PLANAR IMAGES OF DECOMPOSABLE CONTINUA

CHARLES LEMUEL HAGOPIAN
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A nondegenerate metric space that is both compact and connected is called a continuum. In this paper it is proved that if $M$ is a continuum with the property that for each indecomposable subcontinuum $H$ of $M$ there is a continuum $K$ in $M$ containing $H$ such that $K$ is connected im kleinen at some point of $H$ and if $f$ is a continuous function on $M$ into the plane, then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable. Consequently, if $M$ is a continuum in Euclidean $n$-space that does not contain an indecomposable continuum in its boundary, then no planar continuous image of $M$ has an indecomposable continuum in the boundary of one of its complementary domains.

For a given set $Z$, the closure and the boundary of $Z$ are denoted by $\text{Cl} \, Z$ and $\text{Bd} \, Z$ respectively. The union of the elements of $Z$ is denoted by $\text{St} \, Z$.

**Theorem 1.** If $X$ is a continuum in a 2-sphere $S$ and $I$ is an indecomposable subcontinuum of $X$ that is contained in the boundary of a complementary domain of $X$, then every subcontinuum of $X$ that contains a nonempty open subset of $I$ contains $I$.

**Proof.** Let $D$ be a complementary domain of $X$ such that $I \subset \text{Bd} \, D$, and let $X' = S - D$. By Theorem 1 of [1], every subcontinuum of $X'$, and hence every subcontinuum of $X$, which contains a nonempty open subset of $I$ contains $I$.

**Definition.** An indecomposable subcontinuum $I$ of a continuum $X$ is said to be *terminal* in $X$ if there exists a composant $C$ of $I$ such that each subcontinuum of $X$ that meets both $C$ and $X - I$ contains $I$.

**Theorem 2.** Suppose $X$ is a plane continuum, $I$ is an indecomposable subcontinuum of $X$, and each subcontinuum of $X$ that contains a nonempty open subset of $I$ contains $I$. Then $I$ is terminal in $X$.

**Proof.** Suppose there exists a collection $E$ of continua in $X$ such that for each composant $C$ of $I$ there is an element of $E$ that meets both $C$ and $X - I$ and does not contain $I$. Let $\{U_n\}$ be the elements of a countable base (for the topology on the plane) that intersect $I$. For each positive integer $n$, let $P_n$ be the set consisting of all components $Q$ of $I - U_n$ such that $Q$ meets an element of $E$ that is con-
tained in \( X - \text{Cl} \, U_n \). Since \( I = \bigcup_{n=1}^{\infty} \text{St} \, P_n \), for some integer \( n \), the set \( \text{St} \, P_n \) is a second category subset of \( I \). Let \( L \) be the set consisting of all elements \( B \) of \( P_n \) such that there exists a subcontinuum \( F \) of an element of \( E \) contained in \( X - \text{Cl} \, U_n \) with the property that \( F \) meets both \( B \) and \( X - I \) and does not intersect \( I - B \). According to a theorem of Kuratowski's [3], \( \text{St} \, L \) is a first category subset of \( I \). Let \( J \) denote the set of all elements \( H \) of \( E \) such that \( H \) is contained in \( X - \text{Cl} \, U_n \) and meets an element of \( P_n - L \). Define \( R \) to be the union of all components of \( \text{St} \,(J \cup P_n) \) that intersect the set \( \text{St} \, J \). Each element of \( J \) meets three elements of \( P_n \). Hence each component of \( R \) contains a triod. It follows that the components of \( R \) are countable. Since \( \text{St} \,(P_n - L) \) is a second category subset of \( I \) that is contained in \( R \), there exists a component \( T \) of \( R \) such that \( \text{Cl} \, T \) contains a nonempty open subset of \( I \). But since \( \text{Cl} \, T \) is a continuum in \( X - U_n \), this is a contradiction. Hence \( I \) is terminal in \( X \).

**Theorem 3.** Suppose \( M \) is a continuum with the property that for each indecomposable subcontinuum \( H \) of \( M \) there is a continuum \( K \) in \( M \) containing \( H \) such that \( K \) is connected im kleinen at some point of \( H \) and \( f \) is a continuous function on \( M \) into the plane. Then the boundary of each complementary domain of \( f(M) \) is hereditarily decomposable.

**Proof.** Suppose a complementary domain of \( f(M) \) contains an indecomposable continuum \( I \) in its boundary. According to Theorems 1 and 2, \( I \) is terminal in \( f(M) \). Hence there exists a composant \( C \) of \( I \) such that each subcontinuum of \( f(M) \) that meets both \( C \) and \( f(M) - I \) contains \( I \). Let \( p \) be a point of \( f^{-1}(C) \). Define \( Z \) to be the \( p \)-component of \( f^{-1}(I) \). As in the proof of Theorem 2 of [2], \( f(Z) = I \).

Let \( A \) be a composant of \( I \) distinct from \( C \). There exists a continuum \( H \) in \( Z \) such that \( f(H) \) meets \( A \) and \( C \), and no proper subcontinuum of \( H \) has an image under \( f \) that meets both \( A \) and \( C \). Note that \( f(H) = I \) and \( H \) is indecomposable. There is a continuum \( K \) in \( M \) containing \( H \) that is connected im kleinen at some point of \( H \). Hence there exists a continuum \( W \) in \( K \) whose interior (relative to \( K \)) meets \( H \) such that \( f(W) \) does not contain \( I \). Each composant of \( H \) meets \( W \).

Let \( x \) be a point of \( H \cap f^{-1}(C) \). Since the \( x \)-composant of \( H \) intersects \( W \), it follows that \( f(W) \) is contained in \( C \). Let \( y \) be a point of \( H \cap f^{-1}(A) \). There exists a proper subcontinuum \( Y \) of \( H \) that contains \( y \) and meets \( W \). Since \( f(Y) \) meets both \( A \) and \( C \), this is a contradiction. Hence the boundary of each complementary domain of \( f(M) \) is hereditarily decomposable.
**COROLLARY 1.** If a continuous image of a hereditarily decomposable continuum lies in the plane, then the boundary of each of its complementary domains is hereditarily decomposable.

**COROLLARY 2.** If $M$ is a continuum in Euclidean $n$-space that does not contain an indecomposable continuum in its boundary and $f$ is a continuous function on $M$ into the plane, then the boundary of each complementary domain of $f(M)$ is hereditarily decomposable.

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