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STRONG HEREDITY IN RADICAL CLASSES

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In a recent paper, W. G. Leavitt has called a radical class \mathcal{P} in a universal class \mathcal{W} of not necessarily associative rings strongly hereditary if $\mathcal{P}(I) = I \cap \mathcal{P}(R)$ for all ideals I of any ring $R \in \mathcal{W}$. In this paper, strongly hereditary radicals are investigated and a new construction is provided for the minimal strongly hereditary radical containing a given class in \mathcal{W} . Nonassociative versions of some results of E. P. Armendariz on semisimple classes are proved, including a characterization of semisimple classes corresponding to strongly hereditary radicals.

Unless otherwise indicated, \mathcal{W} is assumed to be a universal class of not necessarily associative rings. If \mathcal{P} is any radical class in \mathcal{W} , we denote the class of \mathcal{P} -semisimple rings in \mathcal{W} by $\mathcal{S}\mathcal{P}$. We use the notation $I \leq R$ to denote that I is an ideal of R . For any class \mathcal{M} we denote by $\mathcal{H}\mathcal{M}$ and $\mathcal{I}\mathcal{M}$, respectively, the homomorphic closure and ideal closure of \mathcal{M} .

For any radical class $\mathcal{P} \subseteq \mathcal{W}$, Leavitt in [7] has defined $\mathcal{C}\mathcal{P} = \{J' \mid J \leq I \leq R, J \in \mathcal{P}, \text{ and } J' \text{ is the ideal of } R \text{ generated by } J\}$. Radical classes \mathcal{P} for which $\mathcal{P} = \mathcal{C}\mathcal{P}$ are said to satisfy property (a). Theorem 1 of [7] states that a hereditary radical class \mathcal{P} is strongly hereditary if and only if \mathcal{P} satisfies property (a). In [8], it is shown that any subclass \mathcal{M} of \mathcal{W} is contained in a unique minimal radical class satisfying property (a).

Some preliminary results are required.

LEMMA 1.1. [2]. *Let \mathcal{P} be any radical class in \mathcal{W} . Then $\mathcal{S}\mathcal{P}$ is hereditary if and only if for each $R \in \mathcal{W}$ with $I \leq R$ we have $\mathcal{P}(I) \subseteq (R)$.*

LEMMA 1.2. *Let \mathcal{P} be any radical class. Then \mathcal{P} is strongly hereditary if and only if both \mathcal{P} and $\mathcal{S}\mathcal{P}$ are hereditary.*

Proof. If \mathcal{P} is strongly hereditary, $\mathcal{P}(I) = I \cap \mathcal{P}(R)$ for each $I \leq R$, so \mathcal{P} and $\mathcal{S}\mathcal{P}$ are hereditary. Suppose \mathcal{P} and $\mathcal{S}\mathcal{P}$ are hereditary and let $I \leq R$. By Lemma 1.1 $\mathcal{P}(I) \subseteq I \cap \mathcal{P}(R)$. Also since $\mathcal{P}(R) \in \mathcal{S}$ and \mathcal{P} is hereditary, $I \cap \mathcal{P}(R) \in \mathcal{P}$. Since $I \cap \mathcal{P}(R) \leq I$, we have $I \cap \mathcal{P}(R) \subseteq \mathcal{P}(I)$.

LEMMA 1.3. *Let \mathcal{P} be a radical class satisfying property (a). Then $\mathcal{S}\mathcal{P}$ is hereditary. If \mathcal{P} is hereditary, \mathcal{P} satisfies property*

(a) if and only if \mathcal{SP} is hereditary.

Proof. Suppose \mathcal{P} has property (a). Let $J \leq I \leq R$ where $J \in \mathcal{P}$. Then J' , the ideal of R generated by J , belongs to \mathcal{P} , so $J' \subseteq \mathcal{P}(R)$. Thus $J \subseteq \mathcal{P}(R)$ and $\mathcal{P}(R)$ contains all the \mathcal{P} -ideals of I , so $\mathcal{P}(I) \subseteq \mathcal{P}(R)$. By Lemma 1.1, this means \mathcal{SP} is hereditary. If \mathcal{P} and \mathcal{SP} are hereditary, then \mathcal{P} is strongly hereditary by Lemma 1.2, so \mathcal{P} satisfies property (a) by Theorem 1 of [7].

The semisimple class \mathcal{SP} may be hereditary even when \mathcal{P} does not satisfy property (a). To see this, let R be the ring generated over $GF(2)$ by the nonassociative symbols $\{x, y, z\}$, subject to the relations $x^2 = xy = yx = xz = x$, $yz = zy = zx = y$, $z^2 = z$, $y^2 = 0$. Then the only proper nonzero ideal of R is $I = \{0, x, y, x + y\}$ and the only proper ideal of I is $J = \{0, x\}$. R/I and J are isomorphic simple idempotent rings, and I/J is simple and nilpotent. Let $\mathcal{W} = \{0, R, J, I, I/J\}$ and $\mathcal{P} = \{0, R, J\}$. Then $\mathcal{P} \neq \mathcal{SP}$ but $\mathcal{SP} = \{0, I/J\}$ is hereditary.

2. Radical classes. In Theorem 2 of [2], it is shown that if \mathcal{P} is a radical class in an alternative class \mathcal{W} , and if $R \in \mathcal{W}$ with $I \leq R$, then $\mathcal{P}(I) \leq R$. The following theorem shows that radicals satisfying property (a) in an arbitrary universal class have the same property.

THEOREM 2.1. *Let \mathcal{P} be any radical class in \mathcal{W} . The \mathcal{P} satisfies property (a) if and only if for each $R \in \mathcal{W}$, $I \leq R$ implies $\mathcal{P}(I) \leq R$.*

Proof. Suppose $I \leq R$ implies $\mathcal{P}(I) \leq R$. Then $\mathcal{P}(I) \subseteq \mathcal{P}(R)$ so \mathcal{SP} is hereditary by Lemma 1.1. If \mathcal{P} does not have property (a), we have some $J \leq I \leq R$ with $J \in \mathcal{P}$ and $J' \notin \mathcal{P}$, where J' is the ideal of R generated by J . This means $\mathcal{P}(J') \neq J'$. Since \mathcal{SP} is hereditary and $J \leq J'$, we have $\mathcal{P}(J) = J \subseteq \mathcal{P}(J')$. Since $\mathcal{P}(J') \leq R$, we have J contained in an ideal of R properly smaller than J' , contradicting the definition of J' .

Conversely, suppose \mathcal{P} has property (a) and let $I \leq R$. Then $\mathcal{P}(I) \leq I \leq R$, so $\mathcal{P}(I) \in \mathcal{P}$ and from $\mathcal{P}(I) \leq I$ it follows that $\mathcal{P}(I) \subseteq \mathcal{P}(I)$. Thus $\mathcal{P}(I) = \mathcal{P}(I) \leq R$.

In [3] (Lemma 5) a result is proved which may be restated for our purposes as

LEMMA 2.2. *If \mathcal{P} is a radical class in an alternative class, then \mathcal{P} satisfies property (a).*

This lemma, when applied with Theorem 2.1, shows that in alter-

native classes $I \leq R$ implies $\mathcal{P}(I) \leq R$, thus providing another proof of Theorem 2 of [2].

We next note that property (a) may be satisfied by possibly non-radical classes of rings. For an example of this, let \mathcal{M} be the class of nilpotent rings in the universal class of all associative rings. Then if $J \leq I \leq R$ with $J \in \mathcal{M}$, we have $(J')^3 \subseteq J$ so that $J' \in \mathcal{M}$. The following lemma shows that such classes, if also homomorphically closed, are only one step removed from being radical.

LEMMA 2.3. *If \mathcal{M} is homomorphically closed and satisfies property (a), then $\mathcal{LM} = \mathcal{M}_2$ in the lower radical construction. (For details of this construction see [5]).*

Proof. Let $R \in \mathcal{M}_3$ and let R_1 be an arbitrary homomorphic image of R . Then R_1 has an ideal $I \in \mathcal{M}_2$ and I has a nonzero ideal $J \in \mathcal{M}$. Since \mathcal{M} satisfies property (a), J' , the ideal of R_1 generated by J , is in \mathcal{M} . Thus each image of R has a nonzero ideal in \mathcal{M} , which means $\mathcal{M}_3 \subseteq \mathcal{M}_2$ so $\mathcal{LM} = \mathcal{M}_2$.

Using Lemma 2.3, we next prove that property (a) is preserved by passing to the lower radical.

THEOREM 2.4. *If \mathcal{M} is homomorphically closed and satisfies property (a), then \mathcal{LM} satisfies property (a).*

Proof. Let $\mathcal{P} = \mathcal{LM} = \mathcal{M}_2$, and suppose \mathcal{P} does not have property (a). Then by Theorem 2.1 there is some R with $I \leq R$ but $\mathcal{P}(I)$ not an ideal of R . Since the union of a chain of \mathcal{P} -ideals of R is again a \mathcal{P} -ideal of R (see [1]), we may select by Zorn's Lemma a \mathcal{P} -ideal F of R which is maximal among those \mathcal{P} -ideals of R which are contained in $\mathcal{P}(I)$ (F may of course be zero). We claim $\mathcal{P}(I)/F = \mathcal{P}(I/F)$. Clearly $\mathcal{P}(I)/F \subseteq \mathcal{P}(I/F)$. Let K be the ideal of I such that $K/F = \mathcal{P}(I/F)$. Then $\mathcal{P}(I) \subseteq K$ and $(K/F)/(\mathcal{P}(I)/F) \cong K/\mathcal{P}(I) \in \mathcal{P}$, forcing $K \in \mathcal{P}$, which means $K \subseteq \mathcal{P}(I)$ so $\mathcal{P}(I)/F = \mathcal{P}(I/F)$. Now if R/F has a nonzero \mathcal{P} -ideal $P/F \subseteq \mathcal{P}(I)/F$, we would have $P \in \mathcal{P}$ violating the maximality of F .

Thus by passing if necessary to a homomorphic image, we may assume $\mathcal{P}(I) \subseteq I \leq R$, and $\mathcal{P}(I)$ contains no nonzero \mathcal{P} -ideal of R . Since $\mathcal{P}(I) \in \mathcal{M}_2$, $\mathcal{P}(I)$ has a nonzero ideal $J_1 \in \mathcal{M}$, and the ideal J of I generated by J_1 is also in \mathcal{M} . Thus we have $J_1 \subseteq J \subseteq \mathcal{P}(I) \subseteq I \subseteq R$ where $J \in \mathcal{M}$ and $J \leq I$. Also J' , the ideal of R generated by J , is contained in I , and $J' \in \mathcal{M} \subseteq \mathcal{P}$. Thus $J' \subseteq \mathcal{P}(I)$, contradicting the assumption that $\mathcal{P}(I)$ contains no nonzero \mathcal{P} -ideals of R .

The example following the proof of Lemma 1.3 may be used to show that in the nonassociative case the requirement in Theorem 2.4

that \mathcal{M} be homomorphically closed cannot be dropped. Let \mathcal{W} be as in the example and $\mathcal{M} = \{R\}$. Then \mathcal{M} has property (a) but $\mathcal{H}\mathcal{M} = \mathcal{P}$ does not.

It is shown in [8] that an arbitrary class is contained in a unique minimal radical class satisfying property (a) and in a unique minimal strongly hereditary radical class. The next few results provided countable construction which are at most one (Kuroš) step from these classes.

THEOREM 2.5. *Let $\mathcal{M} \subseteq \mathcal{W}$ with \mathcal{W} any universal class. There exists a unique minimal class in \mathcal{W} containing \mathcal{M} which is homomorphically closed and has property (a).*

Proof. Define $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_{n+1} = \mathcal{G}\mathcal{H}\mathcal{M}_n$ for each $n \geq 1$. Then set $\mathcal{M}^* = \bigcup \mathcal{M}_n$, the union being taken over all positive integers n . \mathcal{M}^* is easily seen to be homomorphically closed. Also \mathcal{M}^* satisfies property (a), for if $J \leq I \leq R$ with $J \in \mathcal{M}^*$, then $J \in \mathcal{M}_n$ for some n so that $J' \in \mathcal{M}_{n+1} \subseteq \mathcal{M}^*$.

If \mathcal{A} is any homomorphically closed class containing \mathcal{M} and satisfying property (a), an easy induction shows $\mathcal{M}_n \subseteq \mathcal{A}$ for each n so that $\mathcal{M}^* \subseteq \mathcal{A}$.

COROLLARY [Leavitt]. *For $\mathcal{M} \subseteq \mathcal{W}$ with \mathcal{W} any universal class there is a unique minimal radical class in \mathcal{W} containing \mathcal{M} which satisfies property (a).*

Proof. This is immediate from Theorems 2.4 and 2.5.

Note that by Lemma 2.2 the radical $\mathcal{L}\mathcal{M}^*$ coincides with the lower radical $\mathcal{L}\mathcal{M}$ in alternative classes, and thus for such classes the above construction may be regarded as an alternate lower radical construction.

THEOREM 2.6. *Let \mathcal{W} be a universal class $\mathcal{M} \subseteq \mathcal{W}$. There is a unique minimal class $\mathcal{M}' \supseteq \mathcal{M}$ which is homomorphically closed, hereditary, and satisfies property (a).*

Proof. Define $\mathcal{M}_1 = \mathcal{M}$ and for $n \geq 1$ let $\mathcal{M}_{n+1} = \mathcal{G}\mathcal{F}\mathcal{H}(\mathcal{M}_n)$. Now set $\mathcal{M}' = \bigcup \mathcal{M}_n$, the union being taken over all positive integers.

As in the proof of Theorem 2.5, \mathcal{M}' is homomorphically closed, hereditary, and has property (a). Also as before (induction) $\mathcal{M}' \subseteq \mathcal{A}$ where \mathcal{A} is any homomorphically closed hereditary class with property (a) containing \mathcal{M} .

COROLLARY 2.7. *If \mathcal{M} is any class, $\mathcal{L}\mathcal{M}'$ is the unique minimal strongly hereditary radical class containing \mathcal{M} .*

Proof. Since $\mathcal{M} \subseteq \mathcal{M}'$, $\mathcal{M} \subseteq \mathcal{L}\mathcal{M}'$. Since \mathcal{M}' is homomorphically closed, hereditary, and satisfies property (a), $\mathcal{L}\mathcal{M}'$ has the same properties by Theorem 2.4 together with Theorem 2 of [6]. Now let $\mathcal{M} \subseteq \mathcal{P}$ where \mathcal{P} is a strongly hereditary radical class. Then \mathcal{P} is homomorphically closed and hereditary, hence satisfies property (a) by Lemmas 1.2 and 1.3. Hence by Theorem 2.6 $\mathcal{M}' \subseteq \mathcal{P}$ and therefore $\mathcal{L}\mathcal{M}' \subseteq \mathcal{P}$.

3. Semisimple classes. Using Theorem 2.1, nonassociative versions of certain theorems concerning semisimple classes can be given. In [4], semisimple classes of associative rings are characterized as those classes \mathcal{Q} satisfying the following four properties:

- (1) \mathcal{Q} is hereditary
- (2) \mathcal{Q} is closed under subdirect sums
- (3) \mathcal{Q} is extension closed
- (4) If $I \leq R$ and $0 \neq I/B \in \mathcal{Q}$ for some ideal B of I , there is an ideal A of R with $A \subseteq I$ and $0 \neq I/A \in \mathcal{Q}$.

For possibly nonassociative classes, we have

THEOREM 3.1. *\mathcal{Q} is a semisimple class for a radical class \mathcal{P} satisfying property (a) if and only if \mathcal{Q} satisfies properties (1), (2), (3), and (4).*

Proof. If $\mathcal{Q} = \mathcal{S}\mathcal{P}$ where \mathcal{P} has property (a), then the proof of (1), (2), (3), and (4) go through as in the associative case (see [4]) using Theorem 2.1 and Lemma 1.3. Conversely, suppose \mathcal{Q} satisfies (1), (2), (3), and (4). Then again as in the associative case \mathcal{Q} is semisimple for some radical \mathcal{P} . Suppose \mathcal{P} does not satisfy property (a). Then by Theorem 2.1 there is some R with an ideal I for which $\mathcal{P}(I)$ is not an ideal of R . Let T be the ideal of R generated by $\mathcal{P}(I)$, then $\mathcal{P}(I) \leq T \leq I \leq R$. Then by (1) and two applications of Lemma 1.1, $\mathcal{P}(\mathcal{P}(I)) = \mathcal{P}(I) \subseteq \mathcal{P}(T) \subseteq \mathcal{P}(I)$ so $\mathcal{P}(I) = \mathcal{P}(T) \neq T \leq R$, and T is the ideal of R generated by $\mathcal{P}(T)$. Also $T/\mathcal{P}(T) \in \mathcal{Q}$ so by (4) there is an ideal K of R with $K \subseteq T$ so that T/K is nonzero in \mathcal{Q} . Thus $K \geq \mathcal{P}(T)$ so K is an ideal of R containing $\mathcal{P}(T)$, and K is proper in T , a contradiction which proves the theorem.

In [4], an ideal I of a ring R is said to be large in R if I has nonzero intersection with every nonzero ideal of R . It is proved there that a radical class \mathcal{P} in an associative universal class is hereditary if and only if $\mathcal{S}\mathcal{P}$ satisfies property (λ): If $I \leq R$ with $I \in \mathcal{S}\mathcal{P}$ and I large in R , then $R \in \mathcal{S}\mathcal{P}$. The same proof given there proves the

following theorem, which is valid in an arbitrary universal class.

THEOREM 3.2. *Let \mathcal{P} be a radical class satisfying property (a). Then \mathcal{P} is hereditary if and only if \mathcal{SP} satisfies property (λ).*

Theorem 3.1 and 3.2 may be combined to give the following characterization of semisimple classes for strongly hereditary radicals:

THEOREM 3.3. *\mathcal{Q} is a semisimple class for a strongly hereditary radical if and only if \mathcal{Q} satisfies properties (1), (2), (3), (4), and (λ).*

Proof. Suppose $\mathcal{Q} = \mathcal{SP}$ where \mathcal{P} is strongly hereditary. Then \mathcal{P} is hereditary and has property (a) so \mathcal{Q} satisfies (1), (2), (3), (4), and (λ). Conversely, if \mathcal{Q} satisfies (1), (2), (3), (4), and (λ), then $\mathcal{Q} = \mathcal{SP}$ for a radical \mathcal{P} satisfying property (a) by Theorem 3.1 and \mathcal{P} is hereditary by Theorem 3.2. Thus \mathcal{P} is strongly hereditary by Lemma 1.2.

The following proposition and its corollary show that certain semisimple classes of associative rings satisfy property (a).

PROPOSITION 3.4. *If \mathcal{M} is a class of associative rings which is hereditary, extension closed, and contains all nilpotent associative rings, then \mathcal{M} satisfies property (a).*

Proof. Suppose $J \subseteq I \subseteq R$ where $J \in \mathcal{M}$, and let J' be the ideal of R generated by J . Then $(J')^3 \subseteq J$ so $(J')^3 \in \mathcal{M}$. Also $J'/(J')^3 \in \mathcal{M}$, so $J' \in \mathcal{M}$ since \mathcal{M} is extension closed.

COROLLARY 3.5. *If \mathcal{P} is a radical in an associative universal class such that $R^2 = R$ for all $R \in \mathcal{P}$, then \mathcal{SP} has property (a).*

Proof. \mathcal{SP} is easily verified to satisfy the hypotheses of Proposition 3.4.

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