensions from split faces</i>	1
David Marion Arnold, <i>A duality for quotient divisible abelian groups of finite rank</i>	11
Donald Pollard Ballou, <i>Shock sets for first order nonlinear hyperbolic equations</i>	17
Leon Brown and Lowell J. Hansen, <i>On the range sets of H^p functions</i>	27
Alexander Munro Davie and Arne Stray, <i>Interpolation sets for analytic functions</i>	33
M. G. Deshpande, <i>Structure of right subdirectly irreducible rings. II</i>	39
Barry J. Gardner, <i>Some closure properties for torsion classes of abelian groups</i>	45
Paul Daniel Hill, <i>Primary groups whose subgroups of smaller cardinality are direct sums of cyclic groups</i>	63
Richard Allan Holzsager, <i>When certain natural maps are equivalences</i>	69
Donald William Kahn, <i>A note on H-equivalences</i>	77
Joong Ho Kim, <i>R-automorphisms of $R[t][[X]]$</i>	81
Shin'ichi Kinoshita, <i>On elementary ideals of polyhedra in the 3-sphere</i>	89
Andrew T. Kitchen, <i>Watts cohomology and separability</i>	99
Vadim Komkov, <i>A technique for the detection of oscillation of second order ordinary differential equations</i>	105
Charles Philip Lanski and Susan Montgomery, <i>Lie structure of prime rings of characteristic 2</i>	117
Andrew Lenard, <i>Some remarks on large Toeplitz determinants</i>	137
Kathleen B. Levitz, <i>A characterization of general Z.P.I.-rings. II</i>	147
Donald A. Lutz, <i>On the reduction of rank of linear differential systems</i>	153
David G. Mead, <i>Determinantal ideals, identities, and the Wronskian</i>	165
Arunava Mukherjea, <i>A remark on Tonelli's theorem on integration in product spaces</i>	177
Hyo Chul Myung, <i>A generalization of the prime radical in nonassociative rings</i>	187
John Piepenbrink, <i>Rellich densities and an application to unconditionally nonoscillatory elliptic equations</i>	195
Michael J. Powers, <i>Lefschetz fixed point theorems for a new class of multi-valued maps</i>	211
Aribindi Satyanarayan Rao, <i>On the absolute matrix summability of a Fourier series</i>	221
T. S. Ravisankar, <i>On Malcev algebras</i>	227
William Henry Ruckle, <i>Topologies on sequences spaces</i>	235
Robert C. Shock, <i>Polynomial rings over finite dimensional rings</i>	251
Richard Tangeman, <i>Strong heredity in radical classes</i>	259
B. R. Wenner, <i>Finite-dimensional properties of infinite-dimensional spaces</i>	267