

Pacific Journal of Mathematics

ON THE DENSITY OF CERTAIN COHESIVE BASIC SEQUENCES

DONALD GOLDSMITH

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It has been shown in previous investigations of the combinatorial properties of basic sequences that any cohesive basic sequence \mathcal{B} which is contained in \mathcal{M} (the set of all pairs of relatively prime positive integers) must be large in some sense. To be precise, it has been proved that if \mathcal{B} is a cohesive basic sequence and $\mathcal{B} \subset \mathcal{M}$, then $C_{\mathcal{B}}(p)$ is infinite for every prime p , where $C_{\mathcal{B}}(p)$ is the set of prime companions of p in primitive pairs in \mathcal{B} . While this implies that \mathcal{B} must contain a great many primitive pairs, no specific statement has been made about the density of \mathcal{B} . It is reasonable to ask, therefore, whether there are cohesive basic sequences \mathcal{B} , contained in \mathcal{M} , with density $\delta(\mathcal{B}) = 0$.

It is shown here that such basic sequences do exist, and a method is given for the construction of a large class of these sequences.

A proof that $C_{\mathcal{B}}(p)$ is infinite when \mathcal{B} is cohesive and $\mathcal{B} \subset \mathcal{M}$ may be found in [2].

A basic sequence \mathcal{B} is a set of pairs (a, b) of positive integers satisfying

- (i) $(1, k) \in \mathcal{B}$ ($k = 1, 2, \dots$),
- (ii) $(a, b) \in \mathcal{B}$ if and only if $(b, a) \in \mathcal{B}$,
- (iii) $(a, bc) \in \mathcal{B}$ if and only if $(a, b) \in \mathcal{B}$ and $(a, c) \in \mathcal{B}$.

A pair (a, b) of positive integers is called a *primitive pair* if both a and b are primes. If $a \neq b$, the pair is a *type I* primitive pair; if $a = b$, the pair is a *type II* primitive pair. If Φ is a set of pairs (primitive or not) of positive integers, the basic sequence *generated* by Φ is defined to be

$$\Gamma[\Phi] = \bigcap \mathcal{D},$$

where the intersection is taken over all basic sequences \mathcal{D} which contain Φ .

A basic sequence \mathcal{B} is *cohesive* if for each positive integer k there is an integer $a > 1$ such that $(k, a) \in \mathcal{B}$.

Finally, we recall that the *density* of a basic sequence \mathcal{B} is defined by

$$(1.1) \quad \delta(\mathcal{B}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \frac{^*B_k}{d(k)}$$

if the limit exists, where $d(k)$ is the number of positive divisors of k , and $*B_k$ is the number of members (m, n) of \mathcal{B} for which $mn = k$.

2. The main theorem. We will use the following notation.

$$P = \{p_1, p_2, \dots\}$$

is the sequence of all primes, written in order of increasing magnitude;

$$Q = \{q_1, q_2, \dots\}$$

is any sequence of primes, also written in order of increasing size; and

$$Q_i = \{q_i, q_{i+1}, q_{i+2}, \dots\} \quad (i = 1, 2, \dots).$$

We define \mathcal{B}_Q to be the basic sequence generated by the primitive pairs

$$\{(p_1, q) \mid q \in Q_1\} \cup \{(p_2, q) \mid q \in Q_2\} \cup \dots.$$

REMARK 1. \mathcal{B}_Q is cohesive. For suppose $k > 1$, so that $k = p_1^{i_1} p_2^{i_2} \dots p_M^{i_M}$ where $i_1 < i_2 < \dots < i_M$. Then $(q_{i_M}, p_{i_j}) \in \mathcal{B}_Q$ for $j = 1, 2, \dots, M$, so $(q_{i_M}, k) \in \mathcal{B}_Q$.

REMARK 2. $\mathcal{B}_Q \subset \mathcal{M}$ if $q_1 \geq 3$. For if $q_1 \geq 3 (= p_2)$ then $q_i > p_i$ for every i , and \mathcal{B}_Q will contain no type II primitive pairs.

THEOREM. If $\sum_{i=1}^{\infty} 1/q_i$ converges, then $\delta(\mathcal{B}_Q) = 0$.

Proof. Let L be a (large) fixed, but arbitrary positive integer which will be determined later. Decompose the set Z^+ of positive integers as follows:

- (a) $X' = \{k \mid *B_k = 2\}$,
- (b) $X'' = \{k \mid k \notin X' \text{ and } k \text{ has less than } 4L \text{ different prime divisors}\}$,
- (c) $Y = \{k \mid k \notin (X \cup X'')\}$.

In order to prove that $\delta(\mathcal{B}_Q) = 0$, let us consider

$$(2.1) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in S}}^N \frac{*B_k}{d(k)},$$

where $S = X', X''$ and Y .

By Lemma 3.2 in [1], we have

$$(2.2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{k=1 \\ k \in X'}}^N \frac{*B_k}{d(k)} \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \frac{2}{d(k)} = 0,$$

while by Theorem 11.8 in [3] we have

$$(2.3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{k=1 \\ k \in X''}}^N \frac{^*B_k}{d(k)} \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{k=1 \\ k \in X''}}^N 1 = 0.$$

It remains to estimate the sum in (2.1) when $S = Y$. Since

$$(2.4) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N \frac{^*B_k}{d(k)} \leq \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N 1,$$

we will find an upper bound for the number of elements of Y which do not exceed N . Our estimate will depend on the following

LEMMA. *Every integer in Y is divisible by at least one of the primes q_i with $i \geq L$.*

Proof of the Lemma. Let k be an element of Y . Then $^*B_k > 2$, so there are integers u, v such that

$$k = uv, u > 1, v > 1, (u, v) \in \mathcal{B}_Q.$$

Suppose that u and v are expressed canonically as products of prime powers:

$$u = p_{i_1}^{a_1} p_{i_2}^{a_2} \cdots p_{i_r}^{a_r}, \quad v = p_{j_1}^{b_1} p_{j_2}^{b_2} \cdots p_{j_s}^{b_s},$$

where $r \geq 1, s \geq 1, p_{i_1} < p_{i_2} < \cdots < p_{i_r}, p_{j_1} < p_{j_2} < \cdots < p_{j_s}$. Since k is divisible by at least $4L$ distinct primes, we have $r + s \geq 4L$. At least one of the numbers r, s must be $\geq 2L$, say

$$r \geq 2L.$$

If $p_{j_1} \in Q$, then every prime divisor of u is in Q since every primitive pair in \mathcal{B}_Q contains at least one member from Q . Hence $p_{i_r} = q_i$ (for some q_i in Q) and $q_i \geq q_r \geq q_{2L}$.

Suppose, on the other hand, that p_{j_1} is in Q . Now separate the primes p_{i_1}, \dots, p_{i_r} into two classes, depending on whether or not they are in Q . Let x_1, \dots, x_λ be those not in Q , written in order of ascending size, and let y_1, \dots, y_ν be those in Q , also given in ascending order. Thus

$$u = x_1^{c_1} \cdots x_\lambda^{c_\lambda} y_1^{d_1} \cdots y_\nu^{d_\nu},$$

with

$$(2.5) \quad \lambda + \nu = r \geq 2L.$$

It follows from (2.5) that either $\lambda \geq L$ or $\nu \geq L$.

If $\lambda \geq L$, then $x_\lambda = p_m$ for some $m \geq L$. Since $p_m \in Q$, only

the primes in Q_m appear as companions of p_m in primitive pairs of \mathcal{B}_Q . In particular, since $(p_m, p_{j_1}) \in \mathcal{B}_Q$, we have

$$p_{j_1} \in Q_m \subset Q_L.$$

Thus $p_{j_1} \in Q$, $p_{j_1} \geq q_L$, and $p_{j_1} | k$.

If $\nu \geq L$, then $y_\nu \in Q$, $y_\nu \geq q_L$, and $y_\nu | k$.

That proves the Lemma.

We return to the estimation of the second sum in (2.4). As a consequence of the Lemma we have

$$\begin{aligned} \sum_{\substack{k=1 \\ k \in Y}}^N 1 &\leq \sum_{\substack{k=1 \\ q_i | k \text{ for some } i \geq L}}^N 1 \\ &\leq \sum_{i=L}^{\infty} \left[\frac{N}{q_i} \right] \\ &\leq N \sum_{i=L}^{\infty} \frac{1}{q_i}, \end{aligned}$$

and this together with (2.4) gives

$$(2.6) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N \frac{\#B_k}{d(k)} \leq \sum_{i=L}^{\infty} \frac{1}{q_i}.$$

Now let $\varepsilon > 0$ be given and choose L large enough so that

$$\sum_{i=L}^{\infty} \frac{1}{q_i} < \frac{\varepsilon}{3}$$

(L depends only on ε and Q). Then from (2.6) we have

$$(2.7) \quad \frac{1}{N} \sum_{\substack{k=1 \\ k \in Y}}^N \frac{\#B_k}{d(k)} < \frac{\varepsilon}{3},$$

and it follows from (2.2), (2.3) and (2.7) that there is an integer $N_0(\varepsilon)$ such that

$$\frac{1}{N} \sum_{k=1}^N \frac{\#B_k}{d(k)} = \frac{1}{N} \left(\sum_{\substack{k=1 \\ k \in X'}}^N + \sum_{\substack{k=1 \\ k \in X''}}^N + \sum_{\substack{k=1 \\ k \in Y}}^N \right) \frac{\#B_k}{d(k)} < \varepsilon$$

when $N \geq N_0(\varepsilon)$.

That proves $\delta(\mathcal{B}_Q) = 0$, and completes the proof of the Theorem.

By Remarks 1 and 2 and the Theorem, each sequence Q of distinct odd primes such that $\sum 1/q_j$ converges leads to a cohesive basic sequence \mathcal{B}_Q in \mathcal{H} such that $\delta(\mathcal{B}_Q) = 0$.

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Received April 19, 1971. This reseach was supported in part by Western Michigan University under a Faculty Research Fellowship.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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