

Pacific Journal of Mathematics

**ON THE DENSITY OF CERTAIN COHESIVE BASIC
SEQUENCES**

DONALD GOLDSMITH

ON THE DENSITY OF CERTAIN COHESIVE BASIC SEQUENCES

DONALD L. GOLDSMITH

It has been shown in previous investigations of the combinatorial properties of basic sequences that any cohesive basic sequence \mathcal{B} which is contained in \mathcal{M} (the set of all pairs of relatively prime positive integers) must be large in some sense. To be precise, it has been proved that if \mathcal{B} is a cohesive basic sequence and $\mathcal{B} \subset \mathcal{M}$, then $C_{\mathcal{B}}(p)$ is infinite for every prime p , where $C_{\mathcal{B}}(p)$ is the set of prime companions of p in primitive pairs in \mathcal{B} . While this implies that \mathcal{B} must contain a great many primitive pairs, no specific statement has been made about the density of \mathcal{B} . It is reasonable to ask, therefore, whether there are cohesive basic sequences \mathcal{B} , contained in \mathcal{M} , with density $\delta(\mathcal{B}) = 0$.

It is shown here that such basic sequences do exist, and a method is given for the construction of a large class of these sequences.

A proof that $C_{\mathcal{B}}(p)$ is infinite when \mathcal{B} is cohesive and $\mathcal{B} \subset \mathcal{M}$ may be found in [2].

A basic sequence \mathcal{B} is a set of pairs (a, b) of positive integers satisfying

- (i) $(1, k) \in \mathcal{B}$ ($k = 1, 2, \dots$),
- (ii) $(a, b) \in \mathcal{B}$ if and only if $(b, a) \in \mathcal{B}$,
- (iii) $(a, bc) \in \mathcal{B}$ if and only if $(a, b) \in \mathcal{B}$ and $(a, c) \in \mathcal{B}$.

A pair (a, b) of positive integers is called a *primitive pair* if both a and b are primes. If $a \neq b$, the pair is a *type I* primitive pair; if $a = b$, the pair is a *type II* primitive pair. If Φ is a set of pairs (primitive or not) of positive integers, the basic sequence *generated* by Φ is defined to be

$$\Gamma[\Phi] = \bigcap \mathcal{D},$$

where the intersection is taken over all basic sequences \mathcal{D} which contain Φ .

A basic sequence \mathcal{B} is *cohesive* if for each positive integer k there is an integer $a > 1$ such that $(k, a) \in \mathcal{B}$.

Finally, we recall that the *density* of a basic sequence \mathcal{B} is defined by

$$(1.1) \quad \delta(\mathcal{B}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \frac{*B_k}{d(k)}$$

if the limit exists, where $d(k)$ is the number of positive divisors of k , and *B_k is the number of members (m, n) of \mathcal{B} for which $mn = k$.

2. The main theorem. We will use the following notation.

$$P = \{p_1, p_2, \dots\}$$

is the sequence of all primes, written in order of increasing magnitude;

$$Q = \{q_1, q_2, \dots\}$$

is any sequence of primes, also written in order of increasing size; and

$$Q_i = \{q_i, q_{i+1}, q_{i+2}, \dots\} \quad (i = 1, 2, \dots).$$

We define \mathcal{B}_Q to be the basic sequence generated by the primitive pairs

$$\{(p_1, q) \mid q \in Q_1\} \cup \{(p_2, q) \mid q \in Q_2\} \cup \dots.$$

REMARK 1. \mathcal{B}_Q is cohesive. For suppose $k > 1$, so that $k = p_{i_1}^{t_1} p_{i_2}^{t_2} \dots p_{i_M}^{t_M}$ where $i_1 < i_2 < \dots < i_M$. Then $(q_{i_j}, p_{i_j}) \in \mathcal{B}_Q$ for $j = 1, 2, \dots, M$, so $(q_{i_M}, k) \in \mathcal{B}_Q$.

REMARK 2. $\mathcal{B}_Q \subset \mathcal{M}$ if $q_1 \geq 3$. For if $q_1 \geq 3 (= p_2)$ then $q_i > p_i$ for every i , and \mathcal{B}_Q will contain no type II primitive pairs.

THEOREM. If $\sum_{i=1}^{\infty} 1/q_i$ converges, then $\delta(\mathcal{B}_Q) = 0$.

Proof. Let L be a (large) fixed, but arbitrary positive integer which will be determined later. Decompose the set \mathbf{Z}^+ of positive integers as follows:

- (a) $X' = \{k \mid {}^*B_k = 2\}$,
- (b) $X'' = \{k \mid k \notin X' \text{ and } k \text{ has less than } 4L \text{ different prime divisors}\}$,
- (c) $Y = \{k \mid k \notin (X \cup X'')\}$.

In order to prove that $\delta(\mathcal{B}_Q) = 0$, let us consider

$$(2.1) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in S}}^N \frac{{}^*B_k}{d(k)},$$

where $S = X', X''$ and Y .

By Lemma 3.2 in [1], we have

$$(2.2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{k=1 \\ k \in X'}}^N \frac{{}^*B_k}{d(k)} \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \frac{2}{d(k)} = 0,$$

while by Theorem 11.8 in [3] we have

$$(2.3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{k=1 \\ k \in X'}}^N \frac{^*B_k}{d(k)} \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{k=1 \\ k \in X'}}^N 1 = 0 .$$

It remains to estimate the sum in (2.1) when $S = Y$. Since

$$(2.4) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N \frac{^*B_k}{d(k)} \leq \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N 1 ,$$

we will find an upper bound for the number of elements of Y which do not exceed N . Our estimate will depend on the following

LEMMA. *Every integer in Y is divisible by at least one of the primes q_i with $i \geq L$.*

Proof of the Lemma. Let k be an element of Y . Then $^*B_k > 2$, so there are integers u, v such that

$$k = uv, u > 1, v > 1, (u, v) \in \mathcal{B}_Q .$$

Suppose that u and v are expressed canonically as products of prime powers:

$$u = p_{i_1}^{a_1} p_{i_2}^{a_2} \cdots p_{i_r}^{a_r}, \quad v = p_{j_1}^{b_1} p_{j_2}^{b_2} \cdots p_{j_s}^{b_s},$$

where $r \geq 1, s \geq 1, p_{i_1} < p_{i_2} < \cdots < p_{i_r}, p_{j_1} < p_{j_2} < \cdots < p_{j_s}$. Since k is divisible by at least $4L$ distinct primes, we have $r + s \geq 4L$. At least one of the numbers r, s must be $\geq 2L$, say

$$r \geq 2L .$$

If $p_{j_1} \in Q$, then every prime divisor of u is in Q since every primitive pair in \mathcal{B}_Q contains at least one member from Q . Hence $p_{i_r} = q_i$ (for some q_i in Q) and $q_i \geq q_r \geq q_{2L}$.

Suppose, on the other hand, that p_{j_1} is in Q . Now separate the primes p_{i_1}, \cdots, p_{i_r} into two classes, depending on whether or not they are in Q . Let x_1, \cdots, x_λ be those not in Q , written in order of ascending size, and let y_1, \cdots, y_ν be those in Q , also given in ascending order. Thus

$$u = x_1^{c_1} \cdots x_\lambda^{c_\lambda} y_1^{d_1} \cdots y_\nu^{d_\nu},$$

with

$$(2.5) \quad \lambda + \nu = r \geq 2L .$$

It follows from (2.5) that either $\lambda \geq L$ or $\nu \geq L$.

If $\lambda \geq L$, then $x_\lambda = p_m$ for some $m \geq L$. Since $p_m \in Q$, only

the primes in Q_m appear as companions of p_m in primitive pairs of \mathcal{B}_Q . In particular, since $(p_m, p_{j_1}) \in \mathcal{B}_Q$, we have

$$p_{j_1} \in Q_m \subset Q_L .$$

Thus $p_{j_1} \in Q$, $p_{j_1} \geq q_L$, and $p_{j_1} | k$.

If $\nu \geq L$, then $y_\nu \in Q$, $y_\nu \geq q_L$, and $y_\nu | k$.

That proves the Lemma.

We return to the estimation of the second sum in (2.4). As a consequence of the Lemma we have

$$\begin{aligned} \sum_{\substack{k=1 \\ k \in Y}}^N 1 &\leq \sum_{\substack{k=1 \\ q_i | k \text{ for some } i \geq L}}^N 1 \\ &\leq \sum_{i=L}^{\infty} \left[\frac{N}{q_i} \right] \\ &\leq N \sum_{i=L}^{\infty} \frac{1}{q_i} , \end{aligned}$$

and this together with (2.4) gives

$$(2.6) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N \frac{^*B_k}{d(k)} \leq \sum_{i=L}^{\infty} \frac{1}{q_i} .$$

Now let $\varepsilon > 0$ be given and choose L large enough so that

$$\sum_{i=L}^{\infty} \frac{1}{q_i} < \frac{\varepsilon}{3}$$

(L depends only on ε and Q). Then from (2.6) we have

$$(2.7) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N \frac{^*B_k}{d(k)} < \frac{\varepsilon}{3} ,$$

and it follows from (2.2), (2.3) and (2.7) that there is an integer $N_0(\varepsilon)$ such that

$$\frac{1}{N} \sum_{k=1}^N \frac{^*B_k}{d(k)} = \frac{1}{N} \left(\sum_{\substack{k=1 \\ k \in Y'}}^N + \sum_{\substack{k=1 \\ k \in Y''}}^N + \sum_{\substack{k=1 \\ k \in Y}}^N \right) \frac{^*B_k}{d(k)} < \varepsilon$$

when $N \geq N_0(\varepsilon)$.

That proves $\delta(\mathcal{B}_Q) = 0$, and completes the proof of the Theorem.

By Remarks 1 and 2 and the Theorem, each sequence Q of distinct odd primes such that $\sum 1/q_j$ converges leads to a cohesive basic sequence \mathcal{B}_Q in \mathcal{N} such that $\delta(\mathcal{B}_Q) = 0$.

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Received April 19, 1971. This research was supported in part by Western Michigan University under a Faculty Research Fellowship.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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