ON THE DENSITY OF CERTAIN COHESIVE BASIC SEQUENCES

DONALD GOLDSMITH
ON THE DENSITY OF CERTAIN COHESIVE BASIC SEQUENCES

DONALD L. GOLDSMITH

It has been shown in previous investigations of the combinatorial properties of basic sequences that any cohesive basic sequence $\mathcal{B}$ which is contained in $\mathcal{M}$ (the set of all pairs of relatively prime positive integers) must be large in some sense. To be precise, it has been proved that if $\mathcal{B}$ is a cohesive basic sequence and $\mathcal{B} \subset \mathcal{M}$, then $C_\mathcal{B}(p)$ is infinite for every prime $p$, where $C_\mathcal{B}(p)$ is the set of prime companions of $p$ in primitive pairs in $\mathcal{B}$. While this implies that $\mathcal{B}$ must contain a great many primitive pairs, no specific statement has been made about the density of $\mathcal{B}$. It is reasonable to ask, therefore, whether there are cohesive basic sequences $\mathcal{B}$, contained in $\mathcal{M}$, with density $\delta(\mathcal{B}) = 0$.

It is shown here that such basic sequences do exist, and a method is given for the construction of a large class of these sequences.

A proof that $C_\mathcal{B}(p)$ is infinite when $\mathcal{B}$ is cohesive and $\mathcal{B} \subset \mathcal{M}$ may be found in [2].

A basic sequence $\mathcal{B}$ is a set of pairs $(a, b)$ of positive integers satisfying

(i) $(1, k) \in \mathcal{B}$ $(k = 1, 2, \cdots)$,
(ii) $(a, b) \in \mathcal{B}$ if and only if $(b, a) \in \mathcal{B}$,
(iii) $(a, bc) \in \mathcal{B}$ if and only if $(a, b) \in \mathcal{B}$ and $(a, c) \in \mathcal{B}$.

A pair $(a, b)$ of positive integers is called a primitive pair if both $a$ and $b$ are primes. If $a \neq b$, the pair is a type I primitive pair; if $a = b$, the pair is a type II primitive pair. If $\Phi$ is a set of pairs (primitive or not) of positive integers, the basic sequence generated by $\Phi$ is defined to be

$$\Gamma[\Phi] = \bigcap \mathcal{D},$$

where the intersection is taken over all basic sequences $\mathcal{D}$ which contain $\Phi$.

A basic sequence $\mathcal{B}$ is cohesive if for each positive integer $k$ there is an integer $a > 1$ such that $(k, a) \in \mathcal{B}$.

Finally, we recall that the density of a basic sequence $\mathcal{B}$ is defined by

$$\delta(\mathcal{B}) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{iB_k}{d(k)}$$

(1.1)
if the limit exists, where \( d(k) \) is the number of positive divisors of \( k \), and \( \ast B_k \) is the number of members \((m, n)\) of \( \mathcal{B} \) for which \( mn = k \).

2. The main theorem. We will use the following notation.

\[
P = \{p_1, p_2, \cdots\}
\]
is the sequence of all primes, written in order of increasing magnitude;

\[
Q = \{q_1, q_2, \cdots\}
\]
is any sequence of primes, also written in order of increasing size; and

\[
Q_i = \{q_i, q_{i+1}, q_{i+2}, \cdots\} \quad (i = 1, 2, \cdots).
\]

We define \( \mathcal{B}_q \) to be the basic sequence generated by the primitive pairs

\[
\{(p_i, q) \mid q \in Q_i\} \cup \{(p_j, q) \mid q \in Q_2\} \cup \cdots.
\]

**Remark 1.** \( \mathcal{B}_q \) is cohesive. For suppose \( k > 1 \), so that \( k = p_{i_1} p_{i_2}^2 \cdots p_{i_M}^l \) where \( i_1 < i_2 < \cdots < i_M \). Then \((q_{i,j}, p_{i,j}) \in \mathcal{B}_q\) for \( j = 1, 2, \cdots, M \), so \((q_{i,j}, k) \in \mathcal{B}_q\).

**Remark 2.** \( \mathcal{B}_q \subseteq \mathcal{M} \) if \( q_1 \geq 3 \). For if \( q_1 \geq 3 \) (\( = p_2 \)) then \( q_i > p_i \) for every \( i \), and \( \mathcal{B}_q \) will contain no type II primitive pairs.

**Theorem.** If \( \sum_{i=1}^{\infty} 1/q_i \) converges, then \( \delta(\mathcal{B}_q) = 0 \).

**Proof.** Let \( L \) be a (large) fixed, but arbitrary positive integer which will be determined later. Decompose the set \( \mathbb{Z}^+ \) of positive integers as follows:

(a) \( X' = \{k \mid \ast B_k = 2\} \),

(b) \( X'' = \{k \mid k \in X' \text{ and } k \text{ has less than } 4L \text{ different prime divisors}\} \),

(c) \( Y = \{k \mid k \in (X \cup X'')\} \).

In order to prove that \( \delta(\mathcal{B}_q) = 0 \), let us consider

\[
\frac{1}{N} \sum_{k \in S} \ast B_k / d(k),
\]

where \( S = X', X'' \) and \( Y \).

By Lemma 3.2 in [1], we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k \in X'} \ast B_k / d(k) \leq \lim_{N \to \infty} \frac{1}{N} \sum_{k \in X'} 2 / d(k) = 0,
\]

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k \in Y} \ast B_k / d(k) \leq \lim_{N \to \infty} \frac{1}{N} \sum_{k \in Y} 2 / d(k) = 0.
\]
while by Theorem 11.8 in [3] we have

\[(2.3) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{\delta \text{B}_k}{d(k)} \leq \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} 1 = 0.\]

It remains to estimate the sum in (2.1) when \(S = Y\). Since

\[(2.4) \quad \frac{1}{N} \sum_{k \in Y} \frac{\delta \text{B}_k}{d(k)} \leq \frac{1}{N} \sum_{k \in Y} 1 ,\]

we will find an upper bound for the number of elements of \(Y\) which do not exceed \(N\). Our estimate will depend on the following

**Lemma.** Every integer in \(Y\) is divisible by at least one of the primes \(q_i\) with \(i \geq L\).

**Proof of the Lemma.** Let \(k\) be an element of \(Y\). Then \(\delta \text{B}_k > 2\), so there are integers \(u, v\) such that

\[k = uv, u > 1, v > 1, (u, v) \in \mathcal{B}_Q.\]

Suppose that \(u\) and \(v\) are expressed canonically as products of prime powers:

\[u = p_1^{s_1} \cdots p_r^{s_r}, \quad v = p_1^{t_1} \cdots p_s^{t_s},\]

where \(r \geq 1, s \geq 1, p_1 \leq \cdots \leq p_r, p_1 \leq \cdots \leq p_s\). Since \(k\) is divisible by at least \(4L\) distinct primes, we have \(r + s \geq 4L\). At least one of the numbers \(r, s\) must be \(\geq 2L\), say

\[r \geq 2L.\]

If \(p_j \in Q\), then every prime divisor of \(u\) is in \(Q\) since every primitive pair in \(\mathcal{B}_Q\) contains at least one member from \(Q\). Hence \(p_i = q_i\) (for some \(q_i\) in \(Q\)) and \(q_i \leq q_r \leq q_{2L}\).

Suppose, on the other hand, that \(p_j \notin Q\). Now separate the primes \(p_1, \cdots, p_r\) into two classes, depending on whether or not they are in \(Q\). Let \(x_1, \cdots, x_s\) be those not in \(Q\), written in order of ascending size, and let \(y_1, \cdots, y_t\) be those in \(Q\), also given in ascending order. Thus

\[u = x_1^{x_1} \cdots x_s^{x_s} y_1^{y_1} \cdots y_t^{y_t},\]

with

\[(2.5) \quad \lambda + \nu = r \geq 2L .\]

It follows from (2.5) that either \(\lambda \geq L\) or \(\nu \geq L\).

If \(\lambda \geq L\), then \(x_i = p_m\) for some \(m \geq L\). Since \(p_m \in Q\), only
the primes in $Q_m$ appear as companions of $p_m$ in primitive pairs of $\mathcal{B}_q$. In particular, since $(p_m, p_{j_1}) \in \mathcal{B}_q$, we have

$$p_{j_1} \in Q_m \subset Q_L.$$ 

Thus $p_{j_1} \in Q$, $p_{j_1} \geq q_L$, and $p_{j_1} \mid k$.

If $\nu \geq L$, then $y_{\nu} \in Q$, $y_{\nu} \geq q_L$, and $y_{\nu} \mid k$.

That proves the Lemma.

We return to the estimation of the second sum in (2.4). As a consequence of the Lemma we have

$$\sum_{k=1}^{N} \frac{1}{q_k} \leq \sum_{k=1}^{N} \frac{1}{q_{j_1}^k \text{ for some } i \geq L} \leq \sum_{i=L}^{\infty} \left[ \frac{N}{q_i} \right] \leq N \sum_{i=L}^{\infty} \frac{1}{q_i},$$

and this together with (2.4) gives

$$(2.6) \quad \frac{1}{N} \sum_{k=1}^{N} \frac{zB_{k}}{d(k)} \leq \sum_{i=L}^{\infty} \frac{1}{q_i}.$$

Now let $\varepsilon > 0$ be given and choose $L$ large enough so that

$$\sum_{i=L}^{\infty} \frac{1}{q_i} < \frac{\varepsilon}{3}$$

($L$ depends only on $\varepsilon$ and $Q$). Then from (2.6) we have

$$(2.7) \quad \frac{1}{N} \sum_{k=1}^{N} \frac{zB_{k}}{d(k)} < \frac{\varepsilon}{3},$$

and it follows from (2.2), (2.3) and (2.7) that there is an integer $N_0(\varepsilon)$ such that

$$\frac{1}{N} \sum_{k=1}^{N} \frac{zB_{k}}{d(k)} = \frac{1}{N} \left( \sum_{k=1}^{N} \frac{zB_{k}}{d(k)} + \sum_{k=1}^{N} \frac{zB_{k}}{d(k)} + \sum_{k=1}^{N} \frac{zB_{k}}{d(k)} \right) < \varepsilon$$

when $N \geq N_0(\varepsilon)$.

That proves $\delta(\mathcal{B}_0) = 0$, and completes the proof of the Theorem.

By Remarks 1 and 2 and the Theorem, each sequence $Q$ of distinct odd primes such that $\sum 1/q_j$ converges leads to a cohesive basic sequence $\mathcal{B}_0$ in $\mathcal{B}$ such that $\delta(\mathcal{B}_0) = 0.$
REFERENCES


Received April 19, 1971. This research was supported in part by Western Michigan University under a Faculty Research Fellowship.

WESTERN MICHIGAN UNIVERSITY
Stephen Richard Bernfeld, *The extendability of solutions of perturbed scalar differential equations* ................................................................. 277
James Edwin Brink, *Inequalities involving* $f_p$ *and* $f^{(n)}_q$ *for* $f$ *with* $n$ *zeros* ........................................................................................................ 289
Orrin Frink and Robert S. Smith, *On the distributivity of the lattice of filters of a groupoid* .......................................................... 313
Donald Goldsmith, *On the density of certain cohesive basic sequences* ........ 323
Charles Lemuel Hagopian, *Planar images of decomposable continua* ......... 329
W. N. Hudson, *A decomposition theorem for biadditive processes* .......... 333
W. N. Hudson, *Continuity of sample functions of biadditive processes* .... 343
Masako Izumi and Shin-ichi Izumi, *Integrability of trigonometric series. II* . 359
H. M. Ko, *Fixed point theorems for point-to-set mappings and the set of fixed points* ............................................................. 369
Gregers Louis Krabbe, *An algebra of generalized functions on an open interval: two-sided operational calculus* .............................. 381
Thomas Latimer Kriete, III, *Complete non-selfadjointness of almost selfadjoint operators* ................................................................. 413
Shiva Narain Lal and Siya Ram, *On the absolute Hausdorff summability of a Fourier series* ............................................................ 439
Ronald Leslie Lipsman, *Representation theory of almost connected groups* ......................................................................................... 453
James R. McLaughlin, *Integrated orthonormal series* .................................. 469
H. Minc, *On permanents of circulants* .......................................................... 477
Akihiro Okuyama, *On a generalization of* $\Sigma$-spaces ................................ 485
Norberto Salinas, *Invariant subspaces and operators of class* $(S)$ ............... 497
James D. Stafney, *The spectrum of certain lower triangular matrices as operators on the* $l_p$ *spaces* ................................................. 515
Arne Stray, *Interpolation by analytic functions* ........................................... 527
Li Pi Su, *Rings of analytic functions on any subset of the complex plane* .... 535
R. J. Tondra, *A property of manifolds compactly equivalent to compact manifolds* ............................................................................ 539