ON THE ABSOLUTE MATRIX SUMMABILITY OF A FOURIER SERIES

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In this paper, the author gives sufficient conditions for a Fourier series at an arbitrary but fixed point to be absolutely matrix summable.

1. Introduction. Let \( \sum_{n=0}^{\infty} u_n \) be an infinite series with partial sums \( s_n \), and let \( A = (a_{nk}) \) be a triangular infinite matrix of real numbers (see Hardy [2]). The series \( \sum u_n \) is said to be absolutely summable \( A \), or summable \( |A| \), if

\[
\sum_{n=1}^{\infty} |\tau_n - \tau_{n-1}| < \infty ,
\]

where

\[
\tau_n = \sum_{k=0}^{n} a_{nk}s_k .
\]

Let \( f(t) \) be a Lebesgue-integrable function of period \( 2\pi \), with Fourier series

(1.1) \[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t) .
\]

With a fixed point \( x \), we set

(1.2) \[
\phi(t) = \phi_x(t) = \frac{1}{2} [f(x + t) + f(x - t)] ,
\]

(1.3) \[
\Phi(t) = \int_{t_0}^{t} |\phi(u)| \, du .
\]

We establish the following theorem for the absolute matrix summability of the Fourier series (1.1) of \( f(t) \) at \( t = x \).

**Theorem.** Let \( A = (a_{nk}) \) be a triangular infinite matrix of real numbers such that \( Aa_{nk} = a_{nk} - a_{n,k+1} \) is monotonic with respect to \( n \geq k \) for each fixed \( k \geq 0 \).

Let \( \alpha(t) \) be a positive function such that \( t^{r/\alpha(t)} \), for some \( r \) with \( 0 < r < 1 \), is nondecreasing for \( t \geq t_0 \). Suppose that

(1.4) \[
\sum_{n=1}^{\infty} \frac{n|a_{nk}|}{\alpha(n)} < \infty ,
\]
Further, let

\[ \Phi(t) = O\left[ \frac{t}{\alpha(1/t)} \right] \quad \text{as } t \to 0^+. \]

If all of the above conditions hold, then the Fourier series (1.1) of \( f(t) \) at \( t = x \) is summable \( |A| \).

We shall require the following lemmas.

**Lemma 1.** If \( \alpha(t) \) is defined as in the theorem, then

\[ \int_{t_0}^{t} \frac{du}{\alpha(u)} = O\left[ \frac{t}{\alpha(t)} \right] \quad \text{for all } t \geq t_0. \]

**Proof.**

\[
\int_{t_0}^{t} \frac{du}{\alpha(u)} = \int_{t_0}^{t} \frac{u^r}{\alpha(u)} \cdot \frac{du}{u^r} \\
\leq \frac{t^r}{\alpha(t)} \int_{t_0}^{t} \frac{du}{u^r} \leq \frac{t^r}{\alpha(t)} \cdot \frac{t^{-r+1}}{1-r} = O\left[ \frac{t}{\alpha(t)} \right].
\]

**Lemma 2.** If \( A = (a_{nk}) \) is defined as in the theorem and if

\begin{align*}
(2.2) & \sum_{n=0}^{\infty} |t_n| \cdot |a_{nn}| < \infty, \\
(2.3) & \sum_{n=1}^{n-1} |t_n| \cdot |\Delta a_{nn}| = O(1) \quad \text{as } m \to \infty,
\end{align*}

where

\[ t_n = \sum_{k=0}^{n} s_k, \]

then \( \sum u_n \) is summable \( |A| \).

**Proof.** By Abel’s transformation,

\[ \tau_m - \tau_{m-1} = \sum_{k=0}^{m} (a_{nk} - a_{n-1,k})s_k \\
= \sum_{k=0}^{m} (\Delta a_{nk} - \Delta a_{n-1,k})t_k + a_{nn}t_n. \]

Now

\[
\sum_{n=0}^{m} \sum_{k=0}^{n-1} |\Delta a_{nk} - \Delta a_{n-1,k}| \cdot |t_k| \\
= \sum_{k=0}^{m} |t_k| \cdot \left( \sum_{n=k+1}^{m} |\Delta a_{nk} - \Delta a_{n-1,k}| \right) = \sum_{k=0}^{m-1} |t_k| \cdot |\Delta a_{mk} - a_{kk}|.
\]
Thus,
\[
\sum_{n=1}^{m} |\tau_n - \tau_{n-1}| \leq \sum_{n=0}^{m-1} |t_n| \cdot |\Delta a_{mn}| + 2 \sum_{n=0}^{m} |t_n| \cdot |a_{nn}| = O(1)
\]
as \( m \to \infty \), by (2.2) and (2.3).

This completes the proof of the lemma.

3. **Proof of the Theorem.** We write
\[
s_n(x) = \sum_{0}^{n} A_k(x), t_n(x) = \sum_{0}^{n} s_k(x).
\]

By (1.6), there exists \( \delta (0 < \delta < 1) \) such that
\[
(3.1) \quad \Phi(t) \leq K \frac{t}{\alpha(1/t)} \quad \text{for } 0 < t \leq \delta,
\]
where \( K \) is a positive constant (not necessarily the same at each occurrence). Now, for \( n > \delta^{-1} \),
\[
\pi t_n(x) = \int_{0}^{\pi} \phi(t) \left[ \frac{\sin (n + 1)(t/2)}{\sin (t/2)} \right] dt
\]
\[
= \int_{0}^{\delta} + \int_{\delta}^{n-1} + \int_{n}^{\pi} = I_1 + I_2 + I_3, \text{ say}.
\]

We observe that
\[
(3.3) \quad \left[ \frac{\sin (n + 1) \cdot (t/2)}{\sin (t/2)} \right]^2 = \begin{cases} O(n^2) & \text{for } \sin t/2 \neq 0 \text{ and } n \geq 1, \\ O(1/t^4) & \text{for } 0 < t \leq \pi. \end{cases}
\]

So, by (3.1),
\[
(3.4) \quad |I_1| \leq Kn^2 \int_{0}^{\delta} |\phi(t)| dt \leq K \frac{n}{\alpha(n)}.
\]

Further, assuming \( t'/\alpha(t) \) nondecreasing for \( t \geq \delta^{-1} \),
\[
|I_2| \leq K \int_{\delta}^{\pi} \frac{|\phi(t)|}{t^2} dt
\]
\[
= K \left\{ \int_{\delta}^{\pi} \frac{\Phi(t)}{t^2} dt + 2 \int_{\delta}^{\pi} \frac{\Phi(t)}{t^3} dt \right\}
\]
\[
\leq K \left[ \frac{\Phi(\delta)}{\delta^2} + \int_{\delta}^{\pi} \frac{dt}{t^2 \alpha(1/t)} \right]
\]
\[
= K \left[ \frac{\Phi(\delta)}{\delta^2} + \int_{\delta}^{\pi} \frac{du}{u^{-1} \alpha(u)} \right]
\]
\[
\leq K \frac{n}{\alpha(n)} \quad \text{as } n \to \infty, \text{ by (2.1)}.
\]
Obviously,

\[ I_3 = O(1) . \]  

From (3.2), (3.4)-(3.6), it follows that

\[ t_n(x) = O\left[ \frac{n}{\alpha(n)} \right] \]  

as \( n \to \infty \).

Hence

\[ \sum_n |t_k(x)| \cdot |a_{kk}| = O\left[ \sum_n \frac{k}{\alpha(k)} |a_{kk}| \right] = o(1) \]

as \( n \to \infty \), by (1.4).

Moreover,

\[ \sum_{n=1}^{m-1} |t_n(x)| \cdot |\Delta a_{mn}| = |t_0(x)| \cdot |\Delta a_{m0}| + O\left[ \sum_{n=1}^{m-1} \frac{n}{\alpha(n)} \cdot |\Delta a_{mn}| \right] \]

\[ = O(1) \]  

as \( m \to \infty \), by (1.5).

Now the theorem follows from Lemma 2.

4. Note. Let \( A = (a_{nk}) \) be a triangular infinite matrix of real numbers such that \( a_{nn} \geq 0 \) for all \( n \geq 0 \) and \( \Delta a_{nk} \) is nondecreasing with respect to \( n \geq k \) for each fixed \( k \geq 0 \). Let \( \alpha(t) \) be defined as in the theorem, and let

\[ \Delta a_{m,0} + \sum_{n=1}^{m} \frac{n(\Delta a_{mn})}{\alpha(n)} = O(1) \]  

as \( m \to \infty \).

Then, if the condition (1.6) holds, the Fourier series (1.1) of \( f(t) \) at \( t = x \) is summable \( |A| \).

**Proof.** Let

\[ \tau_n(x) = \sum_{k=0}^{n} a_{nk}s_k(x) . \]

Then

\[ \sum_{n=1}^{m} |\tau_n(x) - \tau_{n-1}(x)| \]

\[ \leq \sum_{n=1}^{m} \sum_{k=0}^{n} |\Delta a_{nk} - \Delta a_{n-1,k}| \cdot |t_k(x)| \]

\[ = \sum_{k=1}^{m} |t_k(x)| \left( \sum_{n=k}^{m} |\Delta a_{nk} - \Delta a_{n-1,k}| \right) + |t_0(x)| \sum_{n=1}^{m} |\Delta a_{n0} - \Delta a_{n-1,0}| \]

\[ = \sum_{k=1}^{m} |t_k(x)| (\Delta a_{mk}) + |t_0(x)|(\Delta a_{m0} - a_{00}) \]  

(4.2)
\[ \leq |t_{0}(x)| (\Delta a_{m,0}) + O\left[ \sum_{k=1}^{\infty} \frac{k}{\alpha(k)} (\Delta a_{m,k}) \right], \quad \text{by (3.7)} \]

\[ = 0(1) \quad \text{as } m \to \infty, \quad \text{by (4.1)}. \]

So the required result follows.

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