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**A GENERALIZATION OF THE PRIME RADICAL IN
NONASSOCIATIVE RINGS**

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In [5] Tsai defined the Brown-McCoy prime radical for Jordan rings in terms of the quadratic operation and proved basic results for the radical. In this paper we give a definition of the prime radical for arbitrary nonassociative rings in terms of a $*$ -operation defined on the family of ideals and of a function f of the ring into the family of ideals in the ring. The prime radical for Jordan or standard rings is obtained by a particular choice of the $*$ -operation and the function f . We also extend the results for the Jordan case to weakly W -admissible rings which include the generalized standard rings and therefore alternative and standard rings as well as Jordan rings.

1. Let K be any nonassociative ring and let $\mathcal{I}(K)$ denote the family of ideals of K .

DEFINITION 1. We define a $*$ -operation as a mapping of $\mathcal{I}(K) \times \mathcal{I}(K)$ into the family of additive subgroups of K such that

(*1) for A, B, C , and D in $\mathcal{I}(K)$ if $A \subseteq C$ and $B \subseteq D$, then $A*B \subseteq C*D$,

(*2) $(0)*A = B*(0) = (0)$ for all A, B in $\mathcal{I}(K)$,

(*3) $\overline{A*B} = \overline{A}* \overline{B}$ for any homomorphic images \overline{A} and \overline{B} of A and B in $\mathcal{I}(K)$.

If K is a Jordan ring, let $U_x \equiv 2R_x^2 - R_{x^2}$ be the quadratic operation and AU_B be the additive subgroup of K generated by xU_y , $x \in A$ and $y \in B$. Then the U -operation satisfies the conditions above. If the characteristic is not 2, it is shown in [5] that $AU_A = AA^2$ and is an ideal of K for A in $\mathcal{I}(K)$.

For any ring K and A, B in $\mathcal{I}(K)$, if we define $A*B$ as the additive subgroup $AB^2 + B^2A + (AB)B + (BA)B$, then $A*B$ also satisfies the conditions in Definition 1. In case K is a standard ring, it is shown in [6] that $A*B$ is an ideal of K for A, B in $\mathcal{I}(K)$. If K is commutative or anticommutative, then $A*B = AB^2 + (AB)B$. In particular, if K is a Lie ring, $A*B$ is an ideal of K . Since A^2 is not in general an ideal of K for A in $\mathcal{I}(K)$, but there are considerably broad classes of nonassociative rings in which $A^3 \equiv AA^2 + A^2A$ is an ideal of K for every ideal A , this example will be particularly interesting.

We recall that a noncommutative Jordan ring K is one satisfying

the flexible law $(x, y, x) = 0$ and the Jordan identity $(x, y, x^2) = 0$ for all x, y in K , where $(x, y, z) = (xy)z - x(yz)$. Most of the well known nonassociative rings are included in the class of noncommutative Jordan rings. Recently Thedy [4] defined a considerably broad class of algebras that generalizes many of the well known algebras.

DEFINITION 2. A noncommutative Jordan ring K is called weakly W -admissible if it satisfies

$$[(a, b, c), c] - ([a, c], c, b) = 0,$$

and

$$\begin{aligned} & ([a, b], d, c) + ([b, c], d, a) + ([c, a], d, b) \\ & = p[(a, b, c), d] + q[S(a, b, c), d] + r[d, [b, [a, c]]] \end{aligned}$$

for some integers p, q, r such that either $m(p, q, r) \equiv 3 + 2p + 6q - 4r \neq 0$, or $n(p, r) \equiv p + 4r \neq 0$, where $[a, b] = ab - ba$ and $S(a, b, c) = (a, b, c) + (b, c, a) + (c, a, b)$.

Thedy called a noncommutative Jordan algebra over a field W -admissible if it satisfies the identity $[a, (a, a, b)] = 0$ and the two identities above for p, q, r in the field such that either $m(p, q, r) \neq 0$ or $n(p, r) \neq 0$. He proved that if the characteristic is not 2, then any generalized standard ring of Schafer [2] is W -admissible with $p = -2$ and $q = r = 0$. Therefore, weakly W -admissible rings include generalized standard rings and hence alternative and standard rings as well as Jordan rings. In case the characteristic is not 2, it is also shown in [4, p. 192] that in any weakly W -admissible ring K , A^3 is an ideal of K for A in $\mathcal{S}(K)$.

LEMMA 1.1. *Let K be any ring. Then the conditions (*2) and (*3) imply*

(i) $(A + C)*(B + C) \subseteq A*B + C$, and

(ii) $A*B \subseteq A \cap B$

for ideals A, B, C of K .

Proof. Consider the quotient ring $\bar{K} = K/C$, then by (*3) $\overline{(A + C)*(B + C)} = \bar{A}*\bar{B} = \overline{A*B}$, and hence (i). Let $\bar{K} = K/A$, then $\bar{A}*\bar{B} = \bar{A}*\bar{B} = (\bar{0})*\bar{B} = (\bar{0})$ by (*2) and so $A*B \subseteq A$. Similarly $A*B \subseteq B$ and $A*B \subseteq A \cap B$.

DEFINITION 3. Let K be any ring. Then f is defined as a function of K into $\mathcal{S}(K)$ such that for every a in K

(f 1) $a \in f(a)$,

(f 2) if $x \in f(a)$, then $f(x) \subseteq f(a)$,

(f 3) $\overline{f(a)} = f(\bar{a})$, where \bar{a} is a homomorphic image of a .

The principal ideal (a) generated by a in K is an example of $f(a)$. Now let S be a subset of K and define $f(a)$ to be the ideal (a, S) generated by a and S . Then f satisfies the conditions above. A similar function to f has been defined in [1] for the associative case and in [3].

Henceforth we assume that f denotes a function of K into $\mathcal{S}(K)$ satisfying (f 1), (f 2), and (f 3). Then clearly $(a) \subseteq f(a)$. For an ideal A of K , we denote the ideal $\sum_{a \in A} f(a)$ by $f(A)$. Then $A \subseteq f(A)$ and $f(A) \subseteq f(B)$ if $A \subseteq B$, and also $f((a)) = f(a)$. But in general $f(A) \neq A$ as shown by the example $f(a) = (a, S)$ for a subset S of K . Let $\mathcal{S}'(K)$ denote the family of ideals $f(A)$ for A in $\mathcal{S}(K)$. Then $\mathcal{S}'(K) \subseteq \mathcal{S}(K)$ and in particular, if f is such that $f(a) = (a)$ for all a in K , then $f(A) = A$ and $\mathcal{S}'(K) = \mathcal{S}(K)$.

2. In this section we give a definition of the prime radical for any ring in terms of the $*$ -operation and the function f .

LEMMA 2.1. *Let K be any ring where the $*$ -operation and the function f are defined. For an ideal P of K , the following are equivalent:*

- (i) *If $f(A)*f(B) \subseteq P$ for A, B in $\mathcal{S}(K)$, then either $f(A) \subseteq P$ or $f(B) \subseteq P$.*
- (ii) *If we have $f(A) \cap c(P) \neq \emptyset$ and $f(B) \cap c(P) \neq \emptyset$, then $f(A)*f(B) \cap c(P) \neq \emptyset$.*
- (iii) *If a and b are in $c(P)$, then $f(a)*f(b) \cap c(P) \neq \emptyset$.*

Proof. We need only to show that (ii) and (iii) are equivalent. Let a and b be in $c(P)$, then $f(a) \cap c(P) \neq \emptyset$ and $f(b) \cap c(P) \neq \emptyset$. Hence (ii) implies (iii). Now let A and B be ideals of K with $f(A) \cap c(P) \neq \emptyset$ and $f(B) \cap c(P) \neq \emptyset$. Let $a \in f(A) \cap c(P)$ and $b \in f(B) \cap c(P)$. Assuming (iii), we get $f(a)*f(b) \cap c(P) \neq \emptyset$ and by (*1) $f(A)*f(B) \cap (P) \neq \emptyset$, thus (ii) holds.

DEFINITION 4. (i) An ideal P of K is called f^* -prime if it satisfies any one of Lemma 2.1. A nonempty subset M of K is called an f^* -system if, for A, B in $\mathcal{S}(K)$, $f(A) \cap M \neq \emptyset$ and $f(B) \cap M \neq \emptyset$ imply $f(A)*f(B) \cap M \neq \emptyset$.

(ii) An ideal P of K is called f^* -semiprime if, for any ideal A of K , $f(A)*f(A) \subseteq P$ implies $f(A) \subseteq P$. A nonempty subset M of K is called an sf^* -system if, for A in $\mathcal{S}(K)$, $f(A) \cap M \neq \emptyset$ implies $f(A)*f(A) \cap M \neq \emptyset$.

An ideal P is f^* -prime if and only if $c(P)$ is an f^* -system. Similarly, an ideal P is f^* -semiprime if and only if $c(P)$ is an sf^* -

system. Let K be a Jordan or standard ring. If we define $A*B$ as AU_B or as $AB^2 + B^2A + (AB)B + (BA)B$ and define $f(a)$ as (a) for every a in K , then the definition of f^* -prime and f^* -semiprime ideals coincide with those in [5] or in [6].

DEFINITION 5. For A in $\mathcal{S}(K)$, $A^* = \{x \in K \mid \text{any } f^*\text{-system containing } x \text{ meets } A\}$ is called the f^* -radical of A . Similarly, $A_* = \{y \in K \mid \text{any } sf^*\text{-system containing } y \text{ meets } A\}$ is called the sf^* -radical of A .

THEOREM 2.2. *Let A be an ideal of K . Then*

- (i) A^* is the intersection of all the f^* -prime ideals P_i containing A .
- (ii) A_* is the intersection of all f^* -semiprime ideals containing A .
- (iii) A_* is an f^* -semiprime ideal of K .
- (iv) A is f^* -semiprime if and only if $A = A_*$.

Proof. The proofs are essentially the same as in [5]. But to emphasize use of the $*$ -operation and the function f we prove only (i). Let $\bigcap_i P_i$ be the intersection of all the f^* -prime ideals P_i of K containing A . If $a \notin P_i$ for some i , then $a \in c(P_i)$, being an f^* -system, and $c(P_i) \cap A = \emptyset$. Hence $a \notin A^*$ and $A^* \subseteq \bigcap_i P_i$. Conversely, if $a \notin A^*$, then there exists an f^* -system M with $a \in M$ but $A \cap M = \emptyset$. By Zorn's lemma we find a maximal ideal P such that $P \supseteq A$ but $P \cap M = \emptyset$. Let B, C be ideals of K such that $f(B) \cap c(P) \neq \emptyset$ and $f(C) \cap c(P) \neq \emptyset$. By the maximality of P , $(f(B) + P) \cap M \neq \emptyset$ and $(f(C) + P) \cap M \neq \emptyset$. Since M is an f^* -system, $\emptyset \neq (f(B) + P)*(f(C) + P) \cap M \subseteq (f(B)*f(C) + P) \cap M$ by Lemma 1.1 (i), thus $f(B)*f(C) \cap c(P) \neq \emptyset$. Hence P is f^* -prime and $a \in P$.

LEMMA 2.3. *Let a be an element of K and S be an sf^* -system containing a . Then there exists an f^* -system M such that $a \in M$ and $M \subseteq S$.*

Proof. Let $a_1 = a$, then $a_1 \in f(a_1) \cap S$ and so $f(a_1)*f(a_1) \cap S \neq \emptyset$. Hence we obtain a set $M = \{a_1, a_2, \dots, a_n, \dots\}$ such that $a_{k+1} \in f(a_k) \cap S$ and $M \subseteq S$. By Lemma 1.1 (ii) we note that $a_{k+1} \in f(a_k)*f(a_k) \subseteq f(a_k)$ and so $f(a_{k+1}) \subseteq f(a_k)$. Let $p = \max(i, j)$, then $a_{p+1} \in f(a_p)*f(a_p) \cap S \subseteq f(a_i)*f(a_j) \cap S$. Hence $f(a_i)*f(a_j) \cap M \neq \emptyset$ and M is an f^* -system.

Therefore, as in [5], we have

THEOREM 2.4. *For any ideal A of K , $A^* = A_*$. A^* is called the f^* -prime radical of A .*

DEFINITION 6. The f^* -prime radical, $R^*(K)$, of K is the f^* -prime radical of the ideal (0) . A ring K is said to be f^* -semisimple if $R^*(K) = (0)$.

LEMMA 2.5. Let \bar{K} be a homomorphic image of K . If M is an f^* -system of K , then so is \bar{M} in \bar{K} .

Proof. Let \bar{A}, \bar{B} be ideals of \bar{K} such that $f(\bar{A}) \cap \bar{M} \neq \emptyset$ and $f(\bar{B}) \cap \bar{M} \neq \emptyset$, where A and B are ideals in K containing the kernel. Recalling (f 3) and $A \subseteq f(A)$, these imply $f(A) \cap M \neq \emptyset$ and $f(B) \cap M \neq \emptyset$. Since M is an f^* -system, by (*3) and (f 3) we see that $f(\bar{A}) * f(\bar{B}) \cap \bar{M} \neq \emptyset$.

Therefore, by Lemma 2.3 we easily see that any homomorphic image of an f^* -prime ideal containing the kernel is also f^* -prime. Hence we obtain

THEOREM 2.6. Let K be a ring and $R^*(K)$ be the f^* -prime radical of K , then $R^*(K/R^*(K)) = (0)$, that is, $K/R^*(K)$ is f^* -semisimple.

DEFINITION 7. A ring K is called an f^* -prime ring if (0) is an f^* -prime ideal in K .

Clearly, an f^* -prime ring is f^* -semisimple. Since any homomorphic image of an f^* -prime ideal is f^* -prime, if P is an f^* -prime ideal in K then K/P is an f^* -prime ring. Let $\bar{K} = K/P$ be an f^* -prime ring and let $f(A) * f(B) \subseteq P$, then $f(\bar{A}) * f(\bar{B}) \subseteq (\bar{0})$ and so $f(A) \subseteq P$ or $f(B) \subseteq P$, thus P is f^* -prime in K . Hence P is an f^* -prime ideal of K and only if K/P is an f^* -prime ring. Therefore, as for Jordan rings, we obtain

THEOREM 2.7. A ring K is isomorphic to a subdirect sum of f^* -prime rings if and only if K is f^* -semisimple.

3. Throughout this section we assume that the $*$ -operation satisfies the following additional condition:

$$(*4) \quad A * A = A^3 \text{ and } A * A \text{ is an ideal of } K \text{ for } A \text{ in } \mathcal{S}(K).$$

We recall that if K is a weakly W -admissible or Lie ring then $A * B = AB^2 + B^2A + (AB)B + (BA)B$ satisfies (*4).

THEOREM 3.1. Let A be an ideal of a ring K and $r \in A_*$. Then a power of r belongs to A . Furthermore if K is power-associative, then the f^* -radical $R^*(K)$ is a nil ideal in K .

Proof. Let M be the multiplicatively closed system generated

by r in K . Then it follows from (*4) that M is an sf^* -system containing r . Hence $M \cap A \neq \emptyset$. If K is power-associative and $r \in R^*(K)$, then $r^k \in (0)$ for some k and so $R^*(K)$ is nil.

Therefore, the f^* -radical $R^*(K)$ is contained in the nil radical $N(K)$ (the maximal nil ideal in K).

Let $\mathcal{S}'(K)$ denote the set of ideals $f(A)$ for A in $\mathcal{S}(K)$. Then $\mathcal{S}'(K) \subseteq \mathcal{S}(K)$.

THEOREM 3.2. *A ring K is f^* -semisimple if and only if $\mathcal{S}'(K)$ contains no nonzero nilpotent ideal.*

Proof. It follows from Theorem 2.2 (iv) that K is f^* -semisimple if and only if the ideal (0) is f^* -semiprime. If $f(A)$ is a nonzero nilpotent ideal for A in $\mathcal{S}(K)$, there exist positive integers $u = 3^t$ and $v = 3^{t-1}$ such that $f(A)^u = (0)$ but $f(A)^v \neq (0)$. But then since $f(A)^v * f(A)^v \subseteq f(A)^{3v} = f(A)^u = (0)$, (0) is not f^* -semiprime. Conversely, if (0) is not f^* -semiprime, then there exists an ideal $f(A) \neq (0)$ such that $f(A) * f(A) = f(A)^3 = (0)$, thus $f(A)$ is nilpotent.

COROLLARY 3.3. *The f^* -radical $R^*(K)$ contains all the nilpotent ideals in $\mathcal{S}'(K)$.*

Proof. Let $f(A)$ be a nilpotent ideal in $\mathcal{S}'(K)$ and $\bar{K} = K/R^*(K)$, then $\overline{f(A)} = f(\bar{A}) \in \mathcal{S}'(\bar{K})$, and $f(\bar{A})$ is nilpotent in \bar{K} . Since \bar{K} is f^* -semisimple, by Theorem 3.2 $f(\bar{A}) = (\bar{0})$, thus $f(A) \subseteq R^*(K)$.

THEOREM 3.4. *If K is a ring and $\mathcal{S}'(K)$ contains a maximal nilpotent ideal $S'(K)$, then $R^*(K) = S'(K)$.*

Proof. By Corollary 3.3, $S'(K) \subseteq R^*(K)$. Let $\bar{K} = K/S'(K)$, then $\mathcal{S}'(\bar{K})$ contains no nonzero nilpotent ideal and by Theorem 3.2 $R^*(\bar{K}) = (0)$. If $r \notin S'(K)$, then $\bar{r} \neq \bar{0}$ and so there exists an f^* -prime ideal \bar{P} in \bar{K} with $\bar{r} \notin \bar{P}$. From (*3) and (f3) it follows that the inverse image P of \bar{P} is an f^* -prime ideal in K . But since $\bar{r} \notin \bar{P}$, $r \notin P$ and so $r \notin R^*(K)$, thus $R^*(K) \subseteq S'(K)$.

Now suppose that $f(a) = (a)$ for every element a in K . Then $\mathcal{S}(K) = \mathcal{S}'(K)$. Hence by Theorem 3.2 K is f^* -semisimple if and only if K has no nonzero nilpotent ideal, and $R^*(K)$ contains all nilpotent ideals of K . In this case the ideal $S'(K)$ is a maximal nilpotent ideal $S(K)$ in K and by Theorem 3.4 $R^*(K) = S(K)$.

Let K now be a finite dimensional W -admissible or Lie algebra over a field. Let $f(a) = (a)$ for all a in K . If K is W -admissible, then it is shown in [4] that the nil radical $N(K)$ is nilpotent and so the

unique maximal nilpotent ideal $S(K)$. Hence by Theorem 3.4 $R^*(K) = N(K) = S(K)$. If K is a Lie algebra, it is well known that K has a maximal nilpotent ideal $S(K)$ and hence $R^*(K) = S(K)$.

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