Pacific Journal of Mathematics

PLANAR IMAGES OF DECOMPOSABLE CONTINUA

Charles Lemuel Hagopian

Vol. 42, No. 2 February 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 2, 1972

PLANAR IMAGES OF DECOMPOSABLE CONTINUA

CHARLES L. HAGOPIAN

A nondegenerate metric space that is both compact and connected is called a continuum. In this paper it is proved that if M is a continuum with the property that for each indecomposable subcontinuum H of M there is a continuum K in M containing H such that K is connected im kleinen at some point of H and if f is a continuous function on M into the plane, then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable. Consequently, if M is a continuum in Euclidean n-space that does not contain an indecomposable continuum in its boundary, then no planar continuous image of M has an indecomposable continuum in the boundary of one of its complementary domains.

For a given set Z, the closure and the boundary of Z are denoted by Cl Z and Bd Z respectively. The union of the elements of Z is denoted by St Z.

THEOREM 1. *If X is a continuum in a 2-sphere S and I is an indecomposable subcontinuum of X that is contained in the boundary of a complementary domain of X, then every subcontinuum of X that contains a nonempty open subset of I contains I.*

Proof. Let D be a complementary domain of X such that $I \subset \mathrm{Bd}\, D$, and let $X' = S - D$. By Theorem 1 of [1], every subcontinuum of X', and hence every subcontinuum of X, which contains a nonempty open subset of I contains I.

DEFINITION. An indecomposable subcontinuum I of a continuum X is said to be *terminal* in X if there exists a composant C of I such that each subcontinuum of X that meets both C and $X - I$ contains I.

THEOREM 2. *Suppose X is a plane continuum, I is an indecomposable subcontinuum of X, and each subcontinuum of X that contains a nonempty open subset of I contains I. Then I is terminal in X.*

Proof. Suppose there exists a collection E of continua in X such that for each composant C of I there is an element of E that meets both C and $X - I$ and does not contain I. Let $\{U_n\}$ be the elements of a countable base (for the topology on the plane) that intersect I. For each positive integer n, let P_n be the set consisting of all components Q of $I - U_n$ such that Q meets an element of E that is contained in $X - \mathrm{Cl}\, U_n$. Since $I = \bigcup_{n=1}^{\infty} \mathrm{St}\, P_n$, for some integer n, the set $\mathrm{St}\, P_n$ is a second category subset of I. Let L be the set consisting of all elements B of P_n such that there exists a subcontinuum F of an element of E contained in $X - \mathrm{Cl}\, U_n$ with the property that F meets both B and $X - I$ and does not intersect $I - B$. According to a theorem of Kuratowski's [3], $\mathrm{St}\, L$ is a first category subset of I. Let J denote the set of all elements H of E such that H is contained in $X - \mathrm{Cl}\, U_n$ and meets an element of $P_n - L$. Define R to be the union of all components of $\mathrm{St}\,(J \cup P_n)$ that intersect the set $\mathrm{St}\, J$. Each element of J meets three elements of P_n. Hence each component of R contains a triod. It follows that the components of R are countable. Since $\mathrm{St}\,(P_n - L)$ is a second category subset of I that is contained in R, there exists a component T of R such that $\mathrm{Cl}\, T$ contains a nonempty open subset of I. But since $\mathrm{Cl}\, T$ is a continuum in $X - U_n$, this is a contradiction. Hence I is terminal in X.

THEOREM 3. *Suppose M is a continuum with the property that for each indecomposable subcontinuum H of M there is a continuum K in M containing H such that K is connected im kleinen at some point of H and f is a continuous function on M into the plane. Then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable.*

Proof. Suppose a complementary domain of $f(M)$ contains an indecomposable continuum I in its boundary. According to Theorems 1 and 2, I is terminal in $f(M)$. Hence there exists a composant C of I such that each subcontinuum of $f(M)$ that meets both C and $f(M) - I$ contains I. Let p be a point of $f^{-1}(C)$. Define Z to be the p-component of $f^{-1}(I)$. As in the proof of Theorem 2 of [2], $f(Z) = I$.

Let A be a composant of I distinct from C. There exists a continuum H in Z such that $f(H)$ meets A and C, and no proper subcontinuum of H has an image under f that meets both A and C. Note that $f(H) = I$ and H is indecomposable. There is a continuum K in M containing H that is connected im kleinen at some point of H. Hence there exists a continuum W in K whose interior (relative to K) meets H such that $f(W)$ does not contain I. Each composant of H meets W.

Let x be a point of $H \cap f^{-1}(C)$. Since the x-composant of H intersects W, it follows that $f(W)$ is contained in C. Let y be a point of $H \cap f^{-1}(A)$. There exists a proper subcontinuum Y of H that contains y and meets W. Since $f(Y)$ meets both A and C, this is a contradiction. Hence the boundary of each complementary domain of $f(M)$ is hereditarily decomposable.

COROLLARY 1. *If a continuous image of a hereditarily decomposable continuum lies in the plane, then the boundary of each of its complementary domains is hereditarily decomposable.*

COROLLARY 2. *If M is a continuum in Euclidean n-space that does not contain an indecomposable continuum in its boundary and f is a continuous function on M into the plane, then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable.*

The author gratefully acknowledges conversations about these results with Professors E. E. Grace and F. B. Jones.

REFERENCES

1. C. L. Hagopian, *A fixed point theorem for plane continua*, Bulletin Amer. Math. Soc., **77** (1971), 351-354.
2. ———, *λ connected plane continua*, to appear.
3. K. Kuratowski, *Sur une condition qui caractérise les continus indecomposables*, Fundamenta Math., **14** (1929), 116-117.

Received June 17, 1971 and in revised form September, 7, 1971.

SACRAMENTO STATE COLLEGE
AND
ARIZONA STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 42, No. 2 February, 1972