CONTINUITY OF SAMPLE FUNCTIONS OF BIADDITIVE PROCESSES

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Let \( \{X(s, t); 0 \leq s, t \leq 1\} \) be a stochastic process which has independent increments (second differences). Necessary and sufficient conditions are established to ensure the existence of a version with the property that almost every sample function is continuous. A corollary to these results is the existence of a class of measures on Wiener-Yeh space. The conditions are analogous to the usual case of additive processes \( Z(t) \) indexed by one time parameter.

\( X(s, t) \) will be said to have independent "increments" (second differences) if whenever \( 0 \leq s_0 < s_1 < \cdots < s_m \leq 1 \) and \( 0 \leq t_0 < t_1 < \cdots < t_n \leq 1 \) the random variables \( X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1}) \) \( i = 1, \ldots, m, j = 1, \ldots, n \) are independent. If \( X(s, t) \) has independent increments and \( X(0, t) = X(s, 0) = 0 \), then \( X(s, t) \) will be called biadditive. Let \( m(s, t) = E[X(s, t)] \) and \( v(s, t) = \text{var} \{X(s, t)\} \). The following result is proved below:

There is a version of a biadditive process \( X(s, t) \) with the property that almost every sample function is continuous if and only if \( X(s, t) \) is Gaussian, \( m(s, t) \) and \( v(s, t) \) are continuous, and \( v(s, t) \) is the distribution function of a Lebesgue-Stieltjes measure on \([0, 1] \times [0, 1]\).

A special case of this result occurs when \( m(s, t) = 0 \) and \( v(s, t) = st \). This process is realized when the space \( C_2 \) of continuous functions of two variables on \([0, 1] \times [0, 1]\) is assigned the Wiener-Yeh measure and \( X(s, t) \) is defined by \( X(s, t)(f) = f(s, t) \) where \( f \in C_2 \). Theorem 2 will imply the existence of a class of Wiener-Yeh measures on \( C_2 \) corresponding to the choices of a pair of continuous functions \( m(s, t) \) and \( v(s, t) \).

The conditions on \( m(s, t) \) and \( v(s, t) \) are analogous to the well-known conditions for the usual case of a stochastic process indexed by one time parameter. The case for a process indexed by \( n \)-time parameters is similar. The proof here is probabilistic in nature, unlike the analytic proof given by Yeh in [2] for the special case above.

2. Statement of main results.

THEOREM 1. Let \( X(s, t) \) be a biadditive process having the property that almost every sample function is continuous. Then \( X(s, t) \) is Gaussian and the increments of \( X(s, t) \) are Gaussian. Furthermore the functions \( m(s, t) = E[X(s, t)] \) and \( v(s, t) = \text{var} \{X(s, t)\} \) are continuous
and determine the distribution of the process.

The following corollary is easy and its proof will be omitted.

**COROLLARY.** Let \( X(s, t) \) be as in Theorem 1. If the increments of \( X(s, t) \) are stationary, that is, if the distribution of \( X(s + h_1, t + h_2) - X(s + h_1, t) + X(s, t) \) depends only on \( h_1 \) and \( h_2 \), then there are constants \( c_1 \) and \( c_2 \) such that

\[
m(s, t) = \mathbb{E}X(s, t) = c_1st
\]

\[
v(s, t) = \text{var}(X(s, t)) = c_2st.
\]

**THEOREM 2.** Let \( m(s, t) \) and \( v(s, t) \) be continuous functions on \([0, 1] \times [0, 1]\) such that \( m(s, 0) = 0 = m(0, t) \) and \( v(s, 0) = 0 = v(0, t) \) for \( 0 \leq s, t \leq 1 \). Suppose that \( v(s, t) \) satisfies the condition

\[
(A) \quad v(s'', t'') - v(s'', t') - v(s', t'') + v(s', t') \geq 0
\]

whenever

\[
0 \leq s' \leq s'' \leq 1 \quad \text{and} \quad 0 \leq t' < t'' \leq 1.
\]

Then there is a biadditive Gaussian process \( X(s, t), 0 \leq s, t \leq 1 \), such that

(i) \( \mathbb{E}X(s, t) = m(s, t) \) and \( \text{var}(X(s, t)) = v(s, t) \) and

(ii) almost every sample function of \( X(s, t) \) is continuous on \([0, 1] \times [0, 1]\).

The distribution of \( X(s, t) \) is determined by \( m(s, t) \) and \( v(s, t) \).

3. **Proof of Theorem 1.** We prove first that \( X(s, t) \) is Gaussian.

**LEMMA 3.1.** If almost every sample function of \( X(s, t) \) is continuous on \([0, 1] \times [0, 1]\), then \( X(s, t) \) and its increments are normally distributed.

**Proof.** We show that the version of the central limit theorem in reference [1] (Theorem 2, p. 197) applies. Let \((s, t)\) be a fixed point in \([0, 1] \times [0, 1]\) and define \( s_i = s(i/n), t_i = t(i/n) \), and

\[
\Delta_i(n) = X(s_i, t_j) - X(s_i, t_{j-1}) - X(s_{i-1}, t_j) + X(s_{i-1}, t_{j-1}).
\]

Let \( \varepsilon > 0 \) be given and let \( A_n = \{ \max_{i,j=1,2,...,n} |\Delta_i(n)| \geq \varepsilon \} \). Then almost every sample function of \( X(s, t) \) is uniformly continuous on \([0, 1] \times [0, 1]\), and consequently

\[
P(\limsup_{n \to \infty} A_n) = 0.
\]
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Hence \( \limsup_{n \to \infty} P(A_n) = 0 \).

Now \( X(s, t) \) is the sum of independent random variables, that is,

\[
X(s, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}(n) .
\]

The \( A_{ij}(n) \) form an infinitesimal system because

\[
\max_{i, j=1, 2, \ldots, n} P\{ | A_{ij}(n) | \geq \varepsilon \} \leq \max_{i, j=1, 2, \ldots, n} P\{ | A_{ij}(n) | \geq \varepsilon \}
\]

and since

\[
\limsup_{n \to \infty} P(A_n) = 0 ,
\]

\[
\lim_{n \to \infty} \max_{i, j=1, 2, \ldots, n} P\{ | A_{ij}(n) | \geq \varepsilon \} = 0 .
\]

It follows that \( X(s, t) \) is normally distributed.

To show that the increments of \( X(s, t) \) are normally distributed, let \( s_0 \) and \( t_0 \) be fixed and for \( s \geq s_0 , t \geq t_0 \) consider the process

\[
Y(s, t) = X(s, t) - X(s_0, t) - X(s, t_0) + X(s_0, t_0) .
\]

It is biadditive and has continuous sample functions a.s. The above argument shows that \( Y(s, t) \) is Gaussian and hence the increments of \( X(s, t) \) are Gaussian.

To complete the proof of Theorem 1 we need to check that \( m(s, t) \) and \( v(s, t) \) are continuous and determine the distribution of the process. Since \( X(s, t) \) is biadditive, we have for \( s' < s'' \) and \( t' < t'' \)

\[
\begin{align*}
\text{var} \left( X(s'', t'') \right) &= \text{var} \left( X(s'', t') - X(s', t'') - X(s'', s') + X(s', t') \right) \\
&= \text{var} \left( X(s', t'') - X(s', t') \right) + \text{var} \left( X(s'', t') \right) \\
&- X(s', t')) + \text{var} \left( X(s', t') \right) \\
&= \text{var} \left( X(s', t') - X(s', t') \right) + \text{var} \left( X(s', t') \right) = \text{var} \left( X(s', t'') \right) \\
&= \text{var} \left( X(s', t') - X(s', t') + \text{var} \left( X(s', t') \right) \right) \\
&= v(s', t') \cdot v(s', t') + v(s', t') .
\end{align*}
\]

From these equations using \( v(s, t) = \text{var} \left( X(s, t) \right) \) we obtain

\[
\text{var} \left( X(s'', t'') - X(s', t') - X(s', t'') + X(s', t') \right) = v(s'', t'') - v(s', t') - v(s', t') + v(s', t') .
\]

Since a similar relation holds for \( m(s, t) = EX(s, t) \), the fact that the increments are Gaussian and \( X(s, t) \) is biadditive implies that the distribution of \( X(s, t) \) is determined by \( m(s, t) \) and \( v(s, t) \).

Since almost every sample function is continuous,

\[
\lim_{h_1, h_2 \to 0} X(s + h_1, t + h_2) = X(s, t) .
\]
Let $\varphi(h_1, h_2, u)$ denote the characteristic function of $X(s + h_1, t + h_2)$. Then

$$\varphi(h_1, h_2, u) = \exp \left\{ ium(s + h_1, t + h_2) - \frac{u^2}{2}v(s + h_1, t + h_2) \right\}$$

and hence

$$v(s, t) = -2 \log |\varphi(0, 0, 1)|$$
$$= -2 \lim_{h_1, h_2 \to 0} \log |\varphi(h_1, h_2, 1)|$$
$$= \lim_{h_1, h_2 \to 0} v(s + h_1, t + h_2)$$

so $v(s, t)$ is continuous. To show $m(s, t)$ is continuous, we use Chebychev's inequality.

$$\lim_{h_1, h_2 \to 0} P \left[ |X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t)| \geq \varepsilon \right]$$

$$\leq \lim_{h_1, h_2 \to 0} \frac{v(s + h_1, t + h_2) - v(s, t)}{\varepsilon^2} = 0$$

so that

$$X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t) \xrightarrow{P} 0 .$$

Since $X(s + h_1, t + h_2) \to X(s, t)$, it follows that $m(s, t)$ is continuous.

4. **Lemmas for Theorem 2.** In §3, we have shown that any biadditive stochastic process with almost all its sample functions continuous is Gaussian with continuous mean and variance functions. The next task is to show that given a pair of continuous functions $m(s, t)$ and $v(s, t)$ where $v(s, t)$ is a normalized distribution function for a Lebesgue-Stieltjes measure on $[0, 1] \times [0, 1]$, there is a biadditive process $X(s, t)$ such that $EX(s, t) = m(s, t)$ and $\text{var}(X(s, t)) = v(s, t)$. For this proof a few preparatory results are needed. In the following Lemma, $*$ denotes convolution.

**Lemma 4.1.** Suppose there is a system of probability distributions

$$\{\Phi(a_i, b_i, a_2, b_2) | 0 \leq a_i < a_2 \leq 1, 0 \leq b_i < b_2 \leq 1 \}$$

such that for any $\alpha > 0$ and $\beta > 0$

(1) $\Phi(a_i, b_i, a_2 + \alpha, b_2) = \Phi(a_i, b_i, a_2, b_2) * \Phi(a_2, b_i, a_2 + \alpha, b_2)$

(2) $\Phi(a_i, b_i, a_2, b_2 + \beta) = \Phi(a_i, b_i, a_2, b_2) * \Phi(a_i, b_2, a_2, b_2 + \beta)$.

Then there is a biadditive process $X(s, t)$ such that the increment

$$X(a_2, b_2) - X(a_1, b_2) - X(a_2, b_1) + X(a_1, b_1)$$
has the probability distribution \( \Phi(a_1, b_1, a_2, b_2) \) for \( 0 \leq a_1 < a_2 \leq 1 \) and \( 0 \leq b < b_2 \leq 1 \).

Proof. The proof uses the Daniell-Kolmogorov extension theorem in the usual manner and is therefore omitted. Conditions (1) and (2) guarantee the consistency of the system.

**Lemma 4.2.** (Ottaviani’s Inequality). Let \( \{X_1, X_2, \ldots, X_n\} \) be independent random variables and let \( S_k = \sum_{i=1}^{k} X_i \). If for some \( \varepsilon > 0 \),

\[
P[|S_k - S_n| > \varepsilon] \leq \frac{1}{2} \quad \text{for} \quad k = 0, 1, 2, \ldots n,
\]

where \( S_0 \equiv 0 \), then

\[
P\left[ \max_{k=1,2,\ldots,n} |S_k| > 2\varepsilon \right] \leq 2P[|S_n| > \varepsilon].
\]

Proof. The proof may be found in reference [3]. It is very similar to the following lemma which will be proved in full.

**Lemma 4.3.** (An extended version of Ottaviani’s Inequality). Let \( s_0 < s_1 < \cdots < s_m \) and \( t_0 < t_1 < t_2 < \cdots < t_n \). Define

\[
A_{ij} = X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})
\]

where \( X(s, t) \) is a biadditive process on \( D = [0,1] \times [0,1] \). Let \( R_l = \sum_{i=1}^{m} \sum_{j=1}^{l} A_{ij} \) and \( Q_m = \sum_{i=k+1}^{n} A_{ij} \). If for all \( k = 1, 2, \ldots, m \) and \( l = 0, 1, \ldots, n \)

\[
P\left[ |R_l| > \frac{\varepsilon}{2} \right] \leq 1 - \sqrt{\frac{1}{2}}
\]

and

\[
P\left[ |Q_m| > \frac{\varepsilon}{2} \right] \leq 1 - \sqrt{\frac{1}{2}},
\]

then

\[
P\left[ \max_{k=1,2,\ldots,m} \max_{l=1,2,\ldots,n} |S_{kl}| > 2\varepsilon \right] \leq 2P[|S_{mn}| > \varepsilon].
\]

Proof. Let \( A_{ij} \) be defined for \( i = 1, 2, \ldots, m \), and \( j = 1, 2, \ldots, n \) by

\[
A_{ij} = \begin{cases} |S_{ij}| \leq 2\varepsilon & \text{for } l < j \text{ and } k \leq m, \quad |S_{ij}| \leq 2\varepsilon & \text{for } k < i, \quad |S_{ij}| > 2\varepsilon \\
A_{1i} = |S_{1i}| > 2\varepsilon \end{cases}
\]

Let \( T = \{(i, j): 1 = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\} \). It is clear that
and the $A_{ij}$'s are disjoint. Now let

$$B_{kl} \equiv \left[ |R_i| < \frac{\varepsilon}{2}, |Q_{kl}| < \frac{\varepsilon}{2} \right].$$

Then,

$$A_{kl} \cap B_{kl} \subset [|S_{mn}| > \varepsilon]$$

and so,

$$\bigcup_{i=1}^n \bigcup_{k=1}^m (A_{kl} \cap B_{kl}) \subset [|S_{mn}| > \varepsilon].$$

Since $X(s, t)$ is biadditive, $A_{kl}$ and $B_{kl}$ are independent events, and $R_i$ and $Q_{kl}$ are independent random variables. It follows that

$$P(B_{kl}) = P\left[ |R_i| < \frac{\varepsilon}{2} \right] \cdot P\left[ |Q_{kl}| < \frac{\varepsilon}{2} \right] \geq \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} = \frac{1}{2}.$$ 

Hence,

$$\frac{1}{2} P\left[ \max_{i,j} |S_{ij}| > 2 \varepsilon \right] = \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n P(A_{ij}) \leq \sum_{i=1}^m \sum_{j=1}^n P(A_{ij} \cap B_{ij})$$

$$\leq P\left( \bigcup_{i=1}^m \bigcup_{j=1}^n A_{ij} \cap B_{ij} \right)$$

$$\leq P(\bigcup_{i=1}^m \bigcup_{j=1}^n A_{ij} \cap B_{ij}) \cdot \frac{1}{2}.$$

**Lemma 4.4.** Let $X(s, t)$ be a biadditive process on a probability space $(\Omega, \mathcal{F}, P)$ with $(s, t) \in D = [0, 1] \times [0, 1]$. Let $m(s, t) \equiv EX(s, t)$ and $v(s, t) \equiv \text{var}(X(s, t))$ be continuous on $D$. Then for any point $(s_0, t_0) \in D$ and for any sequence of points $\{(s_n, t_n)\} \subset D$ such that

$$\lim_{n \to \infty} (s_n, t_n) = (s_0, t_0)$$

$$P\left[ \lim_{n \to \infty} X(s_n, t_n) = X(s_0, t_0) \right] = 1.$$ 

**Proof.** Let $\varepsilon > 0$ be chosen arbitrarily except for the condition $\varepsilon < 1 - \sqrt{1/2} < 1/2$. Chebychev's Inequality and the uniform continuity of $m(s, t)$ and $v(s, t)$ imply that there is a $\delta > 0$ such that for $(s', t')$ and $(s, t) \in [s_0 - \delta, s_0 + \delta] \times [t_0 - \delta, t_0 + \delta]$

$$P\left[ |X(s, t) - X(s', t')| \geq \frac{\varepsilon}{2} \right] < \frac{\varepsilon}{4}.$$ 

Now let $S$ be a countable dense set in $D$ and let $S_1$, $S_2$, $S_3$, and $S_4$
The first part of the proof will show that
\[ P\left[ \sup_{(s, t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] \leq 6\varepsilon. \]

The same kind of argument can be used to show that for \( i = 2, 3, \) and 4
\[ P\left[ \sup_{(s, t) \in S_i} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] \leq 6\varepsilon \]
and so only the case for \( S_1 \) will be done here.

Let the elements of \( S_i \) be numbered in an arbitrary manner so that \( S_i = \{(s_i, t_i): i = 1, 2, \cdots\} \). Then

\[ P\left[ \sup_{(s, t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] = \lim_{n \to \infty} \max_{t_i} \sup_{s_i \in S_1} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon. \]

Thus it suffices to show that
\[ \max_{t_i} \sup_{s_i \in S_1} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon \]
in order to prove (2). Now clearly
\[ P\left[ \max_{t_i} \sup_{s_i \in S_1} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon \right] \leq \max_{t_i} \sup_{s_i \in S_1} |X(s_i, t_i) - X(s_0, t_i) + X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon \]
\[ \leq \max_{t_i} \sup_{s_i \in S_1} |X(s_i, t_i) - X(s_0, t_i) - X(s_i, t_0) + X(s_0, t_0)| > 2\varepsilon \]
\[ + P\left[ \max_{t_i} \sup_{s_i \in S_1} |X(s_i, t_i) - X(s_0, t_i)| > 2\varepsilon \right] \]
\[ + P\left[ \max_{t_i} \sup_{s_i \in S_1} |X(s_i, t_i) - X(s_0, t_i)| > 2\varepsilon \right]. \]

Consider the first \( n \) points \( (s_1, t_1), \cdots, (s_n, t_n) \) in \( S_i \). Let \( \sigma_1, \cdots, \sigma_n \) and \( \tau_1, \cdots, \tau_n \) be rearrangements of \( s_1, \cdots, s_n \) and \( t_1, \cdots, t_n \) respectively so that \( s_0 \leq \sigma_1 \leq s_2 \leq \cdots \leq s_n + \delta \) and \( t_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq t_0 + \delta \). Since \( X(s, t) \) is biadditive,
\begin{align*}
X(\sigma_i, \tau_j) - X(\sigma_i, t_0) - X(s_0, \tau_j) + X(s_0, t_0) \\
= \sum_{m=1}^{i} \sum_{l=1}^{j} \{X(\sigma_m, \tau_l) - X(\sigma_{m-1}, \tau_l) - X(s_0, \tau_{l-1}) + X(s_0, \tau_{l-1})\}
\end{align*}

where

\[X(\sigma_i, t_0) - X(s_0, t_0) = \sum_{m=1}^{i} \{X(\sigma_m, t_0) - X(\sigma_{m-1}, t_0)\}\]

\[X(s_0, \tau_j) - X(s_0, t_0) = \sum_{l=1}^{j} \{X(s_0, \tau_l) - X(s_0, \tau_{l-1})\}\]

are sums of independent random variables. Now if \((s', t')\) and \((s'', t'')\) are any two points in \([s_0 - \delta, s_0 + \delta] \times [t_0 + \delta, t_0 + \delta]\), then using (1) we may verify that the hypotheses of the Ottaviani inequalities, Lemmas 4.2 and 4.3, are satisfied. Thus

\[P[\max_{i=1, \ldots, n} |X(\sigma_i, t_0) - X(s_0, t_0)| > 2\varepsilon] \leq 2P[|X(s_0, t_0) - X(s_0, t_0)| > \varepsilon] \leq 2P[|X(s_0, t_0) - X(s_0, t_0)| > \varepsilon] \]

\[P[\max_{j=1, \ldots, n} |X(s_0, \tau_j) - X(s_0, t_0)| > 2\varepsilon] \leq 2P[|X(s_0, \tau_n) - X(s_0, t_0) + X(s_0, t_0)| > \varepsilon] \]

From the choice of \(\delta\) we see that the right sides of inequalities (7), (8), and (9) are each not greater than \(2\varepsilon\). Since the \(\sigma_i\)'s are \(s_i\)'s and \(\tau_i\)'s are \(t_i\)'s, we have

\[P[\max_{i=1, \ldots, n} |X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon] \leq 2\varepsilon \]

\[P[\max_{j=1, \ldots, n} |X(s_0, \tau_j) - X(s_0, t_0)| > 2\varepsilon] \leq 2\varepsilon \]

and

\[P[\max_{i=1, \ldots, n} |X(s_i, t_i) - X(s_0, t_i) + X(s_0, t_0)| > 2\varepsilon] \leq 2\varepsilon \]

Substituting (10), (11), and (12) into (6) we get (5), i.e.

\[P[\max_{i=1, \ldots, n} |X(s_i, t_i) - X(s_0, t_i)| > 6\varepsilon] \leq 6\varepsilon \]

Then

\[P[\sup_{(s, t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon] \leq 6\varepsilon \]

Since the proof of (2) is similar, it is omitted.
Now let \( V = S_1 \cup S_2 \cup S_3 \cup S_4 \). Then

\[
P \left[ \sup_{(s, t) \in V} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right]
\leq \sum_{i=1}^{4} P \left[ \sup_{(s, t) \in S_i} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right]
\]

and hence

\[
P \left[ \sup_{(s, t) \in V} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] \leq 24\varepsilon .
\]

Taking limits as \( \delta \downarrow 0 \), we obtain

\[
P \left[ \lim_{\delta \downarrow 0} \sup_{(s, t) \in V} |X(s, t) - X(s_0, t_0)| > 6\varepsilon \right] \leq 24\varepsilon .
\]

Now let \( \varepsilon \downarrow 0 \) and take complements to get

\[
P \left[ \lim_{\delta \downarrow 0} \sup_{(s, t) \in V} |X(s, t) - X(s_0, t_0)| = 0 \right] = 1 .
\]

If an arbitrary sequence \((s_n, t_n)\) with \( \lim_{n \to \infty} (s_n, t_n) = (s_0, t_0) \) is given, we extend the point set \( \{s_n, t_n\} \) to a countable dense set \( S \) in \( D \). Then

\[
\left[ \lim_{n \to \infty} X(s_n, t_n) = X(s_0, t_0) \right] \Rightarrow \left[ \lim_{\delta \downarrow 0} \sup_{(s, t) \in V} |X(s, t) - X(s_0, t_0)| = 0 \right]
\]

and by (16)

\[
P \left[ \lim_{n \to \infty} X(s_n, t_n) = X(s_0, t_0) \right] = 1 .
\]

**Lemma 4.5.** Let \( X(s, t) \) be a biadditive process on a probability space \( (\Omega, \mathcal{F}, P) \) with \( (s, t) \in D = [0, 1] \times [0, 1] \). Suppose that \( v(s, t) = \text{var}(X(s, t)) \) is continuous over \( D \). Furthermore, suppose that for any \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{n} P \left[ \left| X\left(\frac{k}{n}, \frac{j}{n}\right) - X\left(\frac{k-1}{n}, \frac{j}{n}\right) - X\left(\frac{k}{n}, \frac{j-1}{n}\right) + X\left(\frac{k-1}{n}, \frac{j-1}{n}\right) \right| > \varepsilon \right] = 0
\]

(1)

\[
\lim_{n \to \infty} \sum_{k=1}^{n} P \left[ \left| X\left(1, \frac{k}{n}\right) - X\left(1, \frac{k-1}{n}\right) \right| > \varepsilon \right] = 0
\]

(2)

and

\[
\lim_{n \to \infty} \sum_{j=1}^{n} P \left[ \left| X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right) \right| > \varepsilon \right] = 0 .
\]

(3)

Then there is a process \( Y(s, t) \) equivalent to \( X(s, t) \) such that almost
every sample function of $Y(s, t)$ is continuous on $D$.

Proof. Let $S$ be the set of all rational numbers in $[0, 1]$ and let $D' = S \times S$. Define $\Omega'$ by $\Omega' = \{\omega \in \Omega: X(s, t)$ is uniformly continuous on $D\}$. In the first part of the proof, we show that $P(\Omega') = 1$.

Let $Z_n$ be defined on $(\Omega, \mathcal{B}, P)$ by

$$Z_n = \sup \left\{ \left| X(s'', t'') - X(s', t') \right| : (s'', t'') \in D', (s', t') \in D' \text{ and } |s'' - s'| < \frac{1}{n}, |t'' - t'| < \frac{1}{n} \right\}.$$

Then $X(s, t)$ is uniformly continuous on $D'$ if and only if $\lim_{n \to \infty} Z_n = 0$. Hence,

$$P(\Omega') = P\left[ \lim_{n \to \infty} Z_n = 0 \right].$$

Let $S_j = S \cap [(j - 1)/n, j/n] \ j = 1, \ldots, n$, and fix $n$. We number the elements of $S_j$ in an arbitrary manner for each $j = 1, \ldots, n$. Let $j$ and $k$ be now fixed and let $s_1, \ldots, s_{m-1}$ and $t_1, \ldots, t_{m-1}$ denote the first $m - 1$ elements of $S_j$ and $S_k$ respectively. Let $\sigma_1, \ldots, \sigma_{m-1}$ and $\tau_1, \ldots, \tau_{m-1}$ be the arrangements of $\{s_1, \ldots, s_{m-1}\}$ and $\{t_1, \ldots, t_{m-1}\}$ respectively in ascending order so that $\sigma_1 < \sigma_2 < \cdots < \sigma_{m-1}$ and $\tau_1 < \tau_2 < \cdots < \tau_{m-1}$. Choose $\sigma_0 = (j - 1)/n$, $\sigma_m = j/n$, $\tau_0 = (k - 1)/n$, and $\tau_m = k/n$, and define $S_{jm} \equiv \{\tau_0, \tau_1, \ldots, \tau_m\}$. We will use the notation:

$$A(s, t, s', t') = X(s', t') - X(s, t') - X(s', t) + X(s, t).$$

Since $X(s, t)$ is biadditive, the three collections of random variables below are systems of independent random variables:

$$\{A(\sigma_{\mu - 1}, \tau_{\gamma - 1}, \sigma_{\gamma}, \tau_{\gamma}) : \mu, \gamma = 1, \ldots, m\}$$

$$\{A\left(\frac{j - 1}{n}, \tau_{\gamma - 1}, \frac{j}{n}, \tau_{\gamma}\right) : \gamma = 1, \ldots, m \text{ and } j = 1, \ldots, n\}$$

$$\{A\left(\sigma_{\mu - 1}, \frac{k - 1}{n}, \sigma_{\mu}, \frac{k}{n}\right) : \mu = 1, \ldots, m \text{ and } k = 1, \ldots, n\}.$$

Let $\varepsilon > 0$ be chosen arbitrarily. Since $v(s, t)$ and $m(s, t)$ are continuous on $D$, they are uniformly continuous and if $n$ is sufficiently large and if $0 < s'' - s' < 1/n$ or $0 < t'' - t' < 1/n$, then from Chebychef's inequality it follows that

$$P\left[ |A(s', t', s'', t'')| > \frac{\varepsilon}{2} \right] \leq 1 - \sqrt{\frac{1}{2}}.$$

Let $Y_{n,j,k} = \sup_{S_j \times S_k} |X(s, t) - X((j - 1)/n, (k - 1)/n)|$. Then from the
triangle inequality we get

\[ Y_n = \max_{j, k = 1, \ldots, n} Y_{n, j, k} \leq \max_{j, k = 1, \ldots, n} \sup_{s, t \in S_k} \left| \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, s, t \right) \right| \]

\[ + \max_{j, k = 1, \ldots, n} \sup_{s, t \in S_k} \left| X \left( s, \frac{k - 1}{n} \right) - X \left( \frac{j - 1}{n}, \frac{k - 1}{n} \right) \right| \]

\[ + \max_{j, k = 1, \ldots, n} \sup_{s, t \in S_k} \left| X \left( \frac{j - 1}{n}, t \right) - X \left( \frac{j - 1}{n}, \frac{k - 1}{n} \right) \right|. \]

Consequently,

\[ P \left[ Y_n > 6 \varepsilon \right] \leq P \left[ \max_{j, k = 1, \ldots, n} \sup_{s, t \in S_k} \left| \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, s, t \right) \right| > 2 \varepsilon \right] \]

\[ + P \left[ \max_{j, k = 1, \ldots, n} \sup_{s, t \in S_k} \left| X \left( s, \frac{k - 1}{n} \right) - X \left( \frac{j - 1}{n}, \frac{k - 1}{n} \right) \right| > 2 \varepsilon \right] \]

\[ + P \left[ \max_{j, k = 1, \ldots, n} \sup_{s, t \in S_k} \left| X \left( \frac{j - 1}{n}, t \right) - X \left( \frac{j - 1}{n}, \frac{k - 1}{n} \right) \right| > 2 \varepsilon \right]. \]

For \((\sigma_p, \tau_r) \in S_{j_m} \times S_{k_m}\), we see that

\[ \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, \sigma_p, \tau_r \right) = \sum_{q=1}^{r} \sum_{p=1}^{n} \Delta (\sigma_{p-1}, \tau_{q-1}, \sigma_p, \tau_q) \]

a sum of independent random variables. Now (5) implies that the hypotheses of the extended Ottaviani's Inequality (Lemma 4.3) are satisfied and consequently

\[ P \left[ \max_{p, r = 1, \ldots, m} \left| \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, \sigma_p, \tau_r \right) \right| > 2 \varepsilon \right] \leq 2 \varepsilon \left[ \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, \frac{j}{n}, \frac{k}{n} \right) > \varepsilon \right]. \]

Letting \(m \to \infty\), it follows that

\[ P \left[ \sup_{s, t \in S_k} \left| \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, s, t \right) \right| > 2 \varepsilon \right] \leq 2 \varepsilon \left[ \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, \frac{j}{n}, \frac{k}{n} \right) > \varepsilon \right] \]

and hence

\[ P \left[ \max_{j, k = 1, \ldots, n} \sup_{s, t \in S_k} \left| \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, s, t \right) \right| > 2 \varepsilon \right]

\[ \leq 2 \sum_{j=1}^{n} \sum_{k=1}^{n} P \left[ \left| \Delta \left( \frac{j - 1}{n}, \frac{k - 1}{n}, \frac{j}{n}, \frac{k}{n} \right) \right| > \varepsilon \right]. \]

Now if \(\sigma_p \in S_{j_m}\), since \(X(\sigma_p, 0) = X(0, (k - 1)/n) = 0\), we have

\[ X \left( \frac{j - 1}{n}, \frac{k - 1}{n} \right) - X \left( \frac{j - 1}{n}, \frac{k - 1}{n} \right) = \sum_{p=1}^{n} \sum_{q=1}^{n} \Delta (\sigma_{p-1}, \frac{q - 1}{n}, \sigma_p, \frac{q}{n}), \]
as before, a sum of independent random variables. Again, (5) allows us to use the extended Ottaviani's Inequality to obtain

\[ P \left[ \max_{k=1, \ldots, m} \max_{t \in S_{k,n}} \left| X\left( s, \frac{k-1}{n} \right) - X\left( \frac{j-1}{n}, \frac{k-1}{n} \right) \right| > 2\varepsilon \right] \]

\[ \leq 2P \left[ \left| X\left( \frac{j}{n}, 1 \right) - X\left( \frac{j-1}{n}, 1 \right) \right| > \varepsilon \right]. \]

Letting \( m \to \infty \), we get

\[ P \left[ \max_{k=1, \ldots, m} \sup_{t \in S_{k,n}} \left| X\left( s, \frac{k-1}{n} \right) - X\left( \frac{j-1}{n}, \frac{k-1}{n} \right) \right| > 2\varepsilon \right] \]

\[ \leq 2P \left[ \left| X\left( \frac{j}{n}, 1 \right) - X\left( \frac{j-1}{n}, 1 \right) \right| > \varepsilon \right] \]

and

\[ P \left[ \max_{j, k=1, \ldots, m} \sup_{t \in S_{k,n}} \left| X\left( s, \frac{k-1}{n} \right) - X\left( \frac{j-1}{n}, \frac{k-1}{n} \right) \right| > 2\varepsilon \right] \]

\[ = P \left[ \bigcup_{j=1}^{\infty} \left( \max_{k=1, \ldots, m} \sup_{t \in S_{k,n}} \left| X\left( s, \frac{k-1}{n} \right) - X\left( \frac{j-1}{n}, \frac{k-1}{n} \right) \right| > 2\varepsilon \right] \]

\[ \leq 2 \sum_{j=1}^{\infty} P \left[ \left| X\left( \frac{j}{n}, 1 \right) - X\left( \frac{j-1}{n}, 1 \right) \right| > \varepsilon \right]. \]

Similarly for \( \tau \in S_{k,n} \)

\[ X\left( \frac{j-1}{n}, \tau \right) - X\left( \frac{j-2}{n}, \frac{k-1}{n} \right) = \sum_{p=1}^{j-1} \sum_{q=1}^{\tau} a \left( \frac{p-1}{n}, \tau_{q-1}, \frac{p}{n}, \tau_{q} \right), \]

a sum of independent random variables, and so by (5) we may again apply the extended Ottaviani's Inequality and take limits as \( m \to \infty \). We get

\[ P \left[ \max_{j, k=1, \ldots, m} \sup_{t \in S_{k,n}} \left| X\left( \frac{j-1}{n}, t \right) - X\left( \frac{j-1}{n}, \frac{k-1}{n} \right) \right| > 2\varepsilon \right] \]

\[ \leq 2 \sum_{k=1}^{\infty} P \left[ \left| X\left( 1, \frac{k}{n} \right) - X\left( 1, \frac{k-1}{n} \right) \right| > \varepsilon \right]. \]

Inserting (9), (10), and (11) into (7) and letting \( n \to \infty \), we see from the hypotheses (1), (2), and (3) that

\[ \lim_{n \to \infty} P[Y_n > 6\varepsilon] = 0. \]

The inequality \( Z_n \leq 4Y_n \) can be checked by successive applications of the triangle inequality. (If \( |s' - s''| < 1/n \) and \( |t' - t''| < 1/n \), \( s', t' \in [(j-1)/n, j/n] \times [(k-1)/n, k/n] \) implies that \( s'', t'' \in [(j-2)/n, (j+1)/n] \times [(k-2)/n, (k+1)/n] \) and it suffices to check each possibility.) It follows that
Since $0 \leq Z_n$ and $Z_{n+1} \leq Z_n$ for all $n$,

$$\lim_{n \to \infty} P[Z_n > 24\varepsilon] = P[\lim_{n \to \infty} Z_n > 24\varepsilon] = 0$$

by (12). Letting $\varepsilon \downarrow 0$, we obtain

$$P\left[\lim_{n \to \infty} Z_n > 0\right] = 0,$$

and since $Z_n \geq 0$, we get

$$P(\Omega') = P\left[\lim_{n \to 0} Z_n = 0\right] = 1,$$

which finishes the first part of the proof.

Now if $x(s, t)$ is any real-valued function uniformly continuous on a set $D$, it has a unique continuous extension to the closure of $D$. Let $Y(s, t, \omega)$ be defined for $\omega \in \Omega'$ by $Y(s, t, \omega) = X(s, t, \omega)$ if $(s, t) \in D'$.

If $(s, t) \notin D'$, choose a sequence of points $(s_n, t_n)$ in $D'$ such that $\lim_{n \to \infty} (s_n, t_n) = (s, t)$ and define $Y(s, t, \omega) = \lim_{n \to \infty} Y(s_n, t_n, \omega)$ for $\omega \in \Omega'$. Since for $\omega \in \Omega'$ $Y(s, t, \omega)$ is uniformly continuous on $D'$ which is dense in $D$, $Y(s, t, \omega)$ is well-defined for $\omega \in \Omega'$. If $\omega \notin \Omega'$, let $Y(s, t, \omega) = 0$. Then for $(s, t) \in D'$,

$$P[Y(s, t) = X(s, t)] \geq P(\Omega') = 1$$

and if $(s, t) \in D$ but $(s, t) \notin D'$,

$$P\left[Y(s, t) = \lim_{n \to \infty} X(s_n, t_n)\right] \geq P(\Omega') = 1$$

for some sequence $\{(s_n, t_n)\}$ in $D'$ such that $\lim_{n \to \infty} (s_n, t_n) = (s, t)$.

But by Lemma 2.6,

$$P\left[X(s, t) = \lim_{n \to \infty} X(s_n, t_n)\right] = 1$$

and hence for any $(s, t) \in D$,

$$P[Y(s, t) = X(s, t)] = 1.$$

That is, $Y(s, t)$ is a process which is equivalent to $X(s, t)$. It follows from the definition of $Y(s, t)$, that its sample functions are continuous on $\Omega'$, a set of probability one.

5. Proof of Theorem 2.

Proof. Let $\Phi(a, b, c, d)$ denote the normal probability distribution
with mean zero and variance $v(c, d) - v(a, d) - v(c, b) + v(a, b)$ where $0 \leq a < c \leq 1$ and $0 \leq b < d \leq 1$. Then since the convolution of normal distributions is a normal distribution whose mean and variance are the respective sums of the means and variances of the original distributions, for any $\alpha > 0$ we have

$$
\Phi(a, b, c + \alpha, d) = \Phi(a, b, c, d) \ast \Phi(c, b, c + \alpha, d)
$$

$$
\Phi(a, b, c, d + \alpha) = \Phi(a, b, c, d) \ast \Phi(a, d, c, d + \alpha)
$$

where "\ast" denotes the operation of convolution.

By Lemma 4.1, there is a biadditive process $Y(s, t)$ such that for $s' < s''$ and $t' < t''$, $Y(s'', t'') - Y(s', t'') - Y(s', t') + Y(s', t'')$ is normally distributed with mean zero and variance $v(s'', t'') - v(s', t'') - v(s', t') + v(s', t'')$. If $Y(s, t)$ satisfies conditions (1), (2), and (3) of Lemma 4.5 there is a process $Y_0(s, t)$ equivalent to $Y(s, t)$ such that almost every sample function of $Y_0(s, t)$ is continuous over $D$. Define $X(s, t) = Y_0(s, t) + m(s, t)$. Then $X(s, t)$ satisfies (i) and (ii) and is biadditive since $Y_0(s, t)$ is. Furthermore almost every sample function of $X(s, t)$ is continuous over $D$.

Let $\Delta_{jk}$ denote the random variable

$$
\Delta_{jk} = Y\left(\frac{j}{n}, \frac{k}{n}\right) - Y\left(\frac{j-1}{n}, \frac{k}{n}\right) - Y\left(\frac{j}{n}, \frac{k-1}{n}\right) + Y\left(\frac{j-1}{n}, \frac{k-1}{n}\right)
$$

where $n$ is a positive integer. Conditions (1), (2), and (3) of Lemma 4.5 are

(1) \[ \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{n} P[|\Delta_{jk}| > \varepsilon] = 0 \]

(2) \[ \lim_{n \to \infty} \sum_{k=1}^{n} P\left[\left|Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] = 0 \]

(3) \[ \lim_{n \to \infty} \sum_{j=1}^{n} P\left[\left|Y\left(\frac{j}{n}, 1\right) - Y\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] = 0 \]

where $\varepsilon > 0$ is chosen in an arbitrary manner. We will use the following inequality which is valid for $\lambda > 0$.

$$
\int_{-\infty}^{\infty} e^{-t^2/2} dt \leq \frac{1}{\lambda} \int_{-\infty}^{\infty} t e^{-t^2/2} dt = \frac{1}{\lambda} e^{-\lambda^2/2} .
$$

For $\varepsilon > 0$ since $\Delta_{jk}$ is normally distributed,

$$
P[|\Delta_{jk}| > \varepsilon] = \frac{2}{\sqrt{2\pi \nu_{jk}}} \int_{\varepsilon}^{\infty} e^{-t^2/(2\nu_{jk})} dt = \frac{2}{\sqrt{2\pi}} \int_{\varepsilon}^{\infty} e^{-t^2} dt
$$
or

\[ P[|A_{jk}| > \varepsilon] \leq \frac{2}{\lambda \sqrt{2\pi}} \exp \left\{ -\frac{\lambda^2}{2} \right\} = \frac{2}{\varepsilon} \sqrt{\frac{v_{jk}}{2\pi}} \exp \left\{ -\frac{\varepsilon^2}{2v_{jk}} \right\} \]

where

\[ v_{jk} = v\left(\frac{j}{n}, \frac{k}{n}\right) - v\left(\frac{j-1}{n}, \frac{k}{n}\right) - v\left(\frac{j}{n}, \frac{k-1}{n}\right) + v\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \]

and \( \lambda = \varepsilon (v_{jk})^{-1/2} \). Since \( v(s, t) \) is uniformly continuous over \( D \), we can choose \( N \) independently of \( j \) and \( k \) such that \( n \geq N \) implies \( v_{jk}/\varepsilon^2 < 1/M_s^2 \) where \( M_s \) is determined as follows. Since \( (1/x) \exp \{-x^2/2\} = o(x^{-\alpha}) \) as \( x \to \infty \), we have for every positive integer \( \delta \), a number \( M_s \) such that \( x > M_s \) implies \( x \exp \{-x^2/2\} < 1/\delta \), that is, for \( x > M_s \),

\[ \frac{1}{x} \exp \left\{ -\frac{x^2}{2} \right\} < \frac{1}{\delta x^\alpha} . \]

Now \( v_{jk}/\varepsilon^2 < 1/M_s^2 \) entails \( \varepsilon/\sqrt{v_{jk}} > M_s \) and with \( x = \varepsilon/\sqrt{v_{jk}} \) we get

\[ \frac{\sqrt{v_{jk}}}{\varepsilon} \exp \left\{ -\frac{\varepsilon^2}{2v_{jk}} \right\} \leq \frac{1}{\delta} \frac{v_{jk}}{\varepsilon^2} . \]

Then for \( n \geq N \)

\[ P[|A_{jk}| > \varepsilon] \leq \frac{2}{\sqrt{2\pi}} \frac{v_{jk}}{\varepsilon^2} . \]

But \( v(1, 1) - v(1, 0) - v(0, 1) + v(0, 0) = v(1, 1) = \sum_{j=1}^{\varepsilon} \sum_{k=1}^{n} v_{jk} \), and so

\[ \sum_{j=1}^{\varepsilon} \sum_{k=1}^{n} P[|A_{jk}| < \varepsilon] \leq \frac{2}{\sqrt{2\pi}} \frac{1}{\delta \varepsilon^2} v(1, 1) . \]

Since we may take \( \delta \) arbitrarily large, choosing \( N \) sufficiently large for each \( \delta \),

\[ \lim_{n \to \infty} \sum_{j=1}^{\varepsilon} \sum_{k=1}^{n} P[|A_{jk}| > \varepsilon] = 0 \]

and (1) holds for \( Y(s, t) \). A similar argument proves (2) and (3).

Since \( Y(s, 0) = Y(0, t) = 0 \) for all \( s, t \) in \( D \), \( Y(1, k/n) - Y(1, (k - 1)/n) \) is normally distributed with mean zero and variance \( v(1, k/n) - v(1, (k - 1)/n) \), and \( Y(j/n, 1) - Y((j - 1)/n, 1) \) is normally distributed with mean 0 and variance \( v(j/n, 1) - v((j - 1)/n, 1) \). Thus

\[ P\left[ \left| Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right) \right| > \varepsilon \right] = \frac{2}{\sqrt{2\pi v_k}} \int_{-\infty}^{\varepsilon} \exp \{-t^2/2v_k\} dt \]

\[ \leq \frac{2\sqrt{v_k}}{\sqrt{2\pi \varepsilon}} \exp \{-\varepsilon^2/2v_k\} \]
and

\[
P\left[ \left| Y\left( \frac{j}{n}, 1 - Y\left( \frac{j-1}{n}, 1 \right) \right) > \varepsilon \right| \right] = \frac{2}{\sqrt{2\pi \nu_j}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{t^2}{2\nu_j} \right\} dt \leq \frac{2\sqrt{\nu_j}}{\sqrt{2\pi\varepsilon}} \exp\left\{ -\frac{\varepsilon^2}{2\nu_j} \right\}
\]

where \( \nu_j = \nu(j/n, 1 - \nu((j-1)/n, 1) \) and \( \nu_k = \nu(1, k/n) - \nu(1, (k-1)/n) \). Again we may choose \( \delta, M_\delta, N', \) and \( N'' \) so that when \( n > N' \) or \( n \geq N'' \), the respective inequalities

\[
\frac{\nu_j}{\varepsilon^2} < \frac{1}{M_\delta^2} \quad \text{or} \quad \frac{\nu_k}{\varepsilon^2} < \frac{1}{M_\delta^2}
\]

hold. Since \( \nu(1, 1) = \sum_{j=1}^{n} \nu_j = \sum_{k=1}^{n} \nu_k \),

\[
\sum_{k=1}^{n} P\left[ \left| X\left( 1, \frac{k}{n} \right) - X\left( 1, \frac{k-1}{n} \right) \right| > \varepsilon \right] \leq \frac{2}{\sqrt{2\pi\varepsilon^2}} \nu(1, 1)
\]

and

\[
\sum_{j=1}^{n} P\left[ \left| X\left( \frac{j}{n}, 1 \right) - X\left( \frac{j-1}{n}, 1 \right) \right| > \varepsilon \right] \leq \frac{2}{\sqrt{2\pi\varepsilon^2}} \nu(1, 1)
\]

when \( n > N'' \) or \( n > N' \) respectively. Thus there is a process \( Y_0(s, t) \) equivalent to \( Y(s, t) \) such that almost every sample function of \( Y_0 \) is continuous over \( D \). Setting \( X(s, t) = Y_0(s, t) + m(s, t) \) we obtain a biadditive process satisfying (i), (ii), and (iii).

REFERENCES

3. ————, Unpublished Lecture Notes.

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