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**A CHARACTERIZATION OF GENERAL Z.P.I.-RINGS. II**

KATHLEEN B. LEVITZ

## A CHARACTERIZATION OF GENERAL Z.P.I.-RINGS II

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**A commutative ring  $R$  is a general Z.P.I.-ring if each ideal of  $R$  can be represented as a finite product of prime ideals. If  $R$  is not a general Z.P.I.-ring, it is still possible that each principal ideal of  $R$  can be represented as a finite product of prime ideals. In this paper, it is shown that if  $R$  is a commutative ring in which each ideal generated by two elements can be written as a finite product of prime ideals, then  $R$  must be a general Z.P.I.-ring.**

Let  $R$  be a commutative ring.  $R$  is a *general Z.P.I.-ring* if each ideal of  $R$  can be represented as a finite product of prime ideals. In a previous paper, we proved that  $R$  is a general Z.P.I.-ring if each finitely-generated ideal of  $R$  can be represented as a finite product of prime ideals [4; Theorem 2.3]. If each ideal of  $R$  generated by  $n$  or fewer elements can be represented as a finite product of prime ideals, then we define  $R$  to be a  $\pi(n)$ -ring. Mori completely characterized the structure of  $\pi(1)$ -rings in a series of four papers [5, 6, 7, 8]. Using his characterization, it is not difficult to construct a  $\pi(1)$ -ring that is not a  $\pi(n)$ -ring for any  $n > 1$ . For this reason it is surprising that the main result of this paper is the following theorem.

**THEOREM.** *Let  $R$  be a commutative ring. Then the following conditions are equivalent:*

- (a)  $R$  is a general Z.P.I.-ring;
- (b) for  $n \geq 2$ ,  $R$  is a  $\pi(n)$ -ring;
- (c)  $R$  is a  $\pi(2)$ -ring.

Throughout this paper,  $R$  denotes a commutative ring and  $n$  denotes an arbitrary positive integer.

2.  $\pi(n)$ -rings without zero-divisors. If  $D$  is an integral domain, we call a prime ideal  $P$  of  $D$  *minimal* if  $P$  is of height one. An integral domain  $D$  with identity is a *Krull domain* if there is a set of rank one discrete valuation rings  $\{V_\alpha\}$  such that  $D = \bigcap_\alpha V_\alpha$  and such that each nonzero element of  $D$  is a non-unit in only finitely many of the  $V_\alpha$ .

**EXAMPLE 2.1.** An integral domain  $D$  with identity is a  $\pi(1)$ -ring if and only if  $D$  is a Krull domain in which each minimal prime ideal

is invertible [4; Theorem 1.2]. If  $Z$  denotes the rational integers, then the polynomial ring in one indeterminate  $Z[x]$  is a  $\pi(1)$ -ring and  $Z[x]$  is not a  $\pi(n)$ -ring for any  $n > 1$ .

Henceforth we refer to  $\pi(n)$ -rings without zero-divisors as  $\pi(n)$ -domains.

**LEMMA 2.2.** *Let  $R$  be a  $\pi(2)$ -domain with identity. Then  $R$  is a Krull domain in which each prime ideal of height one is invertible. Moreover, the prime ideals of height one are pairwise comaximal.*

*Proof.* If  $R$  is a  $\pi(2)$ -domain,  $R$  is a  $\pi(1)$ -domain. It follows from [4; Theorem 1.2] that  $R$  is a Krull domain in which each minimal prime ideal is invertible. Let  $P_1$  and  $Q$  be distinct minimal prime ideals of  $R$ . Let  $a \in P_1 \setminus Q$ . Then

$$(a) = \prod_{i=1}^s P_i^{e_i},$$

where, for each  $i$ ,  $e_i \geq 1$ ,  $P_i \neq Q$ , and  $P_i$  is a minimal prime ideal. Let  $b \in Q \setminus \bigcup_{i=1}^s P_i$ . Then

$$(a, b) = \prod_{j=1}^m R_j; (a, b^2) = \prod_{k=1}^p S_k,$$

where for each  $j$  and  $k$ ,  $R_j$  and  $S_k$  are prime ideals of  $R$ .

If  $bt \in (a)$  for some  $t \in R$ , then  $(bt) \subset \prod_{i=1}^s P_i^{e_i}$ . If for each  $i$ ,  $1 \leq i \leq s$ , we let  $v_i$  denote the valuation on  $R$  with respect to the minimal prime ideal  $P_i$ , then  $v_i(bt) \geq e_i$  while  $v_i(b) = 0$ . Hence  $t \in P_i^{(e_i)}$ , the  $e_i$ th symbolic power of  $P_i$ . Since for each  $i$ ,  $P_i$  is invertible, it follows that  $P_i^{(e_i)} = P_i^{e_i}$  [9; Lemma 21], and so  $t \in P_i^{e_i}$ . Because each  $P_i$  is invertible, we can use an induction argument on  $s$  to conclude that  $t \in \prod_{i=1}^s P_i^{e_i} = (a)$ .

If  $\bar{R} = R/(a)$ , and  $\bar{b}$  is the image of  $b$  in  $\bar{R}$ , the above argument shows that  $\bar{b}$  is a regular element of  $\bar{R}$ . In  $\bar{R}$ ,

$$\begin{aligned} (\bar{b}) &= \prod_{j=1}^m (R_j/(a)) \\ (\bar{b}^2) &= \prod_{k=1}^p (S_k/(a)). \end{aligned}$$

By [1; Theorem 1], the factorization of the ideal  $(\bar{b}^2)$  is unique up to factors of  $\bar{R}$ . It follows that  $p = 2m$ , and that we can index the ideals  $S_k$ ,  $1 \leq k \leq p$ , so that

$$R_j = S_{2j-1} = S_{2j}.$$

Hence  $(a, b^2) = \prod_{k=1}^p S_k = \prod_{j=1}^m (R_j)^2 = (a, b)^2$ . Thus

$$(a) \subset (a, b^2) = (a, b)^2 \subset (a^2, b).$$

If  $x \in (a)$ , then  $x = ra^2 + sb$ , where  $r, s \in R$ . This implies that  $sb \in (a)$ , and, consequently,  $s \in (a)$ . We conclude that

$$(a) \subseteq (a)(a, b).$$

Since the reverse conclusion is always valid,

$$(a) = (a)(a, b).$$

Because  $a \neq 0$ , it follows that

$$R = (a, b) \subseteq (P, Q) \subseteq R.$$

Hence the minimal prime ideals of  $R$  are comaximal. This completes the proof of the lemma.

An integral domain with identity that is a general Z.P.I.-ring is called a *Dedekind domain*.

**THEOREM 2.3.** *Let  $R$  be an integral domain with identity. The following conditions are equivalent:*

- (1)  $R$  is a Dedekind domain,
- (2) for  $n \geq 2$ ,  $R$  is a  $\pi(n)$ -domain;
- (3)  $R$  is a  $\pi(2)$ -domain.

*Proof.* (1  $\rightarrow$  2) By definition of Dedekind domain.

(2  $\rightarrow$  3) By definition of  $\pi(n)$ -ring.

(3  $\rightarrow$  1) By Lemma 2.1,  $R$  is a Krull domain in which prime ideals of height one are invertible. To conclude that  $R$  is a Dedekind domain, it suffices to show that  $R$  is of Krull dimension one [3; Theorem 35.16]. Each non-unit of  $R$  is contained in some minimal prime ideal. Hence, if  $R$  has a unique minimal prime ideal  $P$ ,  $P$  is also the unique maximal ideal of  $R$ , and  $R$  is of Krull dimension one. If  $R$  has more than one minimal prime ideal, then by Lemma 2.1, all these prime ideals are comaximal. If  $Q$  is any nonzero proper prime ideal of  $R$ , there is a minimal prime ideal  $P$  such that  $P \subseteq Q$  [3; Corollary 35.10]. If  $P \neq Q$ , there exists  $b \in Q \setminus P$ .  $(b) = \prod_{i=1}^t S_i$ , where for each  $i$ ,  $S_i$  is a minimal prime ideal of  $R$  and  $S_i \neq P$ . Since  $b \in Q$ , for some  $i$ ,  $1 \leq i \leq t$ ,  $S_i \subset Q$ . But this implies that  $R = (P, S_i) \subseteq Q$ . Hence  $Q = P$ , and  $R$  is of Krull dimension one. This completes the proof of the theorem.

**THEOREM 2.4.** *Let  $R$  be a  $\pi(2)$ -domain without identity. Then  $R$  is a general Z.P.I.-ring.*

*Proof.* Each minimal prime ideal of  $R$  is a principal ideal [8;

Theorem 26]. If  $R$  contains a unique minimal prime ideal  $(p)$ , then it must be the case that  $R = (p)$  [8; Lemma II]. We assume that  $R$  contains two distinct minimal prime ideals,  $(p)$  and  $(q)$ . Using the same argument we did in Lemma 2.2, we can show that

$$(p) = (p)(p, q).$$

Since  $(p)$  is a regular ideal, it follows that  $R$  must have an identity [2; Corollary 5.2]. Therefore, since  $R$  has no identity, it must be the case that  $R$  is the only nonzero prime ideal of itself.

Let  $A$  be a nonzero ideal of  $R$ . Then there is a smallest positive integer  $n$  such that  $R^n \subset A \subseteq R^{n-1}$ . Let  $a \in A \setminus R^n$ . Since  $(a) = R^k$  for some  $k < n$ , it follows that  $R^n \subset (a) = R^k \subseteq A \subseteq R^{n-1}$ . Hence  $A = R^{n-1}$ . Because each ideal of  $R$  is a power of  $R$  it follows that  $R$  is a general Z.P.I.-ring [10; Theorem 2]. This completes the proof of this theorem.

### 3. Main result.

LEMMA 3.1. *Let  $R$  be a  $\pi(2)$ -ring with identity. If  $R$  is the direct sum of finitely many rings,  $R = \sum_{i=1}^k R_i$ , then each direct summand  $R_i$  is also a  $\pi(2)$ -ring.*

*Proof.* Let  $R_j$  be one of the direct summands of  $R$ , and let  $A_j = (a_{1j}, a_{2j})$  be an ideal of  $R_j$  generated by two elements of  $R_j$ . Let  $e_i$  denote the identity of the direct summand  $R_i$ ,  $1 \leq i \leq k$ . Then if  $A$  is the ideal of  $R$  generated by the two elements  $(\sum_{i \neq j} e_i) + a_{1j}$  and  $(\sum_{i \neq j} e_i) + a_{2j}$ , then

$$A = \prod_{r=1}^t P_r$$

where for each  $r$ ,  $1 \leq r \leq t$ ,  $P_r$  is a prime ideal of  $R$ . Then  $A_j = AR_j = (\prod_{r=1}^t P_r)R_j = \prod_{r=1}^t (P_r R_j)$ . Since for each  $r$ ,  $P_r R_j$  is a prime ideal of  $R_j$ ,  $A_j$  can be expressed as a finite product of prime ideals. Hence  $R_j$  is a  $\pi(2)$ -ring.

A principal ideal ring  $R$  with identity is called a *special primary ring* if  $R$  contains only one prime ideal  $M \neq R$  and if  $M^k = (0)$  for some positive integer  $k$ .

THEOREM 3.2. *Let  $R$  be a commutative ring. Then the following conditions are equivalent:*

- (a)  $R$  is a general Z.P.I.-ring;
- (b) for  $n \geq 2$ ,  $R$  is a  $\pi(n)$ -ring;
- (c)  $R$  is a  $\pi(2)$ -ring.

*Proof.* It is clear that (a) implies (b) and that (b) implies (c). We now show that (c) implies (a). We consider three cases: (1)  $R$  is a commutative ring with identity; (2)  $R$  is a commutative ring without identity, but with zero divisors; (3)  $R$  is an integral domain without identity.

If  $R$  is a commutative ring with identity, then  $R$  is a direct sum of  $\pi(1)$ -domain with identity and special primary rings by [7; Hauptsatz]. Using [10; Theorem 2], we can conclude that  $R$  is a general Z.P.I.-ring if any summand  $R_i$  of  $R$  that is a domain is Dedekind. From Lemma 3.1 it follows that each summand of  $R$  is a  $\pi(2)$ -ring. Hence if the summand  $R_i$  is a domain,  $R_i$  is Dedekind by Theorem 2.3. Thus a  $\pi(2)$ -ring with identity is a general Z.P.I.-ring.

If  $R$  is a commutative ring without identity, but with zero-divisors, then  $R = M$  or  $R = M + K$ , where  $K$  is a field and  $M$  is a ring without identity such that each ideal of  $M$  is a power of  $M$  [8; Hauptsatz 11].  $R$  is a general Z.P.I.-ring by [10; Theorem 2].

The last case is settled by Theorem 2.4.

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