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**CONTINUITY OF SAMPLE FUNCTIONS OF BIADDITIVE  
PROCESSES**

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## CONTINUITY OF SAMPLE FUNCTIONS OF BIADDITIVE PROCESSES

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**Let  $\{X(s, t): 0 \leq s, t \leq 1\}$  be a stochastic process which has independent increments (second differences). Necessary and sufficient conditions are established to ensure the existence of a version with the property that almost every sample function is continuous. A corollary to these results is the existence of a class of measures on Wiener-Yeh space. The conditions are analogous to the usual case of additive processes  $Z(t)$  indexed by one time parameter.**

$X(s, t)$  will be said to have independent "increments" (second differences) if whenever  $0 \leq s_0 < s_1 < \dots < s_m \leq 1$  and  $0 \leq t_0 < t_1 < \dots < t_n \leq 1$  the random variables  $X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})$   $i = 1, \dots, m, j = 1, \dots, n$  are independent. If  $X(s, t)$  has independent increments and  $X(0, t) = X(s, 0) = 0$ , then  $X(s, t)$  will be called biadditive. Let  $m(s, t) = E[X(s, t)]$  and  $v(s, t) = \text{var} [X(s, t)]$ . The following result is proved below:

There is a version of a biadditive process  $X(s, t)$  with the property that almost every sample function is continuous if and only if  $X(s, t)$  is Gaussian,  $m(s, t)$  and  $v(s, t)$  are continuous, and  $v(s, t)$  is the distribution function of a Lebesgue-Stieltjes measure on  $[0, 1] \times [0, 1]$ .

A special case of this result occurs when  $m(s, t) = 0$  and  $v(s, t) = st$ . This process is realized when the space  $C_2$  of continuous functions of two variables on  $[0, 1] \times [0, 1]$  is assigned the Wiener-Yeh measure and  $X(s, t)$  is defined by  $X(s, t)(f) = f(s, t)$  where  $f \in C_2$ . Theorem 2 will imply the existence of a class of Wiener-Yeh measures on  $C_2$  corresponding to the choices of a pair of continuous functions  $m(s, t)$  and  $v(s, t)$ .

The conditions on  $m(s, t)$  and  $v(s, t)$  are analogous to the well-known conditions for the usual case of a stochastic process indexed by one time parameter. The case for a process indexed by  $n$ -time parameters is similar. The proof here is probabilistic in nature, unlike the analytic proof given by Yeh in [2] for the special case above.

### 2. Statement of main results.

**THEOREM 1.** *Let  $X(s, t)$  be a biadditive process having the property that almost every sample function is continuous. Then  $X(s, t)$  is Gaussian and the increments of  $X(s, t)$  are Gaussian. Furthermore the functions  $m(s, t) = EX(s, t)$  and  $v(s, t) = \text{var} (X(s, t))$  are continuous*

and determine the distribution of the process.

The following corollary is easy and its proof will be omitted.

**COROLLARY.** *Let  $X(s, t)$  be as in Theorem 1. If the increments of  $X(s, t)$  are stationary, that is, if the distribution of  $X(s + h_1, t + h_2) - X(s, t + h_2) - X(s + h_1, t) + X(s, t)$  depends only on  $h_1$  and  $h_2$ , then there are constants  $c_1$  and  $c_2$  such that*

$$m(s, t) = EX(s, t) = c_1st$$

$$v(x, t) = \text{var}(X(s, t)) = c_2st.$$

**THEOREM 2.** *Let  $m(s, t)$  and  $v(s, t)$  be continuous functions on  $[0, 1] \times [0, 1]$  such that  $m(s, 0) = 0 = m(0, t)$  and  $v(s, 0) = 0 = v(0, t)$  for  $0 \leq s, t \leq 1$ . Suppose that  $v(s, t)$  satisfies the condition*

$$(A) \quad v(s'', t'') - v(s'', t') - v(s', t'') + v(s', t') \geq 0$$

whenever

$$0 \leq s' \leq s'' \leq 1 \quad \text{and} \quad 0 \leq t' < t'' \leq 1.$$

Then there is a biadditive Gaussian process  $X(s, t)$ ,  $0 \leq s, t \leq 1$ , such that

- (i)  $EX(s, t) = m(s, t)$  and  $\text{var}(X(s, t)) = v(s, t)$  and
- (ii) almost every sample function of  $X(s, t)$  is continuous on  $[0, 1] \times [0, 1]$ .

The distribution of  $X(s, t)$  is determined by  $m(s, t)$  and  $v(s, t)$ .

3. *Proof of Theorem 1.* We prove first that  $X(s, t)$  is Gaussian.

**LEMMA 3.1.** *If almost every sample function of  $X(s, t)$  is continuous on  $[0, 1] \times [0, 1]$ , then  $X(s, t)$  and its increments are normally distributed.*

*Proof.* We show that the version of the central limit theorem in reference [1] (Theorem 2, p. 197) applies. Let  $(s, t)$  be a fixed point in  $[0, 1] \times [0, 1]$  and define  $s_i = s(i/n)$ ,  $t_i = t(i/n)$ , and

$$\Delta_{ij}(n) = X(s_i, t_j) - X(s_i, t_{j-1}) - X(s_{i-1}, t_j) + X(s_{i-1}, t_{j-1}).$$

Let  $\varepsilon > 0$  be given and let  $A_n = [\max_{i,j=1,2,\dots,n} |\Delta_{ij}(n)| \geq \varepsilon]$ . Then almost every sample function of  $X(s, t)$  is uniformly continuous on  $[0, 1] \times [0, 1]$ , and consequently

$$P\{\limsup_{n \rightarrow \infty} A_n\} = 0.$$

Hence  $\limsup_{n \rightarrow \infty} P(A_n) = 0$ .

Now  $X(s, t)$  is the sum of independent random variables, that is,

$$X(s, t) = \sum_{i=1}^n \sum_{j=1}^n \Delta_{ij}(n) .$$

The  $\Delta_{ij}(n)$  form an infinitesimal system because

$$\max_{i, j=1, 2, \dots, n} P[|\Delta_{ij}(n)| \geq \varepsilon] \leq P[\max_{i, j=1, 2, \dots, n} |\Delta_{ij}(n)| \geq \varepsilon]$$

and since

$$\limsup_{n \rightarrow \infty} P(A_n) = 0 ,$$

$$\lim_{n \rightarrow \infty} \max_{i, j=1, 2, \dots, n} P[|\Delta_{ij}(n)| \geq \varepsilon] = 0 .$$

It follows that  $X(s, t)$  is normally distributed.

To show that the increments of  $X(s, t)$  are normally distributed, let  $s_0$  and  $t_0$  be fixed and for  $s \geq s_0, t \geq t_0$  consider the process

$$Y(s, t) = X(s, t) - X(s_0, t) - X(s, t_0) + X(s_0, t_0) .$$

It is biadditive and has continuous sample functions a.s. The above argument shows that  $Y(s, t)$  is Gaussian and hence the increments of  $X(s, t)$  are Gaussian.

To complete the proof of Theorem 1 we need to check that  $m(s, t)$  and  $v(s, t)$  are continuous and determine the distribution of the process. Since  $X(s, t)$  is biadditive, we have for  $s' < s''$  and  $t' < t''$

$$\begin{aligned} \text{var}(X(s'', t'')) &= \text{var}(X(s'', t'') - X(s', t'') - X(s'', s') + X(s', t')) \\ &\quad + \text{var}(X(s', t'') - X(s', t')) + \text{var}(X(s'', t') \\ &\quad - X(s', t')) + \text{var}(X(s', t')) \end{aligned}$$

$$\text{var}(X(s', t'') - X(s', t)) + \text{var}(X(s', t)) = \text{var}(X(s', t''))$$

$$\text{var}(X(s'', t') - X(s', t')) + \text{var}(X(s', t)) = \text{var}(X(s'', t')) .$$

From these equations using  $v(s, t) = \text{var}(X(s, t))$  we obtain

$$\begin{aligned} \text{var}(X(s'', t'') - X(s'', t') - X(s', t'') + X(s', t')) \\ = v(s'', t'') - v(s', t'') - v(s'', t') + v(s', t') . \end{aligned}$$

Since a similar relation holds for  $m(s, t) = EX(s, t)$ , the fact that the increments are Gaussian and  $X(s, t)$  is biadditive implies that the distribution of  $X(s, t)$  is determined by  $m(s, t)$  and  $v(s, t)$ .

Since almost every sample function is continuous,

$$\lim_{h_1, h_2 \rightarrow 0} X(s + h_1, t + h_2) = X(s, t) .$$

Let  $\varphi(h_1, h_2, u)$  denote the characteristic function of  $X(s + h_1, t + h_2)$ . Then

$$\varphi(h_1, h_2, u) = \exp \left\{ ium(s + h_1, t + h_2) - \frac{u^2}{2}v(s + h_1, t + h_2) \right\}$$

and hence

$$\begin{aligned} v(s, t) &= -2 \log |\varphi(0, 0, 1)| \\ &= -2 \lim_{h_1, h_2 \rightarrow 0} \log |\varphi(h_1, h_2, 1)| \\ &= \lim_{h_1, h_2 \rightarrow 0} v(s + h_1, t + h_2) \end{aligned}$$

so  $v(s, t)$  is continuous. To show  $m(s, t)$  is continuous, we use Chebychef's inequality.

$$\begin{aligned} \lim_{h_1, h_2 \rightarrow 0} P[|X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t)| \geq \varepsilon] \\ \leq \lim_{h_1, h_2 \rightarrow 0} \frac{v(s + h_1, t + h_2) - v(s, t)}{\varepsilon^2} = 0 \end{aligned}$$

so that

$$X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t) \xrightarrow{P} 0 .$$

Since  $X(s + h_1, t + h_2) \rightarrow X(s, t)$ , it follows that  $m(s, t)$  is continuous.

4. *Lemmas for Theorem 2.* In §3, we have shown that any biadditive stochastic process with almost all its sample functions continuous is Gaussian with continuous mean and variance functions. The next task is to show that given a pair of continuous functions  $m(s, t)$  and  $v(s, t)$  where  $v(s, t)$  is a normalized distribution function for a Lebesgue-Stieltjes measure on  $[0, 1] \times [0, 1]$ , there is a biadditive process  $X(s, t)$  such that  $EX(s, t) = m(s, t)$  and  $\text{var}(X(s, t)) = v(s, t)$ . For this proof a few preparatory results are needed. In the following Lemma, \* denotes convolution.;

LEMMA 4.1. *Suppose there is a system of probability distributions  $\{\Phi(a_1, b_1, a_2, b_2) | 0 \leq a_1 < a_2 \leq 1, 0 \leq b_1 < b_2 \leq 1\}$  such that for any  $\alpha > 0$  and  $\beta > 0$*

$$(1) \quad \Phi(a_1, b_1, a_2 + \alpha, b_2) = \Phi(a_1, b_1, a_2, b_2) * \Phi(a_2, b_1, a_2 + \alpha, b_2)$$

$$(2) \quad \Phi(a_1, b_1, a_2, b_2 + \beta) = \Phi(a_1, b_1, a_2, b_2) * \Phi(a_1, b_2, a_2, b_2 + \beta) .$$

*Then there is a biadditive process  $X(s, t)$  such that the increment*

$$X(a_2, b_2) - X(a_1, b_2) - X(a_2, b_1) + X(a_1, b_1)$$

has the probability distribution  $\Phi(a_1, b_1, a_2, b_2)$  for  $0 \leq a_1 < a_2 \leq 1$  and  $0 \leq b < b_2 \leq 1$ .

*Proof.* The proof uses the Daniell-Kolmogorov extension theorem in the usual manner and is therefore omitted. Conditions (1) and (2) guarantee the consistency of the system.

**LEMMA 4.2.** (Ottaviani's Inequality). *Let  $\{X_1, X_2, \dots, X_n\}$  be independent random variables and let  $S_k \equiv \sum_{i=1}^k X_i$ . If for some  $\varepsilon > 0$ ,*

$$P[|S_n - S_k| > \varepsilon] \leq \frac{1}{2} \text{ for } k = 0, 1, 2, \dots, n,$$

where  $S_0 \equiv 0$ , then

$$P[\max_{k=1,2,\dots,n} |S_k| > 2\varepsilon] \leq 2P[|S_n| > \varepsilon].$$

*Proof.* The proof may be found in reference [3]. It is very similar to the following lemma which will be proved in full.

**LEMMA 4.3.** (An extended version of Ottaviani's Inequality). *Let  $s_0 < s_1 < \dots < s_m$  and  $t_0 < t_1 < t_2 < \dots < t_n$ . Define*

$$A_{ij} \equiv X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})$$

where  $X(s, t)$  is a biadditive process on  $D = [0,1] \times [0,1]$ . Let  $R_l \equiv \sum_{i=1}^m \sum_{j=l+1}^n A_{ij}$  and  $Q_{kl} = \sum_{i=k+1}^m A_{il}$ . If for all  $k = 1, 2, \dots, m$  and  $l = 0, 1, \dots, n$

$$P\left[|R_l| > \frac{\varepsilon}{2}\right] \leq 1 - \sqrt{\frac{1}{2}}$$

and

$$P\left[|Q_{kl}| > \frac{\varepsilon}{2}\right] \leq 1 - \sqrt{\frac{1}{2}},$$

then

$$P\left[\max_{\substack{k=1,2,\dots,m \\ l=1,2,\dots,n}} |S_{kl}| > 2\varepsilon\right] \leq 2P[|S_{mn}| > \varepsilon].$$

*Proof.* Let  $A_{ij}$  be defined for  $i = 1, 2, \dots, m$ , and  $j = 1, 2, \dots, n$  by

$$A_{ij} \equiv [ |S_{kl}| \leq 2\varepsilon \text{ for } l < j \text{ and } k \leq m, |S_{kj}| \leq 2\varepsilon \text{ for } k < i, |S_{ij}| > 2\varepsilon ]$$

$$A_{11} \equiv [ |S_{11}| > 2\varepsilon ].$$

Let  $T \equiv \{(i, j) : i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n\}$ . It is clear that

$$\left[ \max_{(i,j) \in T} |S_{ij}| > 2\varepsilon \right] = \bigcup_{i=1}^m \bigcup_{j=1}^n A_{ij}$$

and the  $A_{ij}$ 's are disjoint. Now let

$$B_{kl} \equiv \left[ |R_l| < \frac{\varepsilon}{2}, |Q_{kl}| < \frac{\varepsilon}{2} \right].$$

Then,

$$A_{kl} \cap B_{kl} \subset [ |S_{mn}| > \varepsilon ]$$

and so,

$$\bigcup_{l=1}^n \bigcup_{k=1}^m (A_{kl} \cap B_{kl}) \subset [ |S_{mn}| > \varepsilon ] .$$

Since  $X(s, t)$  is biadditive,  $A_{kl}$  and  $B_{kl}$  are independent events, and  $R_l$  and  $Q_{kl}$  are independent random variables. It follows that

$$P(B_{kl}) = P\left[ |R_l| < \frac{\varepsilon}{2} \right] \cdot P\left[ |Q_{kl}| < \frac{\varepsilon}{2} \right] \cong \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} = \frac{1}{2} .$$

Hence,

$$\begin{aligned} \frac{1}{2} P\left[ \max |S_{ij}| > 2\varepsilon \right] &= \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n P(A_{ij}) \leq \sum_{i=1}^m \sum_{j=1}^n P(A_{ij} \cap B_{ij}) \\ &= P\left( \bigcup_{i=1}^m \bigcup_{j=1}^n A_{ij} \cap B_{ij} \right) \\ &\leq P[ |S_{mn}| > \varepsilon ] . \end{aligned}$$

LEMMA 4.4. *Let  $X(s, t)$  be a biadditive process on a probability space  $(\Omega, \mathfrak{B}, P)$  with  $(s, t) \in D = [0, 1] \times [0, 1]$ . Let  $m(s, t) \equiv EX(s, t)$  and  $v(s, t) \equiv \text{var}(X(s, t))$  be continuous on  $D$ . Then for any point  $(s_0, t_0) \in D$  and for any sequence of points  $\{(s_n, t_n)\} \subset D$  such that*

$$\lim_{n \rightarrow \infty} (s_n, t_n) = (s_0, t_0)$$

$$P\left[ \lim_{n \rightarrow \infty} X(s_n, t_n) = X(s_0, t_0) \right] = 1 .$$

*Proof.* Let  $\varepsilon > 0$  be chosen arbitrarily except for the condition  $\varepsilon < 1 - \sqrt{1/2} < 1/2$ . Chebychef's Inequality and the uniform continuity of  $m(s, t)$  and  $v(s, t)$  imply that there is a  $\delta > 0$  such that for  $(s', t')$  and  $(s, t) \in [s_0 - \delta, s_0 + \delta] \times [t_0 - \delta, t_0 + \delta]$

$$(1) \quad P\left[ |X(s, t) - X(s', t')| \geq \frac{\varepsilon}{2} \right] < \frac{\varepsilon}{4} .$$

Now let  $S$  be a countable dense set in  $D$  and let  $S_1, S_2, S_3,$  and  $S_4$

denote the sets

$$\begin{aligned} S_1 &\equiv S \cap ([s_0, s_0 + \delta] \times [t_0, t_0 + \delta]) \\ S_2 &\equiv S \cap ([s_0, s_0 + \delta] \times [t_0 - \delta, t_0]) \\ S_3 &\equiv S \cap ([s_0 - \delta, s_0] \times [t_0, t_0 + \delta]) \\ S_4 &\equiv S \cap ([s_0 - \delta, s_0] \times [t_0 - \delta, t_0]) . \end{aligned}$$

The first part of the proof will show that

$$(2) \quad P \left[ \sup_{(s,t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] \leq 6\varepsilon .$$

The same kind of argument can be used to show that for  $i = 2, 3,$  and  $4$

$$(3) \quad P \left[ \sup_{(s,t) \in S_i} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] \leq 6\varepsilon$$

and so only the case for  $S_1$  will be done here.

Let the elements of  $S_1$  be numbered in an arbitrary manner so that  $S_1 = \{(s_i, t_i): i = 1, 2, \dots\}$ . Then

$$(4) \quad \begin{aligned} &P \left[ \sup_{(s,t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] \\ &= \lim_{n \rightarrow \infty} P \left[ \max_{i=1, \dots, n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon \right] . \end{aligned}$$

Thus it suffices to show that

$$(5) \quad P \left[ \max_{i=1, \dots, n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon \right] \leq 6\varepsilon$$

in order to prove (2). Now clearly

$$(6) \quad \begin{aligned} &P \left[ \max_{i=1, \dots, n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon \right] \\ &\leq P \left[ \max_{i=1, \dots, n} |X(s_i, t_i) - X(s_0, t_i) - X(s_i, t_0) + X(s_0, t_0)| > 2\varepsilon \right] \\ &\quad + P \left[ \max_{i=1, \dots, n} |X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon \right] \\ &\quad + P \left[ \max_{i=1, \dots, n} |X(s_0, t_i) - X(s_0, t_0)| > 2\varepsilon \right] . \end{aligned}$$

Consider the first  $n$  points  $(s_1, t_1), \dots, (s_n, t_n)$  in  $S_1$ . Let  $\sigma_1, \dots, \sigma_n$  and  $\tau_1, \dots, \tau_n$  be rearrangements of  $s_1, \dots, s_n$  and  $t_1, \dots, t_n$  respectively so that  $s_0 \leq \sigma_1 \leq \sigma_2 \leq \dots, \leq \sigma_n \leq s_0 + \delta$  and  $t_0 \leq \tau_1 \leq \tau_2 \leq \dots, \leq \tau_n \leq t_0 + \delta$ . Since  $X(s, t)$  is biadditive,



$$\begin{aligned}
& X(\sigma_i, \tau_j) - X(\sigma_i, t_0) - X(s_0, \tau_j) + X(s_0, t_0) \\
&= \sum_{m=1}^i \sum_{l=1}^j \{X(\sigma_m, \tau_l) - X(\sigma_{m-1}, \tau_l) - X(\sigma_m, \tau_{l-1}) + X(\sigma_{m-1}, \tau_{l-1})\} \\
& X(\sigma_i, t_0) - X(s_0, t_0) = \sum_{m=1}^i \{X(\sigma_m, t_0) - X(\sigma_{m-1}, t_0)\} \\
& X(s_0, t_j) - X(s_0, t_0) = \sum_{l=1}^j \{X(s_0, \tau_l) - X(s_0, \tau_{l-1})\}
\end{aligned}$$

are sums of independent random variables. Now if  $(s', t')$  and  $(s'', t'')$  are any two points in  $[s_0 - \delta, s_0 + \delta] \times [t_0 + \delta, t_0 + \delta]$ , then using (1) we may verify that the hypotheses of the Ottaviani inequalities, Lemmas 4.2 and 4.3, are satisfied. Thus

$$(7) \quad P\left[\max_{i=1, \dots, n} |X(\sigma_i, t_0) - X(s_0, t_0)| > 2\varepsilon\right] \leq 2P[|X(\sigma_n, t_0) - X(s_0, t_0)| > \varepsilon]$$

$$(8) \quad P\left[\max_{j=1, \dots, n} |X(s_0, \tau_j) - X(s_0, t_0)| > 2\varepsilon\right] \leq 2P[|X(s_0, \tau_n) - X(s_0, t_0)| > \varepsilon]$$

and

$$(9) \quad P\left[\max_{\substack{i=1, \dots, n \\ j=1, \dots, n}} |X(\sigma_i, \tau_j) - X(s_0, \tau_j) - X(\sigma_i, t_0) + X(s_0, t_0)| > 2\varepsilon\right] \\ \leq 2P[|X(\sigma_n, \tau_n) - X(s_0, \tau_n) - X(\sigma_n, t_0) + X(s_0, t_0)| > \varepsilon].$$

From the choice of  $\delta$  we see that the right sides of inequalities (7), (8), and (9) are each not greater than  $2\varepsilon$ . Since the  $\sigma_i$ 's are  $s_i$ 's and  $\tau_i$ 's are  $t_i$ 's, we have

$$(10) \quad P\left[\max_{i=1, \dots, n} |X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon\right] \leq 2\varepsilon$$

$$(11) \quad P\left[\max_{j=1, \dots, n} |X(s_0, t_j) - X(s_0, t_0)| > 2\varepsilon\right] \leq 2\varepsilon$$

and

$$(12) \quad P\left[\max_{i=1, \dots, n} |X(s_i, t_i) - X(s_0, t_i) - X(s_i, t_0) + X(s_0, t_0)| > 2\varepsilon\right] \leq 2\varepsilon.$$

Substituting (10), (11), and (12) into (6) we get (5), i.e.

$$P\left[\max_{i=1, \dots, n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\right] \leq 6\varepsilon.$$

Then

$$P\left[\sup_{(s,t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\right] \leq 6\varepsilon.$$

Since the proof of (2) is similar, it is omitted.

Now let  $V = S_1 \cup S_2 \cup S_3 \cup S_4$ . Then

$$(13) \quad \begin{aligned} &P\left[\sup_{(s,t) \in V} |X(s,t) - X(s_0,t_0)| > 6\varepsilon\right] \\ &\leq \sum_{i=1}^4 P\left[\sup_{(s,t) \in S_i} |X(s,t) - X(s_0,t_0)| > 6\varepsilon\right] \end{aligned}$$

and hence

$$(14) \quad P\left[\sup_{(s,t) \in V} |X(s,t) - X(s_0,t_0)| > 6\varepsilon\right] \leq 24\varepsilon.$$

Taking limits as  $\delta \downarrow 0$ , we obtain

$$(15) \quad P\left[\limsup_{\delta \downarrow 0} \sup_V |X(s,t) - X(s_0,t_0)| > 6\varepsilon\right] \leq 24\varepsilon.$$

Now let  $\varepsilon \downarrow 0$  and take complements to get

$$(16) \quad P\left[\limsup_{\delta \downarrow 0} \sup_V |X(s,t) - X(s_0,t_0)| = 0\right] = 1.$$

If an arbitrary sequence  $(s_n, t_n)$  with  $\lim_{n \rightarrow \infty} (s_n, t_n) = (s_0, t_0)$  is given, we extend the point set  $\{s_n, t_n\}$  to a countable dense set  $S$  in  $D$ . Then

$$\left[\lim_{n \rightarrow \infty} X(s_n, t_n) = X(s_0, t_0)\right] \supset \left[\limsup_{\delta \downarrow 0} \sup_V |X(s,t) - X(s_0,t_0)| = 0\right]$$

and by (16)

$$P\left[\lim_{n \rightarrow \infty} X(s_n, t_n) = X(s_0, t_0)\right] = 1.$$

LEMMA 4.5. Let  $X(s, t)$  be a biadditive process on a probability space  $(\Omega, \mathfrak{B}, P)$  with  $(s, t) \in D = [0, 1] \times [0, 1]$ . Suppose that  $v(s, t) \equiv \text{var}(X(s, t))$  is continuous over  $D$ . Furthermore, suppose that for any  $\varepsilon > 0$ ,

$$(1) \quad \lim_{n \rightarrow \infty} \sum_{j=1}^n \sum_{k=1}^n P\left[\left|X\left(\frac{k}{n}, \frac{j}{n}\right) - X\left(\frac{k-1}{n}, \frac{j}{n}\right) - X\left(\frac{k}{n}, \frac{j-1}{n}\right) + X\left(\frac{k-1}{n}, \frac{j-1}{n}\right)\right| > \varepsilon\right] = 0$$

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n P\left[\left|X\left(1, \frac{k}{n}\right) - X\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] = 0$$

and

$$(3) \quad \lim_{n \rightarrow \infty} \sum_{j=1}^n P\left[\left|X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] = 0.$$

Then there is a process  $Y(s, t)$  equivalent to  $X(s, t)$  such that almost

every sample function of  $Y(s, t)$  is continuous on  $D$ .

*Proof.* Let  $S$  be the set of all rational numbers in  $[0, 1]$  and let  $D' = S \times S$ . Define  $\Omega'$  by  $\Omega' = \{\omega \in \Omega: X(s, t) \text{ is uniformly continuous on } D'\}$ . In the first part of the proof, we show that  $P(\Omega') = 1$ .

Let  $Z_n$  be defined on  $(\Omega, \mathfrak{B}, P)$  by

$$Z_n = \sup \left\{ \left| X(s'', t'') - X(s', t') \right| : (s'', t'') \in D', (s', t') \in D' \text{ and } |s'' - s'| < \frac{1}{n}, |t'' - t'| < \frac{1}{n} \right\}.$$

Then  $X(s, t)$  is uniformly continuous on  $D'$  if and only if  $\lim_{n \rightarrow \infty} Z_n = 0$ . Hence,

$$(4) \quad P(\Omega') = P\left[\lim_{n \rightarrow \infty} Z_n = 0\right].$$

Let  $S_j \equiv S \cap [(j - 1)/n, j/n]$   $j = 1, \dots, n$ , and fix  $n$ . We number the elements of  $S_j$  in an arbitrary manner for each  $j = 1, \dots, n$ . Let  $j$  and  $k$  be now fixed and let  $s_1, \dots, s_{m-1}$  and  $t_1, \dots, t_{m-1}$  denote the first  $m - 1$  elements of  $S_j$  and  $S_k$  respectively. Let  $\sigma_1, \dots, \sigma_{m-1}$  and  $\tau_1, \dots, \tau_{m-1}$  be the arrangements of  $\{s_1, \dots, s_{m-1}\}$  and  $\{t_1, \dots, t_{m-1}\}$  respectively in ascending order so that  $\sigma_1 < \sigma_2 < \dots < \sigma_{m-1}$  and  $\tau_1 < \tau_2 < \dots < \tau_{m-1}$ . Choose  $\sigma_0 = (j - 1)/n$ ,  $\sigma_m = j/n$ ,  $\tau_0 = (k - 1)/n$ , and  $\tau_m = k/n$ , and define  $S_{jm} \equiv \{\tau_0, \tau_1, \dots, \tau_m\}$ . We will use the notation:

$$\Delta(s, t, s', t') \equiv X(s', t') - X(s, t') - X(s', t) + X(s, t).$$

Since  $X(s, t)$  is biadditive, the three collections of random variables below are systems of independent random variables:

$$\begin{aligned} & \{\Delta(\sigma_{\mu-1}, \tau_{\gamma-1}, \sigma_\mu, \tau_\gamma): \mu, \gamma = 1, \dots, m\} \\ & \left\{ \Delta\left(\frac{j-1}{n}, \tau_{\gamma-1}, \frac{j}{n}, \tau_\gamma\right): \gamma = 1, \dots, m \text{ and } j = 1, \dots, n \right\} \\ & \left\{ \Delta\left(\sigma_{\mu-1}, \frac{k-1}{n}, \sigma_\mu, \frac{k}{n}\right): \mu = 1, \dots, m \text{ and } k = 1, \dots, n \right\}. \end{aligned}$$

Let  $\varepsilon > 0$  be chosen arbitrarily. Since  $v(s, t)$  and  $m(s, t)$  are continuous on  $D$ , they are uniformly continuous and if  $n$  is sufficiently large and if  $0 < s'' - s' < 1/n$  or  $0 < t'' - t' < 1/n$ , then from Chebychef's inequality it follows that

$$(5) \quad P\left[|\Delta(s', t', s'', t'')| > \frac{\varepsilon}{2}\right] \leq 1 - \sqrt{\frac{1}{2}}.$$

Let  $Y_{n,j,k} \equiv \sup_{s \in S_j, t \in S_k} |X(s, t) - X((j - 1)/n, (k - 1)/n)|$ . Then from the

triangle inequality we get

$$\begin{aligned}
 Y_n &\equiv \max_{j,k=1,\dots,n} Y_{n,j,k} \\
 &\leq \max_{j,k=1,\dots,n} \sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| \\
 (6) \quad &+ \max_{j,k=1,\dots,n} \sup_{s \in S_j} \left| X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| \\
 &+ \max_{j,k=1,\dots,n} \sup_{t \in S_k} \left| X\left(\frac{j-1}{n}, t\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right|.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 &P[Y_n > 6\varepsilon] \\
 &\leq P\left[ \max_{j,k=1,\dots,n} \sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon \right] \\
 (7) \quad &+ P\left[ \max_{j,k=1,\dots,n} \sup_{s \in S_j} \left| X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| > 2\varepsilon \right] \\
 &+ P\left[ \max_{j,k=1,\dots,n} \sup_{t \in S_k} \left| X\left(\frac{j-1}{n}, t\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| > 2\varepsilon \right].
 \end{aligned}$$

For  $(\sigma_\mu, \tau_\gamma) \in S_{j_m} \times S_{k_m}$ , we see that

$$\Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \sigma_\mu, \tau_\gamma\right) = \sum_{q=1}^{\gamma} \sum_{p=1}^{\mu} \Delta(\sigma_{p-1}, \tau_{q-1}, \sigma_p, \tau_q)$$

a sum of independent random variables. Now (5) implies that the hypotheses of the extended Ottaviani's Inequality (Lemma 4.3) are satisfied and consequently

$$P\left[ \max_{\mu,\gamma=1,\dots,m} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \sigma_\mu, \tau_\gamma\right) \right| > 2\varepsilon \right] \leq 2P\left[ \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon \right].$$

Letting  $m \rightarrow \infty$ , it follows that

$$P\left[ \sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon \right] \leq 2P\left[ \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon \right]$$

and hence

$$\begin{aligned}
 (9) \quad &P\left[ \max_{j,k=1,\dots,n} \sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon \right] \\
 &\leq 2 \sum_{j=1}^n \sum_{k=1}^n P\left[ \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon \right].
 \end{aligned}$$

Now if  $\sigma_\mu \in S_{i_m}$ , since  $X(\sigma_\mu, 0) = X(0, (k-1)/n) = 0$ , we have

$$X\left(\sigma_\mu, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) = \sum_{p=1}^{\mu} \sum_{q=1}^{k-1} \Delta\left(\sigma_{p-1}, \frac{q-1}{n}, \sigma_p, \frac{q}{n}\right),$$

as before, a sum of independent random variables. Again, (5) allows us to use the extended Ottaviani's Inequality to obtain

$$P\left[\max_{k=1,\dots,n} \max_{t \in S_{jm}} \left|X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right)\right| > 2\varepsilon\right] \leq 2P\left[\left|X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right].$$

Letting  $m \rightarrow \infty$ , we get

$$P\left[\max_{k=1,\dots,n} \sup_{s \in S_j} \left|X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right)\right| > 2\varepsilon\right] \leq 2P\left[\left|X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right]$$

and

$$(10) \quad P\left[\max_{j,k=1,\dots,n} \sup_{s \in S_j} \left|X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right)\right| > 2\varepsilon\right] = P\left\{\bigcup_{j=1}^n \left[\max_{k=1,\dots,n} \sup_{s \in S_j} \left|X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right)\right| > 2\varepsilon\right]\right\} \leq 2 \sum_{j=1}^n P\left[\left|X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right].$$

Similarly for  $\tau_r \in S_{k,m}$

$$X\left(\frac{j-1}{n}, \tau_r\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) = \sum_{p=1}^{j-1} \sum_{q=1}^r d\left(\frac{p-1}{n}, \tau_{q-1}, \frac{p}{n}, \tau_q\right),$$

a sum of independent random variables, and so by (5) we may again apply the extended Ottaviani's Inequality and take limits as  $m \rightarrow \infty$ . We get

$$(11) \quad P\left[\max_{j,k=1,\dots,n} \sup_{t \in S_k} \left|X\left(\frac{j-1}{n}, t\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right)\right| > 2\varepsilon\right] \leq 2 \sum_{k=1}^n P\left[\left|X\left(1, \frac{k}{n}\right) - X\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right].$$

Inserting (9), (10), and (11) into (7) and letting  $n \rightarrow \infty$ , we see from the hypotheses (1), (2), and (3) that

$$(12) \quad \lim_{n \rightarrow \infty} P[Y_n > 6\varepsilon] = 0.$$

The inequality  $Z_n \leq 4Y_n$  can be checked by successive applications of the triangle inequality. (If  $|s' - s''| < 1/n$  and  $|t' - t''| < 1/n$ ,  $(s', t') \in [(j-1)/n, j/n] \times [(k-1)/n, k/n]$  implies that  $(s'', t'') \in [(j-2)/n, (j+1)/n] \times [(k-2)/n, (k+1)/n]$  and it suffices to check each possibility.) It follows that

$$P[Z_n > 24\varepsilon] \leq P[Y_n > 6\varepsilon] .$$

Since  $0 \leq Z_n$  and  $Z_{n+1} \leq Z_n$  for all  $n$ ,

$$\lim_{n \rightarrow \infty} P[Z_n > 24\varepsilon] = P\left[\lim_{n \rightarrow \infty} Z_n > 24\varepsilon\right] = 0$$

by (12). Letting  $\varepsilon \downarrow 0$ , we obtain

$$P\left[\lim_{n \rightarrow \infty} Z_n > 0\right] = 0 ,$$

and since  $Z_n \geq 0$ , we get

$$P(\mathcal{Q}') = P\left[\lim_{n \rightarrow 0} Z_n = 0\right] = 1 ,$$

which finishes the first part of the proof.

Now if  $x(s, t)$  is any real-valued function uniformly continuous on a set  $D$ , it has a unique continuous extension to the closure of  $D$ . Let  $Y(s, t, \omega)$  be defined for  $\omega \in \mathcal{Q}'$  by  $Y(s, t, \omega) = X(s, t, \omega)$  if  $(s, t) \in D'$ .

If  $(s, t) \notin D'$ , choose a sequence of points  $(s_n, t_n)$  in  $D'$  such that  $\lim_{n \rightarrow \infty} (s_n, t_n) = (s, t)$  and define  $Y(s, t, \omega) \equiv \lim_{n \rightarrow \infty} Y(s_n, t_n, \omega)$  for  $\omega \in \mathcal{Q}'$ . Since for  $\omega \in \mathcal{Q}'$   $Y(s, t, \omega)$  is uniformly continuous on  $D'$  which is dense in  $D$ ,  $Y(s, t, \omega)$  is well-defined for  $\omega \in \mathcal{Q}'$ . If  $\omega \notin \mathcal{Q}'$ , let  $Y(s, t, \omega) \equiv 0$ . Then for  $(s, t) \in D'$ ,

$$P[Y(s, t) = X(s, t)] \geq P(\mathcal{Q}') = 1$$

and if  $(s, t) \in D$  but  $(s, t) \notin D'$ ,

$$P\left[Y(s, t) = \lim_{n \rightarrow \infty} X(s_n, t_n)\right] \geq P(\mathcal{Q}') = 1$$

for some sequence  $\{(s_n, t_n)\}$  in  $D'$  such that  $\lim_{n \rightarrow \infty} (s_n, t_n) = (s, t)$ . But by Lemma 2.6,

$$P\left[X(s, t) = \lim_{n \rightarrow \infty} X(s_n, t_n)\right] = 1$$

and hence for any  $(s, t) \in D$ ,

$$P[Y(s, t) = X(s, t)] = 1 .$$

That is,  $Y(s, t)$  is a process which is equivalent to  $X(s, t)$ . It follows from the definition of  $Y(s, t)$ , that its sample functions are continuous on  $\mathcal{Q}'$ , a set of probability one.

### 5. Proof of Theorem 2.

*Proof.* Let  $\Phi(a, b, c, d)$  denote the normal probability distribution

with mean zero and variance  $v(c, d) - v(a, d) - v(c, b) + v(a, b)$  where  $0 \leq a < c \leq 1$  and  $0 \leq b < d \leq 1$ . Then since the convolution of normal distributions is a normal distribution whose mean and variance are the respective sums of the means and variances of the original distributions, for any  $\alpha > 0$  we have

$$\begin{aligned} \Phi(a, b, c + \alpha, d) &= \Phi(a, b, c, d) * \Phi(c, b, c + \alpha, d) \\ \Phi(a, b, c, d + \alpha) &= \Phi(a, b, c, d) * \Phi(a, d, c, d + \alpha) \end{aligned}$$

where “\*” denotes the operation of convolution.

By Lemma 4.1, there is a biadditive process  $Y(s, t)$  such that for  $s' < s''$  and  $t' < t''$ ,  $Y(s'', t'') - Y(s', t'') - Y(s'', t') + Y(s', t')$  is normally distributed with mean zero and variance  $v(s'', t'') - v(s', t'') - v(s'', t') + v(s', t')$ . If  $Y(s, t)$  satisfies conditions (1), (2), and (3) of Lemma 4.5 there is a process  $Y_0(s, t)$  equivalent to  $Y(s, t)$  such that almost every sample function of  $Y_0(s, t)$  is continuous over  $D$ . Define  $X(s, t) = Y_0(s, t) + m(s, t)$ . Then  $X(s, t)$  satisfies (i) and (ii) and is biadditive since  $Y_0(s, t)$  is. Furthermore almost every sample function of  $X(s, t)$  is continuous over  $D$ .

Let  $\Delta_{jk}$  denote the random variable

$$\Delta_{jk} \equiv Y\left(\frac{j}{n}, \frac{k}{n}\right) - Y\left(\frac{j-1}{n}, \frac{k}{n}\right) - Y\left(\frac{j}{n}, \frac{k-1}{n}\right) + Y\left(\frac{j-1}{n}, \frac{k-1}{n}\right)$$

where  $n$  is a positive integer. Conditions (1), (2), and (3) of Lemma 4.5 are

- (1) 
$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \sum_{k=1}^n P[|\Delta_{jk}| > \varepsilon] = 0$$
- (2) 
$$\lim_{n \rightarrow \infty} \sum_{k=1}^n P\left[\left| Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right) \right| > \varepsilon\right] = 0$$
- (3) 
$$\lim_{n \rightarrow \infty} \sum_{j=1}^n P\left[\left| Y\left(\frac{j}{n}, 1\right) - Y\left(\frac{j-1}{n}, 1\right) \right| > \varepsilon\right] = 0$$

where  $\varepsilon > 0$  is chosen in an arbitrary manner. We will use the following inequality which is valid for  $\lambda > 0$ .

$$\int_{\lambda}^{\infty} e^{-t^2/2} dt \leq \frac{1}{\lambda} \int_{\lambda}^{\infty} t e^{-t^2/2} dt = \frac{1}{\lambda} e^{-\lambda^2/2}.$$

For  $\varepsilon > 0$  since  $\Delta_{jk}$  is normally distributed,

$$\begin{aligned} P[|\Delta_{jk}| > \varepsilon] &= \frac{2}{\sqrt{2\pi v_{jk}}} \int_{\varepsilon}^{\infty} e^{-(t^2/2v_{jk})} dt \\ &= \frac{2}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-t^2/2} dt \end{aligned}$$

or

$$P[|A_{jk}| > \varepsilon] \leq \frac{2}{\lambda\sqrt{2\pi}} \exp\left\{-\frac{\lambda^2}{2}\right\} = \frac{2}{\varepsilon} \sqrt{\frac{v_{jk}}{2\pi}} \exp\left\{-\frac{\varepsilon^2}{2v_{jk}}\right\}$$

where

$$v_{jk} \equiv v\left(\frac{j}{n}, \frac{k}{n}\right) - v\left(\frac{j-1}{n}, \frac{k}{n}\right) - v\left(\frac{j}{n}, \frac{k-1}{n}\right) + v\left(\frac{j-1}{n}, \frac{k-1}{n}\right)$$

and  $\lambda = \varepsilon(v_{jk})^{-(1/2)}$ . Since  $v(s, t)$  is uniformly continuous over  $D$ , we can choose  $N$  independently of  $j$  and  $k$  such that  $n \geq N$  implies  $v_{jk}/\varepsilon^2 < 1/M_\delta^2$  where  $M_\delta$  is determined as follows. Since  $(1/x) \exp\{-(x^2/2)\} = o(x^{-2})$  as  $x \rightarrow \infty$ , we have for every positive integer  $\delta$ , a number  $M_\delta$  such that  $x > M_\delta$  implies  $x \exp\{-(x^2/2)\} < 1/\delta$ , that is, for  $x > M_\delta$ ,

$$\frac{1}{x} \exp\left\{-\frac{x^2}{2}\right\} < \frac{1}{\delta x^2}.$$

Now  $v_{jk}/\varepsilon^2 < 1/M_\delta^2$  entails  $\varepsilon/\sqrt{v_{jk}} > M_\delta$  and with  $x = \varepsilon/\sqrt{v_{jk}}$  we get

$$\frac{\sqrt{v_{jk}}}{\varepsilon} \exp\left\{-\frac{\varepsilon^2}{2v_{jk}}\right\} \leq \frac{1}{\delta} \frac{v_{jk}}{\varepsilon^2}.$$

Then for  $n \geq N$

$$P[|A_{jk}| > \varepsilon] \leq \frac{2}{\sqrt{2\pi}} \cdot \frac{v_{jk}}{\varepsilon^2}.$$

But  $v(1, 1) - v(1, 0) - v(0, 1) + v(0, 0) = v(1, 1) = \sum_{k=1}^n \sum_{j=1}^n v_{jk}$ , and so

$$\sum_{j=1}^n \sum_{k=1}^n P[|A_{jk}| < \varepsilon] \leq \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\delta \varepsilon^2} v(1, 1).$$

Since we may take  $\delta$  arbitrarily large, choosing  $N$  sufficiently large for each  $\delta$ ,

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \sum_{k=1}^n P[|A_{jk}| > \varepsilon] = 0$$

and (1) holds for  $Y(s, t)$ . A similar argument proves (2) and (3). Since  $Y(s, 0) = Y(0, t) = 0$  for all  $(s, t)$  in  $D$ ,  $Y(1, k/n) - Y(1, (k-1)/n)$  is normally distributed with mean zero and variance  $v(1, k/n) - v(1, (k-1)/n)$ , and  $Y(j/n, 1) - Y((j-1)/n, 1)$  is normally distributed with mean 0 and variance  $v(j/n, 1) - v((j-1)/n, 1)$ . Thus

$$\begin{aligned} P\left[\left|Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] &= \frac{2}{\sqrt{2\pi v_k}} \int_{\varepsilon}^{\infty} \exp\{-t^2/2v_k\} dt \\ &\leq \frac{2\sqrt{v_k}}{\sqrt{2\pi\varepsilon}} \exp\{-\varepsilon^2/2v_k\} \end{aligned}$$



and

$$\begin{aligned}
 P\left[\left|Y\left(\frac{j}{n}, 1\right) - Y\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] &= \frac{2}{\sqrt{2\pi v_j}} \int_{\varepsilon}^{\infty} \exp\{-t^2/2v\} dt \\
 &\leq \frac{2\sqrt{v_j}}{\sqrt{2\pi\varepsilon}} \exp\{-\varepsilon^2/2v_j\}
 \end{aligned}$$

where  $v_j \equiv v(j/n, 1 - v((j - 1)/n, 1))$  and  $v_k \equiv v(1, k/n) - v(1, (k - 1)/n)$ . Again we may choose  $\delta, M_\delta, N',$  and  $N''$  so that when  $n \geq N'$  or  $n \geq N''$ , the respective inequalities

$$\frac{v_j}{\varepsilon^2} < \frac{1}{M_\delta^2} \quad \text{or} \quad \frac{v_k}{\varepsilon^2} < \frac{1}{M_\delta^2}$$

hold. Since  $v(1, 1) = \sum_{j=1}^n v_j = \sum_{k=1}^n v_k$ ,

$$\sum_{k=1}^n P\left[\left|X\left(1, \frac{k}{n}\right) - X\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] \leq \frac{2}{\sqrt{2\pi\delta\varepsilon^2}} v(1, 1)$$

and

$$\sum_{j=1}^n P\left[\left|X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] \leq \frac{2}{\sqrt{2\pi\delta\varepsilon^2}} v(1, 1)$$

when  $n > N''$  or  $n > N'$  respectively. Thus there is a process  $Y_0(s, t)$  equivalent to  $Y(s, t)$  such that almost every sample function of  $Y_0$  is continuous over  $D$ . Setting  $X(s, t) = Y_0(s, t) + m(s, t)$  we obtain a biadditive process satisfying (i), (ii), and (iii).

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