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A study is made of two means to topologize a space of sequences. The first method rests upon the duality of every sequence space S with the sequence space φ (finitely $\neq 0$) by means of the form

$$((a_j), (b_j)) = \sum_j a_j b_j \quad (a_j) \in S, (b_j) \in \varphi .$$

The second method is a generalization of the Köthe-Toeplitz duality theory. The Köthe dual S^α of a sequence space S consists of all (b_j) such that $(a_j b_j) \in l^1$ (absolutely convergent series) for $(a_j) \in S$. Other spaces may take the role of l^1 in the above definition. A means to construct a topology on S is determined using this generalized dual. Finally, a particularly suitable type of space (the sum space) to play the role of l^1 is defined.

Our motivation is primarily the inexact but nevertheless meaningful question: what is the "natural" topology for an arbitrary space of sequences S. We consider two classes of topologies on S. Both classes include the topologies studied by Köthe and Toeplitz [10] and Garling [3, 4]. Our most important result is Theorem 4.10 which establishes a relationship between these two classes.

The first method of topologizing a space of sequences is based upon the observation that every sequence space S is in duality with the space φ of finitely nonzero sequences by means of the natural pairing

$$((a_j), (b_j)) = \sum_j a_j b_j \quad (a_j) \in S, (b_j) \in \varphi .$$

It is thus possible to define upon S topologies having a neighborhood base at 0 consisting of polars of a subfamily of the collection of all S-bounded subsets of φ. A few basis observations are made concerning this topology in § 3.

The second method is a direct extension of the Köthe-Toeplitz duality theory [10]. The Köthe-Toeplitz dual, S^α, of a sequence space S consists of all sequences (b_j) such that $\sum_{j=1}^{\infty} |a_j b_j| < \infty$ for each (a_j) in S. In other words, S^α consists of all (b_j) such that $(a_j b_j) \in l^1$ (absolutely convergent series) for each $(a_j) \in S$. It is easy to see how S and S^α are in duality. In § 4 we examine the consequences of allowing other spaces to play the role of l^1 in the above alternative definition. Thus for S a sequence space and T a sequence space with a linear topological structure S^T consists of all sequences (b_j) such that

$(a_jb_j) \in T$ for each $(a_j) \in S$. Although S and S^T may not be in duality there is a very natural way to determine topologies on S using S^T and T. Many of the results of § 4 are generalizations of the results of Köthe and Toeplitz to this new setting.

A particularly suitable type of space to use for T is the sum space which is related to the summation of series. The spaces φ, l^1 and cs (convergent series) are all sum spaces, but there are many other examples. Examples of sum spaces and ways to generate them are presented in [13].

2. **Notation and algebraic preliminaries.** The results of this paper apply to both complex and real spaces of sequences. No further distinctions regarding the scalar field will be made. The letters, s, t, u, v with or without subscripts will denote sequences. For s the sequence $\{a_1, a_2, \cdots\}$ $s(j)$ means a_j the jth coordinate of s. If A is any subset of $\{1, 2, \cdots\}$ then $s[A]$ is the sequence for which

$$s[A](j) = \begin{cases} s(j) & j \in A \\ 0 & j \notin A \end{cases}.$$

In particular, if A is the set $\{1, 2, \cdots, n\}$, $s[A]$ is written $s[\leqq n]$. The sequence $s[\{j\}]$ will always be written $s[j]$. The sequence consisting entirely of one's is denoted by e.

The operations of addition, scalar multiplication, and multiplication of sequences are defined coordinatewise. Thus, for instance, $u = st$ means $u(j) = s(j)t(j)$ for each j. The set of all sequences, which will be denoted by ω, is a linear algebra under these operations. A *sequence space* is a set of sequences which is closed under addition and scalar multiplication. If in addition a set is closed under multiplication it is called *a sequence algebra.*

For A and B, sets of sequences, $A + B$ is the set of all $s + t$ with $s \in A, t \in B$; AB is the set of all st with $s \in A$ and $t \in B$; aA is the set of all as with $s \in A$. For s and t sequences such that st is summable the linear form

$$(s, t) = \sum_{j=1}^{\infty} s(j)t(j)$$

is defined.

The following statement is not difficult to verify.

PROPOSITION 2.1. *If* $\{S_\alpha : \alpha \in A\}$ *is a family of sequence spaces (algebras)* $\bigcap_\alpha S_\alpha$ *is a sequence space (algebra). The set of all finite sums* $s_1 + s_2 + \cdots + s_n$ *with* $\{s_1, s_2, \cdots, s_n\} \subseteqq \bigcup_\alpha S_\alpha$ *and* n *arbitrary is a sequence space (not necessarily an algebra) which is equal to*

$\cap \{T: T \text{ is a sequence space containing each } S\}$.

For B a set of sequences and t a sequence tB is the set of all sequences ts as s ranges over B. The set of all sequences s for which $ts \in B$ is denoted by $t^{-1}B$. If $t(j) \neq 0$ for each j and $t'(j) = 1/t(j)$ for each j then $t^{-1}B = t'B$ so that $t(t^{-1}B) = B$. If $t(j)$ is not different from 0 for each j it is still true that $t(t^{-1}B) \subset B$ and $t^{-1}(tB) \supset B$.

If A and B are sets of sequences A^B is the set of all s such that $ts \in B$ for each $t \in A$. In other words

$$A^B = \cap \{t^{-1}B: t \in A\} \tag{2.1}$$

which may be the empty set. However, if S is a sequence space $t^{-1}S$ and tS are clearly sequence spaces. Hence A^S at least contains the zero sequence, and is in fact a sequence space which will be called the *S-dual* of A.

If S is the space l^1 of all sequences s such that $\sum_{j=1}^{\infty} |s(j)| < \infty$ and T is any sequence space, T^S is the Köthe-Toeplitz $(\alpha-)$ dual of T introduced in [10], § 2, Definition 1. If S is the space cs of all sequences s such that $\sum_{j=1}^{\infty} s(j)$ converges, T^S is the space called the "g-dual" of T by Chillingworth in [2] and the β-dual of T by Köthe and others [9], p. 427. For S and T arbitrary sequence spaces T^S is the space called $(T \rightarrow S)$ by G. Goes [5, p. 137 and elsewhere]. For S equal to bs, the space of all sequences s for which $\sup_n |\sum_{j=1}^{n} s(j)| < \infty$, and T arbitrary, T^S corresponds to the γ-dual of T of Garling [4] and others. In relation to α-duality (2.1) corresponds to Satz 1 of [8].

For S and T, sequence spaces, T is called *S-perfect* if T^{SS} (i.e. $(T^S)^S$ is equal to T (cf. [10], § 2, Definition 2).

PROPOSITION 2.2. *Let A, B and C be sets of sequences and S, T and U sequence spaces.*

(a) $\varphi^C = \omega$.
(b) *$A \subset B$ implies* $A^C \supset B^C$.
(c) $A \subset A^{CC}$.
(d) *A^C is C-perfect.*
(e) *If A and B both contain 0,* $(A + B)^U = A^U \cap B^U$.
(f) *$B \subset C$ implies* $A^B \subset A^C$.
(g) $A^{B \cap C} = A^B \cap A^C$.
(h) $(S \cap T)^U \supset (S^U + T^U)^{UU}$ *and equality need not hold.*
(i) *If S and T are U-perfect* $(S \cap T)^U = (S^U + T^U)^{UU}$.
(j) *If S and T are U-perfect so is* $S \cap T$.
(k) *A is always A-perfect.*
(l) $(AB)^C = A^{\{B^C\}} = B^{\{A^C\}}$.

Proof. Statements (a), (b), (c) and (f) follow immediately from the relevant definitions so we omit their proof. We also omit the proofs of statements (d), (e), (g), (i) and (k) which are not hard.

(h) By (b), $(S \cap T)^U \supset (S^U + T^U)$ so by (b) applied twice $(S \cap T)^{UUU} \supset (S^U + T^U)^{UU}$. But $(S \cap T)^{UUU} = (S \cap T)^U$ by (d).

Let φ be the space of all sequences which are finitely nonzero, and let $S = \varphi + [e]$. That is, S consists of all sequences which are eventually constant. If $s(j) = (-1)^j$ for each j,

$$(S \cap sS)^{l^1} = \varphi^{l^1} = \omega$$

while

$$(S^{l^1} + (sS)^{l^1})^{l^1 l^1} = (l^1 + l^1)^{l^1 l^1} = l^1 .$$

(j) By (i), $(S \cap T)^{UU} = (S^U + T^U)^{UUU}$ and by (d) $(S^U + T^U)^{UUU} = (S^U + T^U)^U$ which is $S^{UU} \cap T^{UU} = S \cap T$ by (e) because S and T are U-perfect.

(l) If $v \in (AB)^C$ and $s \in A$, $vs \in B^C$ since for $t \in B$, $vst \in C$. Thus $v \in A^{\{B^C\}}$. On the other hand if $v \in A^{\{B^C\}}$, $s \in A$ and $t \in B$, $vst \in C$ since $vs \in B^C$. Hence $v \in (AB)^C$.

The set A^A is called the *set of multipliers* of A and written $M(A)$. If S is a sequence space $M(S)$ is a sequence algebra called the *multiplier algebra* of S [11].

PROPOSITION 2.3. (a) *If S is a sequence algebra $M(S) \supset S$. If A contains e, $M(A) \subset A$. Thus if S is a sequence algebra containing e, $M(S) = S$.*

(b) *For A and B sets of sequences $M(A^B) = (AA^B)^B$ and is thus B-perfect.*

(c) *$M(A^B)$ contains both $M(A)$ and $M(B)$.*

Proof. (a) Obvious.

(b) By 2.2 (b), $(AA^B)^B = \{A^B\}^{\{A^B\}}$ which is $M(A^B)$.

(c) If $v \in M(A)$ and $t \in A$, $vt \in A$ so that $uvt \in B$ for $u \in A^B$. Thus $vu \in A^B$ for $u \in A^B$ which implies $v \in M(A^B)$.

If $s \in M(B)$, $sut \in B$ for $u \in A^B$ and $t \in A$. Hence $su \in A^B$ for $u \in A^B$ which implies $s \in M(A^B)$.

For A an arbitrary subset of φ, $A^{(\omega)}$, the *polar of A in ω* is the set of all sequences s such that

$$\sup \{|(s, t)| : t \in A\} \leqq 1 .$$

If S is an arbitrary sequence space the *polar of A in S*, $A^{(S)}$ is $A^{(\omega)} \cap S$. For B an arbitrary subset of ω, $B^{(\varphi)}$, the *polar of B in φ* is the set of all $t \in \varphi$ such that

$$\sup \{|(s, t)|: s \in B\} \leqq 1 .$$

If A and B are sets of sequences such that $BA \subset A$, A is called B-invariant. It is clear that A is B-invariant if and only if $B \subset M(A)$ so that $M(A)$ is the maximal set under which A is invariant.

PROPOSITION 2.4. (a) *If A is a subset of φ which is B-invariant then $A^{(\omega)}$ is B-invariant. If S is a sequence space which is B-invariant then $A^{(S)}$ is B-invariant.*

(b) *If A is a subset of ω which is B-invariant, then $A^{(\omega)}$ is invariant under B.*

(c) *If B is a semigroup of sequences AB is B-invariant.*

Proof. (a) Let Λ denote the set of all $s \in \varphi$ such that $|\sum_j s(j)| \leqq 1$. Then it is easy to see that $A^{\Lambda} = A^{(\omega)}$. If A is B-invariant $B \subset M(A)$ and by 2.3(c), $M(A) \subset M(A^{\Lambda})$. The second assertion of (a) is an immediate consequence of the fact that if A_1 and A_2 are B-invariant so is $A_1 \cap A_2$.

The proofs of (b) and (c) are obvious.

3. **φ-Topologies on sequence spaces.** The coordinate functionals are defined by $E_j(s) = s(j)$. A *K-space* is a sequence space S with a locally convex topology on which each E_j is continuous. For S a locally convex sequence space containing φ, S^0 denotes the closure of φ in S, and S^f denotes the space of all sequence $\{f(e[j])\}$ as f ranges over S^*, the topological dual space of S.

PROPOSITION 3.1. *If S is a locally convex sequence space containing φ, $S^f = (S^0)^f$, and $(S^0)^f$ is algebraically isomorphic to $(S^0)^*$ under the correspondence of $f \in (S^0)^*$ to $\{f(e[j])\}$ in $(S^0)^f$.*

Proof. The first assertion is a direct consequence of the Hahn-Banach theorem. The second results from the fact that if $f \in (S^0)^*$ is such that $f(e[j]) = 0$ for each j then $f = 0$.

PROPOSITION 3.2. *Let S be a locally convex sequence space containing φ, and let $\mathscr{P}$ be the family of all continuous seminorms on S. If for each p in $\mathscr{P}$, A_p consists of all s in φ such that $p(s) \leqq 1$ then S^f is precisely equal to*

$$\cup \{A_p^{(\omega)}: p \in \mathscr{P}\} .$$

Proof. Denote the above union by T. If $f \in S^*$ there is $p \in \mathscr{P}$ such that

$$|f(s)| \leqq p(s)$$

for each $s \in S$ and hence for each $s \in \varphi$. If t_f is the sequence defined by $t_g(j) = f(e[j])$ for each j then $f(s) = (s, t_f)$ for each $s \in \varphi$. Thus $t_f \in A_p^{(\omega)}$ which implies that $S^f \subset T$.

Conversely, if $t \in A_p^{(\omega)}$ for some $p \in \mathscr{P}$ then the linear functional

$$f_t(s) = (t, s) \qquad s \in \varphi$$

is continuous on φ given the relative topology of S. This functional can be extended to all of S by the Hahn-Banach theorem. For the extension of f_t to S,

$$f_t(e[j]) = t(j)$$

for each j so that $t \in S^f$.

In 3.2 the hypothesis can be weakened by requiring only that $\mathscr{P}$ be a family of continuous seminorms on S which is directed by the relation $\leqq$ and determines the topology of S.

The bilinear form (s, t), $t \in \varphi$, $s \in S$ provides a duality between φ and each sequence space S. A locally convex topology on a sequence space S is called a φ-topology if there is a fundamental system of neighborhoods of 0 in S having the form $\{A^{(S)} : A \in \varphi\}$ where φ is a family of S-bounded subsets of φ. Note that $A^{(S)}$ is the absolute polar of A with respect to the aforementioned duality.

If S is a sequence space and $\mathscr{B}$ is a family of S-bounded subsets of φ, the $\mathscr{B}$-topology on S is defined to be the coarsest locally convex topology on S for which each $B^{(S)}$, $B \in \mathscr{B}$ is a neighborhood of 0. A collection $\mathscr{B}$ of S-bounded subsets of φ is called T-saturated for T an arbitrary sequence space if (1) $B \in \mathscr{B}$ and $B_1 \subset B$ implies $B_1 \in \mathscr{B}$; (2) $B \in \mathscr{B}$ implies $aB \in \mathscr{B}$ for each scalar a; (3) $B_1, B_2, \cdots, B_n$ in $\mathscr{B}$ implies $(\bigcup_{j=1}^n B_j)^{(T)(\varphi)}$ is in $\mathscr{B}$. By the bipolar theorem $(\bigcup_{j=1}^n B_j)^{(T)(\varphi)}$ consists of the $\sigma(\varphi, T)$ closure of $\Gamma(\bigcup_{j=1}^n B_j)$ in φ where Γ denotes absolutely convex hull. Thus if $\mathscr{B}$ is T_1-saturated and $T_1 \subset T_2$ then $\mathscr{B}$ is T_2-saturated. The T-saturated hull of $\mathscr{B}$ is by definition the smallest T-saturated collection of subsets of φ containing $\mathscr{B}$. If $S \subset T$ then each B is the saturated hull of $\mathscr{B}$ is S-bounded. Hence if φ contains $\mathscr{B}$ and is contained in the S-saturated hull of $\mathscr{B}$, the $\mathscr{B}$-topology and φ-topology coincide on S. The *$\mathscr{B}\varphi$-topology* on S is the $\mathscr{B}$-topology for which $\mathscr{B}$ is the (S-saturated) family of all S-bounded subsets of φ.

The following statement is a direct consequence of 3.2 and its proof need not be given.

PROPOSITION 3.3. *If a sequence space S containing φ has the $\mathscr{B}$-topology where $\mathscr{B}$ is an S-saturated family of S-bounded subsets*

of φ then S^f is precisely equal to

$$\cup \{B^{(\varphi)(\omega)} : B \in \mathscr{B}\} .$$

The hypothesis of 3.3 can be weakened by requiring only that $\mathscr{B}$ have the property that for each B in the S-saturated hull of $\mathscr{B}$ there is B_1 in $\mathscr{B}$ with $B_1 \supset B$.

We mension the following theorem and its corollary since they are analogous to the uniqueness and inclusion theorems of Zeller [15] for FK-spaces which have proven very useful. In the work of Zeller the Closed Graph Theorem is essential while the present results rest upon the Uniform Boundedness Principle. Since there are (pathelogical) FK-spaces which do not have a φ-topology the present work does not include that of Zeller. Conversely since there are K-spaces which have barrelled φ-topologies but are not FK-spaces (e.g. φ with its Mackey topology) the work of Zeller is not more general than this. It is easy to show that if $\{e[n] : n = 1, 2, \cdots\}$ is a basis for a sequence space S then S has a $\mathscr{B}\varphi$-topology. Thus the results of Jones and Retherford [6] on bases in barrelled spaces can be derived from Theorem 3.4.

The proofs of both Theorem 3.4 and its corollary are very easy and we omit them.

THEOREM 3.4. *For a sequence space S there is at most one barrelled φ-topology for which S is a K-space, namely, the $\mathscr{B}\varphi$-topology.*

COROLLARY. *If S and T are both barrelled K-spaces having a φ-topology and $S \subset T$ then the inclusion map of S into T is continuous.*

4. **Duality of a sequence space with respect to a sum space.** Let S be a sequence space with a locally convex topology and T an arbitrary spaces of sequences. For U a subspace of T^S, a locally convex topology is determined on T by the collection of seminorms

$$p_u(s) = p(us) \qquad u \in U$$

where p ranges over the family of continuous seminorms on S (or over a fundamental subfamily). This topology will be called the $\sigma S(T, U)$ topology on T. If S is a K-space and $\varphi \subset U$, then T with the $\sigma S(T, U)$ topology is K-space. Throughout this section it is always assumed that these conditions are fulfilled. The $\sigma S(T, U)$ is thus the projective topology on T with respect to the linear maps $f_u(t) = ut$ as u ranges over U. See [14] p. 51. A net $\{t_\nu\}$ in T converges to t in $\sigma S(T, U)$ if and only if $\{ut_\nu\}$ converges to ut in S for each u in T^S.

EXAMPLES. (a) If $S = l^1$, T^s is the α-dual of T in the sense of Köthe and Toeplitz [10]. They defined a sequence $\{t_n\}$ in T to be convergent (to t) if $\lim_n (t_n, u)$ exists $(=t)$ for each u in T^s. This definition does not explicitly define the intended topology on T because by a well known theorem [1], p. 137 a sequence in l^1 converges weakly if and only if it converges strongly. Thus the topology on T which provides such convergent sequences could have arisen from either the weak or normed topology on l^1. In later work Köthe discussed a variety of topologies on T based upon the natural duality between T and T^{l^1}. See for instance [7] or [8].

(b) Chillingworth [2] studied sequential convergence in T with respect to T^{cs} where cs has its weak topology. The complications involved in such an approach were observed by Köthe and Toeplitz at the end of [10]. They arise primarily because cs with its weak topology is not sequentially complete.

(c) If $S = bs$ with the BK-topology given by the norm

$$\|s\| = \sup_n \left\{ \left| \sum_{j=1}^{n} s(j) \right| \right\}$$

then the $\sigma S(T, U)$ topology is determined by the means of the seminorms

$$p_u(t) = \sup_n \left\{ \left| \sum_{j=1}^{n} u(j)t(j) \right| \right\}$$

where u ranges over U. Thus the $\sigma bs(T, T^{bs})$ topology on T coincides with the $\sigma\gamma(T, T^{\gamma})$-topology studied by Garling [4].

PROPOSITION 4.1. *Let S have AK and $\varphi \subset U \subset T^s$. Then with the topology $\sigma S(T, U)$, T has AK. If T is sequentially complete with the topology $\sigma S(T, U)$ then T is S-perfect and in fact equal to U^s.* (cf. [10] § 3, *Satz* 2, § 4, *Satz* 1; [8], *Satz* 3, p. 74).

Proof. If $t \in T$ and $u \in U$ then $tu[\leqq n] = t[\leqq n]u \to tu$ in S so that $t[\leqq n] \to t$ in T. Thus T has AK.

Let T be $\sigma S(T, U)$ sequentially complete. Since $U \subset T^s$, $U^s \supset T^{ss} \supset T$. If $t \in U^s$, $t[\leqq n]$ is in T for each n and $\{t[\leqq n]\}$ is Cauchy since $t[\leqq n]u \to tu$ for each $u \in U$. Thus $\lim_n t[\leqq n] = t \in T$ so that $U^s \subset T$.

PROPOSITION 4.2. *Let $S \supset \varphi$ be (sequentially) complete.* (a) *If T is S-perfect then T is (sequentially) complete in the topology $\sigma S(T, T^s)$.* (b) *T^s is (sequentially) complete in the topology $\sigma S(T^s, T)$.* (cf. [10] § 3, *Satz* 5; § 4, *Satz* 2; [8] *Satz* 4, p. 74.)

Proof. Let $\{s_\nu\}$ be a Cauchy net (sequence) in T. Then since $\sigma S(T, T^S)$ makes T a K-space $\{s_\nu(j)\}$ converges, say to a_j, for each j. If s is the sequence with $s(j) = a_j$ for each j and $t \in T^S$, then $ts_\nu \in S$ for each ν and $\{ts_\nu\}$ converges necessarily to ts. Hence $ts \in S$ so $s \in T^{SS} = T$, and $\{t_\nu\}$ converges in $\sigma S(T, T^S)$ to s. A similar argument establishes (b).

It is clear that a subset A of T is $\sigma S(T, T^S)$-bounded if and only if uA is bounded in S for each $u \in T^S$. A subset A of T is said to be completely bounded if AB is bounded in S for each $\sigma S(T^S, T)$-bounded set B of T^S. It is obvious that completely bounded sets are bounded.

PROPOSITION 4.3. *If S is sequentially complete and A is $\sigma S(T, T^S)$-bounded in T then A is completely bounded in T.* (cf. [10] § 5, *Satz* 1).

Proof. Suppose A were a $\sigma S(T^S, T)$-bounded subset of S which is not completely bounded. Then there would be a $\sigma S(T^S, T)$-bounded subset B of T^S and a continuous seminorm p on S such that $p(AB)$ is not bounded. Let $\{t_n\}$ be a sequence in A and $\{u_n\}$ a sequence in B such that

$$p(t_n u_n) > 4^n$$

for $n = 1, 2, \cdots$.

An increasing subsequence of integers $k(1), k(2), \cdots$ can be defined by induction such that:

(a) $$2^{-k(m)} \sup \{p(t_n u) : u \in B\} < 2^{-m}$$

(b) $$2^{-k(m)} \sup \{p(t u_n) : u \in A\} < 2^{-m}$$

where $n = 1, 2, \cdots, k(m-1)$. The series $\sum_{j=1}^{\infty} 2^{-k(j)} u_{k(j)}$ converges absolutely to a point u in T^S with respect to the $\sigma S(T^S, T)$ topology. This is because T^S is $\sigma S(T^S, T)$-complete by 4.1(b) and for each $t \in T$ and each continuous seminorm q on S

$$\sum_{j=1}^{\infty} q(2^{-k(j)} u_{k(j)} t) \leqq \sum_{j=1}^{\infty} 2^{-k(j)} \sup \{q(ut) : u \in B\}$$
$$\leqq \sup \{q(ut) : u \in B\} < \infty .$$

But this leads to a contradiction of the boundedness of A in T since

$$p(tu) \geqq p(t_{k(n)} 2^{-k(n)} u_{k(n)}) - \sum_{j=1}^{n-1} 2^{-k(j)} p(t_{k(n)} u_{k(j)})$$
$$- \sum_{j=m+1}^{\infty} 2^{-k(j)} p(t_{k(n)} u_{k(j)}) \geqq 2^n - 1 .$$

This is because

$$\sum_{j=1}^{n-1} 2^{-k(j)} p(t_{k(n)} u_{k(j)}) < \sum_{j=1}^{n-1} 2^{-j} \qquad \text{by (b)}$$

and

$$\sum_{j=n+1}^{\infty} 2^{-k(j)} p(t_{k(n)} u_{k(j)}) < \sum_{j=m+1}^{\infty} 2^{-j} \qquad \text{by (a)} .$$

For U a subspace of T^S, the topology $\beta S(T, U)$ on T is determined by the seminorms

$$p_A(t) = \sup \{p(tu): u \in A\} \tag{4.1}$$

where A ranges over the collection of T-bounded subsets of U and p ranges over the collection of continuous seminorms on S.

PROPOSITION 4.4. *Let S and T contain φ, and suppose T^S with the $\beta S(T^S, T)$ topology has AK. A sequence in T is $\sigma S(T, T^S)$-Cauchy if and only if it is $\sigma S(T, T^S)$-bounded and coordinatewise convergent.* (cf. [10] § 5, *Satz* 6).

Proof. It is clear that a $\sigma S(T, T^S)$-Cauchy sequence in T is $\sigma S(T, T^S)$-bounded and coordinatewise convergent.

Let $\{t_n\}$ be a sequence in T which is $\sigma S(T, T^S)$-bounded and coordinatewise convergent. It will suffice to prove that $\{ut_n\}$ is a Cauchy sequence in S for each $u \in T^S$. Let $\varepsilon > 0$ and p a continuous seminorm be given. Since T^S has AK with the $\beta S(T^S, T)$ topology there is k_0 such that

$$p_B(u - u[\leqq k]) < \varepsilon/3 \qquad \text{for } k \geqq k_0$$

where $B = \{t_n\}$. Since $\{t_n\}$ is coordinatewise convergent there is n_0 such that $p(ut_n[\leqq k_0] - ut_m[\leqq k_0]) < \varepsilon/3$ for $m, n > n_0$. Thus for $n, m > n_0$

$$\begin{aligned} p(ut_n - ut_m) &\leqq p(ut_n - ut_n[\leqq k]) + p(ut_n[\leqq k] - ut_m[\leqq k]) \\ &\quad + p(ut_m[\leqq k] - ut_m) \\ &\leqq p_B(u - u[\leqq k]) + p(ut_n[\leqq k] - ut_m[k]) \\ &\quad + p_B(u - u[\leqq k]) \\ &\leqq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon . \end{aligned}$$

Throughout the remainder of § 4, S will represent a K-space containing φ which has a φ-topology. The (not uniquely determined) family of S-bounded subsets of φ whose polars in S form a fundamental system of neighborhoods of zero in S will be denoted by $\mathscr{B}$.

PROROSITION 4.5. *Let $S \supset \varphi$ be a K-space having a φ topology.*

For each sequence space T both the $\sigma S(T, T^S)$ and $\beta S(T, T^S)$ topologies on T are-φ-topologies. A fundamental system of zero neighborhoods in $\sigma S(T, T^S)$ consists of sets of the form

$$(uB)^{(T)} u \in T^S, B \in \mathscr{B} .$$

A fundamental system of zero-neighborhoods in $\beta(T, T^S)$ consists of sets having the form

$$(AB)^{(S)}$$

where A is a $\sigma S(T, T^S)$ bounded subset of T^S and $B \in \mathscr{B}$.

Proof. A fundamental system of seminorms determining the topology $\sigma S(T, T^S)$ consists of those having the form

$$p_u(t) = p(ut) \qquad u \in T^S$$

where p is defined on S by

$$p(s) = \sup \{|(s, v)|: v \in B\} .$$

Thus

$$p_u(t) = \sup \{|(ut, v)|: v \in B\} = \sup \{|(t, s)|: v \in uB\}$$

so that $p_u(t) \leqq 1$ if and only if $t \in (uB)^{(S)}$.

The topology $\beta S(T, T^S)$ is determined by a system of seminorms each of which has the form

$$p_A(t) = \sup \{p(ut): u \in A\} \qquad t \in T$$

where A is a T-bounded subset of T^S and p is a continuous seminorm on S having the form (5.1). Thus for $t \in T$ we have

$$\begin{aligned} p_A(t) &= \sup \{\sup \{|(ut, v)|: u \in A, v \in B\}\} \\ &= \sup \{|(ut, v)|: u \in A, v \in B\} \\ &= \sup \{|(t, s)|: s \in AB\} . \end{aligned}$$

Consequently $p_A(t) \leqq 1$ if and only if $t \in (AB)^{(S)}$.

PROPOSITION 4.6. *Let $S \supset \varphi$ be a K-space having a φ-topology, namely the $\mathscr{B}$-topology where it is assumed that each member of the T-saturated hull of $\mathscr{B}$ is contained in a member of $\mathscr{B}$, and let T be a sequence space containing φ.*

(a) *If T has the $\sigma S(T, T^S)$-topology*

$$T^f = \cup \{uB^{(\varphi)(\omega)}: B \in \mathscr{B}, u \in T^S\} .$$

(b) *If T has the $\beta S(T, T^S)$-topology*

$$T^f = \cup \{(AB)^{(\varphi)(\omega)} : B \in \mathscr{B},\ A \text{ is a } \sigma S(T^S, T)\text{-bounded subset of } T^S\}\ .$$

Proof. (a) By 4.5 and 3.3

$$T^f = \{(uB)^{(\varphi)(\omega)} : B \in \mathscr{B},\ u \in T^S\}\ .$$

It may be assumed that each B is absolutely convex since $B^{(S)} = B^{(S)(\varphi)(S)}$ and $B^{(S)(\varphi)} \supset B$ is absolutely convex. Since $S \supset \varphi$ and B is S-bounded B and hence $B^{(\varphi)(\omega)}$ is φ-bounded, and hence compact in ω. The operator $t \to ut$ is continuous and linear from ω into ω. Thus $uB^{(\varphi)(\omega)}$ is absolutely convex and compact in ω so that $uB^{(\varphi)(\omega)} \supset (uB)^{(\varphi)(\omega)}$. On the other hand $(uB)^{(\varphi)(\omega)}$ is the closure of uB in ω and $uB^{(\varphi)(\omega)}$ is the image of the closure of B under a continuous map. Thus $(uB)^{(\varphi)(\omega)} \supset uB^{(\varphi)(\omega)}$ so that the two sets are equal. This implies that T^f has the desired form.

(b) This follows immediately from 4.5 and 3.3.

A *sum space* is defined to be a K-space S containing φ on which a φ-topology is defined and such that $S^f = M(S)$. It is easy to see that cs, bs, l^1 and φ (φ having its weak or Mackey topologies) are all sum spaces. The concept of sum space is studied in the paper [13] in which further examples are established e.g., $cs \cap l^p$ for $1 \leqq p < \infty$ and ms the space of mean series summable sequences as well as all rearrangements of these spaces.

THEOREM 4.7. *If S is a sum space and $T \supset \varphi$ is an arbitrary sequence space given the $\sigma S(T, T^S)$ topology $T^f = T^S$.*

Proof. If $v \in T^f$ there is $u \in T^S$ and $t \in B^{(\varphi)(\omega)}$ where $B \in \mathscr{B}$ such that $v = tu$. But $B^{(\varphi)(\omega)} \subset S^f = M(S) \subset M(T^S)$ so that $v = tu \in T^S$.

If $u \in T^S$ define g on T by $g(t) = E(ut)$ where E is any continuous linear functional on S such that $E(e[j]) = 1$ for each j. There is such a function because $e \in S^f$. Then for each j,

$$g(e[j]) = E(e[j]u) = u(j)$$

so $u \in T^f$.

Throughout the remainder of this section, S will denote a sum space which is a BK-space, and A will denote the unit ball of $S^f = M(S)$ when $M(S)$ is interpreted as a BK-algebra of operators on S. Thus

$$A = \{t \in M(S) : tB \subset B\}$$

where B is the unit ball of S, and so A is a multiplicative semigroup containing e.

PROPOSITION 4.8. *Let S denote a BK-sum space and A the unit ball of S^f. If T is S-perfect then T is A-invariant.* (cf. [10] § 3, *Satz* 4).

Proof. If $u \in A$, $t \in T$ and $v \in T^S$ $(ut)v = u(tv) \in S$ since $tv \in S$ and $u \in M(S)$. Thus $ut \in T^{SS} = T$.

PROPOSITION 4.9. *Let S denote a BK-sum space and A the unit ball of S^f. If T is A-invariant then the $\sigma S(T, T^S)$ and $\beta S(T, T^S)$ topologies are locally A-invariant.*

Proof. $\sigma S(T, T^S)$. By 4.5 a fundamental system of zero neighborhoods for $\sigma S(T, T^S)$ consists of T-polars of sets of the form

$$auB^{(\varphi)}u \in T^S,\ a > 0\ .$$

Since B is A-invariant so is $B^{(\varphi)}$, $auB^{(\varphi)}$ for $u \in T^S$ and $a > 0$ as well as $(auB^{(\varphi)})^{(T)}$ since T is A-invariant.

$\beta S(T, T^S)$. An argument essentially the same as the one in the preceding paragraph will show that the $\sigma S(T^S, T)$ topology on T^S is locally A-invariant. Thus if C is a T-bounded subset of T^S, AC is also T-bounded since it is the image of C under an equicontinuous set of operators. Thus a fundamental system of zero-neighborhoods for $\beta S(T, T^S)$ consists of T-polars of sets having the form

$$ACB^{(\varphi)} \qquad C \text{ is } \sigma S(T^S, T)\text{-bounded in } T^S.$$

Since each such set is A-invariant so is its T-polar because T is A-invariant so that $\beta S(T, T^S)$ is locally A-invariant.

COROLLARY. *In an A-invariant space $T \supset \varphi$, the A-invariant hull AB of a $\sigma S(T, T^S)$ bounded set B is itself $\sigma S(T, T^S)$ bounded* (cf. [10] § 5, *Satz* 2).

Proof. This is immediate since AB is the image under B of an equicontinuous family of operators.

THEOREM 4.10. *Let S denote a BK-sum space and A the unit ball of S^f. If T is A-invariant the $\beta S(T, T^S)$ topology on T coincides with the $\beta\varphi$-topology.*

Proof. By 4.5 the $\beta S(T, T^S)$ topology on T is a φ-topology thus weaker than the $\beta\varphi$-topology.

Let C be a $\sigma(\varphi, T)$-bounded subset of φ. For each $t \in T$ define the following seminorm on S^f

$$P_t(v) = \sup \{|(tv, u)|: u \in C\} \qquad v \in S^f .$$

Since C is $\sigma(\varphi, T)$-bounded and T is A-invariant (hence S^f invariant) $p_t(v) < \infty$ for each $v \in S^f$. Furthermore, S^f is a K-space so p_t is the supremum of continuous seminorms on S^f (recall $C \subset \varphi$). Since S^f is a barrelled space p_t is continuous on S^f. There is thus $a_t > 0$ such that

$$p_t(v) < a_t ||v||_f$$

where $||\ ||_f$ is the norm on S^f. Hence, if $t \in T$ and $||\ ||_S$ is the norm on S,

$$\begin{aligned} \sup \{||tu||_S: u \in C\} &= \sup \{\sup |(tu, v)|: v \in A\}, u \in C\} \\ &= \sup \{|(tv, u)|: v \in A, u \in C\} \\ &= \sup \{p_t(v): v \in A\} < a_t . \end{aligned}$$

Thus C is also a $\sigma S(T^s, T)$ bounded subset of T^s. As in the proof of 4.9, AC is also $\sigma S(T^s, T)$-bounded. By 4.5 the polars in T of such subsets of φ form a basis system of zero neighborhoods in the $\beta S(T, T^s)$ topology on T. Since $AC \supset C$, $(AC)^{(T)} \subset C^{(T)}$. Thus the $\beta S(T, T^s)$ topology on T is stronger than the $\beta\varphi$-topology on T so that the two topologies must coincide.

EXAMPLE. The $\beta\varphi$-topology on bv is the normed topology. However, $bv^{l^1} = l^1$, and the $\beta l^1(bv, l^1)$ topology on bv is the relative topology on bv as a subspace of m, which is a weaker topology.

REFERENCES

1. S. Banach, *Theórie des Operation Lineáires*, Warsaw, 1932.
2. H. R. Chillingworth, *Generalized 'dual' sequence spaces*, Nederl. Akad. Wetensch. Indag. Math., **20** (1958), 307-315.
3. D. J. H. Garling, *On topological sequence spaces*, Proc. Camb. Phill. Soc., **63** (1967), 997-1019.
4. ———, *The β and γ-duality of sequence spaces*, ibid., 963-981.
5. G. Goes, *Complementary spaces of Fourier coefficients, convolutions and generalized matrix transformations and operators between BX-spaces*, J. Math. Mech., **10** (1961), 135-157.
6. O. T. Jones and J. R. Retherford, *On similar bases in barrelled spaces*, Proc. Amer. Math. Soc., **18** (1967), 677-680.
7. G. Köthe, *Lösbarkeithabedingungen für Gleichungen mit unendlich vielen unbekannten*, J. für Math., **178** (1938), 193-213.
8. ———, *Neubegrundung der Theorie der vollkommen Räume*, Math. Nachr., **4** (1951), 70-80.
9. ———, *Topologische Lineare Räume* I, 1st edn., Berlin-Heidelberg-New York: Springer 1960.
10. G. Köthe and O. Toeplitz, *Lineare Räume mit unendlichvielen Koordinaten und Ringe unendlicher Matrizen*, J. für Math., **171** (1934) 193-226.

11. R. J. McGivney, and W. H. Ruckle, *Multiplier algebras of biorthogonal systems*, Pacific J. Math., **29** (1969), 375-388.
12. W. H. Ruckle, *Lattices of sequence spaces*, Duke Math. J., **35** (1968), 491-504.
13. ———, *An abstract concept of the sum of a numerical series*, Canad. J. Math., **22** (1970), 863-874.
14. H. H. Schaefer, *Topological Vector Spaces*, 1st edn., New York, MacMillan 1966.
15. K. Zeller, *Allegemeine Eigenschaften von Limitierungsverfahren*, Math. Z., **53** (1951), 473-487.

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