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**ON THE ABSOLUTE MATRIX SUMMABILITY OF A FOURIER
SERIES**

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**In this paper, the author gives sufficient conditions for
 a Fourier series at an arbitrary but fixed point to be absolutely
 matrix summable.**

1. **Introduction.** Let $\sum_0^\infty u_n$ be an infinite series with partial sums s_n , and let $A = (a_{nk})$ be a triangular infinite matrix of real numbers (see Hardy [2]). The series $\sum u_n$ is said to be absolutely summable A , or summable $|A|$, if

$$\sum_1^\infty |\tau_n - \tau_{n-1}| < \infty ,$$

where

$$\tau_n = \sum_{k=0}^n a_{nk} s_k .$$

Let $f(t)$ be a Lebesgue-integrable function of period 2π , with Fourier series

$$(1.1) \quad \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) \equiv \sum_0^\infty A_n(t) .$$

With a fixed point x , we set

$$(1.2) \quad \phi(t) = \phi_x(t) = \frac{1}{2}[f(x+t) + f(x-t)] ,$$

$$(1.3) \quad \Phi(t) = \int_0^t |\phi(u)| du .$$

We establish the following theorem for the absolute matrix summability of the Fourier series (1.1) of $f(t)$ at $t = x$.

THEOREM. *Let $A = (a_{nk})$ be a triangular infinite matrix of real numbers such that $\Delta a_{nk} = a_{nk} - a_{n,k+1}$ is monotonic with respect to $n \geq k$ for each fixed $k \geq 0$.*

Let $\alpha(t)$ be a positive function such that $t^r/\alpha(t)$, for some r with $0 < r < 1$, is nondecreasing for $t \geq t_0$. Suppose that

$$(1.4) \quad \sum_{n=1}^\infty \frac{n|a_{nn}|}{\alpha(n)} < \infty ,$$

$$(1.5) \quad |\Delta a_{m,0}| + \sum_{n=1}^{m-1} \frac{n |\Delta a_{mn}|}{\alpha(n)} = O(1) \quad \text{as } m \rightarrow \infty .$$

Further, let

$$(1.6) \quad \Phi(t) = O\left[\frac{t}{\alpha(1/t)}\right] \quad \text{as } t \rightarrow 0 + .$$

If all of the above conditions hold, then the Fourier series (1.1) of $f(t)$ at $t = x$ is summable $|A|$.

We shall require the following lemmas.

LEMMA 1. If $\alpha(t)$ is defined as in the theorem, then

$$(2.1) \quad \int_{t_0}^t \frac{du}{\alpha(u)} = O\left[\frac{t}{\alpha(t)}\right] \quad \text{for all } t \geq t_0 .$$

Proof.

$$\begin{aligned} \int_{t_0}^t \frac{du}{\alpha(u)} &= \int_{t_0}^t \frac{u^r}{\alpha(u)} \cdot \frac{du}{u^r} \\ &\leq \frac{t^r}{\alpha(t)} \int_{t_0}^t \frac{du}{u^r} \leq \frac{t^r}{\alpha(t)} \cdot \frac{t^{-r+1}}{1-r} = O\left[\frac{t}{\alpha(t)}\right] . \end{aligned}$$

LEMMA 2. If $A = (a_{nk})$ is defined as in the theorem and if

$$(2.2) \quad \sum_{n=0}^{\infty} |t_n| \cdot |a_{nn}| < \infty ,$$

$$(2.3) \quad \sum_{n=0}^{m-1} |t_n| \cdot |\Delta a_{mn}| = O(1) \quad \text{as } m \rightarrow \infty ,$$

where

$$t_n = \sum_{k=0}^n s_k ,$$

then $\sum u_n$ is summable $|A|$.

Proof. By Abel's transformation,

$$\begin{aligned} \tau_n - \tau_{n-1} &= \sum_{k=0}^n (a_{nk} - a_{n-1,k}) s_k \\ &= \sum_{k=0}^{n-1} (\Delta a_{nk} - \Delta a_{n-1,k}) t_k + a_{nn} t_n . \end{aligned}$$

Now

$$\begin{aligned} &\sum_{n=1}^m \sum_{k=0}^{n-1} |\Delta a_{nk} - \Delta a_{n-1,k}| \cdot |t_k| \\ &= \sum_{k=0}^{m-1} |t_k| \cdot \left(\sum_{n=k+1}^m |\Delta a_{nk} - \Delta a_{n-1,k}| \right) = \sum_{k=0}^{m-1} |t_k| \cdot |\Delta a_{mk} - a_{kk}| . \end{aligned}$$

Thus,

$$\sum_{n=1}^m |\tau_n - \tau_{n-1}| \leq \sum_{n=0}^{m-1} |t_n| \cdot |\Delta a_{m_n}| + 2 \sum_{n=0}^m |t_n| \cdot |a_{n_n}| = O(1)$$

as $m \rightarrow \infty$, by (2.2) and (2.3).

This completes the proof of the lemma.

3. *Proof of the Theorem.* We write

$$s_n(x) = \sum_0^n A_k(x), t_n(x) = \sum_0^n s_k(x).$$

By (1.6), there exists $\delta(0 < \delta < 1)$ such that

$$(3.1) \quad \Phi(t) \leq K \frac{t}{\alpha(1/t)} \quad \text{for } 0 < t \leq \delta,$$

where K is a positive constant (not necessarily the same at each occurrence). Now, for $n > \delta^{-1}$,

$$(3.2) \quad \begin{aligned} \pi t_n(x) &= \int_0^\pi \phi(t) \left[\frac{\sin(n+1)(t/2)}{\sin(t/2)} \right]^2 dt \\ &= \int_0^{n^{-1}} + \int_{n^{-1}}^\delta + \int_\delta^\pi = I_1 + I_2 + I_3, \text{ say.} \end{aligned}$$

We observe that

$$(3.3) \quad \left[\frac{\sin(n+1) \cdot (t/2)}{\sin(t/2)} \right]^2 = \begin{cases} O(n^2) & \text{for } \sin t/2 \neq 0 \text{ and } n \geq 1, \\ O(1/t^2) & \text{for } 0 < t \leq \pi. \end{cases}$$

So, by (3.1),

$$(3.4) \quad |I_1| \leq Kn^2 \int_0^{n^{-1}} |\phi(t)| dt \leq K \frac{n}{\alpha(n)}.$$

Further, assuming $t^r/\alpha(t)$ nondecreasing for $t \geq \delta^{-1}$,

$$(3.5) \quad \begin{aligned} |I_2| &\leq K \int_{n^{-1}}^\delta \frac{|\phi(t)|}{t^2} dt \\ &= K \left\{ \left[\frac{\Phi(t)}{t^2} \right]_{n^{-1}}^\delta + 2 \int_{n^{-1}}^\delta \frac{\Phi(t)}{t^3} dt \right\} \\ &\leq K \left[\frac{\Phi(\delta)}{\delta^2} + \int_{n^{-1}}^\delta \frac{dt}{t^2 \alpha(1/t)} \right] \\ &= K \left[\frac{\Phi(\delta)}{\delta^2} + \int_{\delta^{-1}}^n \frac{du}{\alpha(u)} \right] \\ &\leq K \frac{n}{\alpha(n)} \quad \text{as } n \rightarrow \infty, \text{ by (2.1).} \end{aligned}$$

Obviously,

$$(3.6) \quad I_3 = O(1) .$$

From (3.2), (3.4)–(3.6), it follows that

$$(3.7) \quad t_n(x) = O\left[\frac{n}{\alpha(n)}\right] \quad \text{as } n \rightarrow \infty .$$

Hence

$$(3.8) \quad \sum_n |t_k(x)| \cdot |a_{kk}| = O\left[\sum_n \frac{k}{\alpha(k)} |a_{kk}|\right] = o(1) \\ \text{as } n \rightarrow \infty , \text{ by (1.4).}$$

Moreover,

$$(3.9) \quad \sum_0^{m-1} |t_n(x)| \cdot |\Delta a_{mn}| = |t_0(x)| \cdot |\Delta a_{m0}| + O\left[\sum_1^{m-1} \frac{n}{\alpha(n)} \cdot |\Delta a_{mn}|\right] \\ = O(1) \quad \text{as } m \rightarrow \infty , \text{ by (1.5).}$$

Now the theorem follows from Lemma 2.

4. NOTE. Let $A = (a_{nk})$ be a triangular infinite matrix of real numbers such that $a_{nn} \geq 0$ for all $n \geq 0$ and Δa_{nk} is nondecreasing with respect to $n \geq k$ for each fixed $k \geq 0$. Let $\alpha(t)$ be defined as in the theorem, and let

$$(4.1) \quad \Delta a_{m,0} + \sum_{n=1}^m \frac{n(\Delta a_{mn})}{\alpha(n)} = O(1) \quad \text{as } m \rightarrow \infty .$$

Then, if the condition (1.6) holds, the Fourier series (1.1) of $f(t)$ at $t = x$ is summable $|A|$.

Proof. Let

$$\tau_n(x) = \sum_{k=0}^n a_{nk} s_k(x) .$$

Then

$$(4.2) \quad \sum_{n=1}^m |\tau_n(x) - \tau_{n-1}(x)| \\ \leq \sum_{n=1}^m \sum_{k=0}^n |\Delta a_{nk} - \Delta a_{n-1,k}| \cdot |t_k(x)| \\ = \sum_{k=1}^m |t_k(x)| \left(\sum_{n=k}^m |\Delta a_{nk} - \Delta a_{n-1,k}| \right) + |t_0(x)| \sum_{n=1}^m |\Delta a_{n0} - \Delta a_{n-1,0}| \\ = \sum_{k=1}^m |t_k(x)| (\Delta a_{mk}) + |t_0(x)| (\Delta a_{m0} - a_{00})$$

$$\begin{aligned} &\leq |t_0(x)|(\Delta a_{m,0}) + O\left[\sum_{k=1}^m \frac{k}{\alpha(k)}(\Delta a_{mk})\right], \quad \text{by (3.7)} \\ &= 0(1) \quad \text{as } m \rightarrow \infty, \text{ by (4.1).} \end{aligned}$$

So the required result follows.

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