Pacific Journal of Mathematics

CONTINUITY OF SAMPLE FUNCTIONS OF BIADDITIVE PROCESSES

W. N. HUDSON

Vol. 42, No. 2 February 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 2, 1972

CONTINUITY OF SAMPLE FUNCTIONS OF BIADDITIVE PROCESSES

W. N. HUDSON

Let $\{X(s,t): 0 \leqq s, t \leqq 1\}$ be a stochastic process which has independent increments (second differences). Necessary and sufficient conditions are established to ensure the existence of a version with the property that almost every sample function is continuous. A corollary to these results is the existence of a class of measures on Wiener-Yeh space. The conditions are analogous to the usual case of additive processes $Z(t)$ indexed by one time parameter.

$X(s, t)$ will be said to have independent "increments" (second differences) if whenever $0 \leqq s_0 < s_1 < \cdots < s_m \leqq 1$ and $0 \leqq t_0 < t_1 < \cdots < t_n \leqq 1$ the random variables $X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})$ $i = 1, \cdots, m$, $j = 1, \cdots, n$ are independent. If $X(s, t)$ has independent increments and $X(0, t) = X(s, 0) = 0$, then $X(s, t)$ will be called biadditive. Let $m(s, t) = E[X(s, t)]$ and $v(s, t) = \text{var}\,[X(s, t)]$. The following result is proved below:

There is a version of a biadditive process $X(s, t)$ with the property that almost every sample function is continuous if and only if $X(s, t)$ is Gaussian, $m(s, t)$ and $v(s, t)$ are continuous, and $v(s, t)$ is the distribution function of a Lebesgue-Stieltjes measure on $[0, 1] \times [0, 1]$.

A special case of this result occurs when $m(s, t) = 0$ and $v(s, t) = st$. This process is realized when the space C_2 of continuous functions of two variables on $[0, 1] \times [0, 1]$ is assigned the Wiener-Yeh measure and $X(s, t)$ is defined by $X(s, t)(f) = f(s, t)$ where $f \in C_2$. Theorem 2 will imply the existence of a class of Wiener-Yeh measures on C_2 corresponding to the choices of a pair of continuous functions $m(s, t)$ and $v(s, t)$.

The conditions on $m(s, t)$ and $v(s, t)$ are analogous to the well-known conditions for the usual case of a stochastic process indexed by one time parameter. The case for a process indexed by n-time parameters is similar. The proof here is probabilistic in nature, unlike the analytic proof given by Yeh in [2] for the special case above.

2. Statement of main results.

THEOREM 1. *Let $X(s, t)$ be a biadditive process having the property that almost every sample function is continuous. Then $X(s, t)$ is Gaussian and the increments of $X(s, t)$ are Gaussian. Furthermore the functions $m(s, t) = EX(s, t)$ and $v(s, t) = \text{var}\,(X(s, t))$ are continuous* *and determine the distribution of the process.*

The following corollary is easy and its proof will be omitted.

COROLLARY. *Let $X(s, t)$ be as in Theorem 1. If the increments of $X(s, t)$ are stationary, that is, if the distribution of $X(s+h_1, t+h_2) - X(s, t+h_2) - X(s+h_1, t) + X(s, t)$ depends only on h_1 and h_2, then there are constants c_1 and c_2 such that*

$$m(s, t) = EX(s, t) = c_1 st$$

$$v(x, t) = \operatorname{var}(X(s, t)) = c_2 st .$$

THEOREM 2. *Let $m(s, t)$ and $v(s, t)$ be continuous functions on $[0, 1] \times [0, 1]$ such that $m(s, 0) = 0 = m(0, t)$ and $v(s, 0) = 0 = v(0, t)$ for $0 \leqq s, t \leqq 1$. Suppose that $v(s, t)$ satisfies the condition*

$$\text{(A)} \qquad v(s'', t'') - v(s'', t') - v(s', t'') + v(s', t') \geqq 0$$

whenever

$$0 \leqq s' \leqq s'' \leqq 1 \quad \textit{and} \quad 0 \leqq t' < t'' \leqq 1 .$$

Then there is a biadditive Gaussian process $X(s, t)$, $0 \leqq s, t \leqq 1$, such that

(i) *$EX(s, t) = m(s, t)$ and $\operatorname{var}(X(s, t)) = v(s, t)$ and*

(ii) *almost every sample function of $X(s, t)$ is continuous on $[0, 1] \times [0, 1]$.*

The distribution of $X(s, t)$ is determined by $m(s, t)$ and $v(s, t)$.

3. *Proof of Theorem* 1. We prove first that $X(s, t)$ is Gaussian.

LEMMA 3.1. *If almost every sample function of $X(s, t)$ is continuous on $[0, 1] \times [0, 1]$, then $X(s, t)$ and its increments are normally distributed.*

Proof. We show that the version of the central limit theorem in reference [1] (Theorem 2, p. 197) applies. Let (s, t) be a fixed point in $[0, 1] \times [0, 1]$ and define $s_i = s(i/n)$, $t_i = t(i/n)$, and

$$\Delta_{ij}(n) = X(s_i, t_j) - X(s_i, t_{j-1}) - X(s_{i-1}, t_j) + X(s_{i-1}, t_{j-1}) .$$

Let $\varepsilon > 0$ be given and let $A_n = [\max_{i,j=1,2,\cdots,n} |\Delta_{ij}(n)| \geqq \varepsilon]$. Then almost every sample function of $X(s, t)$ is uniformly continuous on $[0, 1] \times [0, 1]$, and consequently

$$P\{\limsup_{n \to \infty} A_n\} = 0 .$$

Hence $\limsup_{n\to\infty} P(A_n) = 0$.

Now $X(s, t)$ is the sum of independent random variables, that is,

$$X(s, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{ij}(n) .$$

The $\Delta_{ij}(n)$ form an infinitesimal system because

$$\max_{i,j=1,2,\cdots,n} P[|\Delta_{ij}(n)| \geqq \varepsilon] \leqq P[\max_{i,j=1,2,\cdots,n} |\Delta_{ij}(n)| \geqq \varepsilon]$$

and since

$$\limsup_{n\to\infty} P(A_n) = 0 ,$$

$$\lim_{n\to\infty} \max_{i,j=1,2,\cdots,n} P[|\Delta_{ij}(n)| \geqq \varepsilon] = 0 .$$

It follows that $X(s, t)$ is normally distributed.

To show that the increments of $X(s, t)$ are normally distributed, let s_0 and t_0 be fixed and for $s \geqq s_0$, $t \geqq t_0$ consider the process

$$Y(s, t) = X(s, t) - X(s_0, t) - X(s, t_0) + X(s_0, t_0) .$$

It is biadditive and has continuous sample functions a.s. The above argument shows that $Y(s, t)$ is Gaussian and hence the increments of $X(s, t)$ are Gaussian.

To complete the proof of Theorem 1 we need to check that $m(s, t)$ and $v(s, t)$ are continuous and determine the distribution of the process. Since $X(s, t)$ is biadditive, we have for $s' < s''$ and $t' < t''$

$$\begin{aligned} \operatorname{var}(X(s'', t'')) = & \operatorname{var}(X(s'', t'') - X(s', t'') - X(s'', s') + X(s', t')) \\ & + \operatorname{var}(X(s', t'') - X(s', t')) + \operatorname{var}(X(s'', t') \\ & - X(s', t')) + \operatorname{var}(X(s', t')) \end{aligned}$$

$$\operatorname{var}(X(s', t'') - X(s', t')) + \operatorname{var}(X(s', t')) = \operatorname{var}(X(s', t''))$$

$$\operatorname{var}(X(s'', t') - X(s', t')) + \operatorname{var}(X(s', t')) = \operatorname{var}(X(s'', t')) .$$

From these equations using $v(s, t) = \operatorname{var}(X(s, t))$ we obtain

$$\begin{aligned} & \operatorname{var}(X(s'', t'') - X(s'', t') - X(s', t'') + X(s', t')) \\ & \quad = v(s'', t'') - v(s', t'') - v(s'', t') + v(s', t') . \end{aligned}$$

Since a similar relation holds for $m(s, t) = EX(s, t)$, the fact that the increments are Gaussian and $X(s, t)$ is biadditive implies that the distribution of $X(s, t)$ is determined by $m(s, t)$ and $v(s, t)$.

Since almost every sample function is continuous,

$$\lim_{h_1, h_2 \to 0} X(s + h_1, t + h_2) = X(s, t) .$$

Let $\varphi(h_1, h_2, u)$ denote the characteristic function of $X(s + h_1, t + h_2)$. Then

$$\varphi(h_1, h_2, u) = \exp\left\{ium(s + h_1, t + h_2) - \frac{u^2}{2}v(s + h_1, t + h_2)\right\}$$

and hence

$$\begin{aligned} v(s, t) &= -2\log|\varphi(0, 0, 1)| \\ &= -2\lim_{h_1, h_2 \to 0} \log|\varphi(h_1, h_2, 1)| \\ &= \lim_{h_1, h_2 \to 0} v(s + h_1, t + h_2) \end{aligned}$$

so $v(s, t)$ is continuous. To show $m(s, t)$ is continuous, we use Chebychef's inequality.

$$\lim_{h_1, h_2 \to 0} P[|X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t)| \geqq \varepsilon]$$

$$\leqq \lim_{h_1, h_2 \to 0} \frac{v(s + h_1, t + h_2) - v(s, t)}{\varepsilon^2} = 0$$

so that

$$X(s + h_1, t + h_2) - X(s, t) - m(s + h_1, t + h_2) + m(s, t) \xrightarrow{P} 0 .$$

Since $X(s + h_1, t + h_2) \to X(s, t)$, it follows that $m(s, t)$ is continuous.

4. *Lemmas for Theorem* 2. In §3, we have shown that any biadditive stochastic process with almost all its sample functions continuous is Gaussian with continuous mean and variance functions. The next task is to show that given a pair of continuous functions $m(s, t)$ and $v(s, t)$ where $v(s, t)$ is a normalized distribution function for a Lebesgue-Stieltjes measure on $[0, 1] \times [0, 1]$, there is a biadditive process $X(s, t)$ such that $EX(s, t) = m(s, t)$ and $\operatorname{var}(X(s, t)) = v(s, t)$. For this proof a few preparatory results are needed. In the following Lemma, * denotes convolution.;

LEMMA 4.1. *Suppose there is a system of probability distributions* $\{\Phi(a_1, b_1, a_2, b_2) \mid 0 \leqq a_1 < a_2 \leqq 1, 0 \leqq b_1 < b_2 \leqq 1\}$ *such that for any* $\alpha > 0$ *and* $\beta > 0$

(1) $$\Phi(a_1, b_1, a_2 + \alpha, b_2) = \Phi(a_1, b_1, a_2, b_2) * \Phi(a_2, b_1, a_2 + \alpha, b_2)$$

(2) $$\Phi(a_1, b_1, a_2, b_2 + \beta) = \Phi(a_1, b_1, a_2, b_2) * \Phi(a_1, b_2, a_2, b_2 + \beta) .$$

Then there is a biadditive process $X(s, t)$ *such that the increment*

$$X(a_2, b_2) - X(a_1, b_2) - X(a_2, b_1) + X(a_1, b_1)$$

has the probability distribution $\Phi(a_1, b_1, a_2, b_2)$ *for* $0 \leqq a_1 < a_2 \leqq 1$ *and* $0 \leqq b < b_2 \leqq 1$.

Proof. The proof uses the Daniell-Kolmogorov extension theorem in the usual manner and is therefore omitted. Conditions (1) and (2) guarantee the consistency of the system.

LEMMA 4.2. (Ottaviani's Inequality). *Let* $\{X_1, X_2, \cdots, X_n\}$ *be independent random variables and let* $S_k \equiv \sum_{i=1}^{k} X_i$. *If for some* $\varepsilon > 0$,

$$P[|S_n - S_k| > \varepsilon] \leqq \frac{1}{2} \quad for \quad k = 0, 1, 2, \cdots n \,,$$

where $S_0 \equiv 0$, *then*

$$P[\max_{k=1,2\cdots n} |S_k| > 2\varepsilon] \leqq 2P[|S_n| > \varepsilon] \,.$$

Proof. The proof may be found in reference [3]. It is very similar to the following lemma which will be proved in full.

LEMMA 4.3. (An extended version of Ottaviani's Inequality). *Let* $s_0 < s_1 < \cdots s_m$ *and* $t_0 < t_1 < t_2 < \cdots < t_n$. *Define*

$$\varDelta_{ij} \equiv X(s_i, t_j) - X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})$$

where $X(s, t)$ *is a biadditive process on* $D = [0,1] \times [0, 1]$. *Let* $R_l \equiv \sum_{i=1}^{m} \sum_{j=l+1}^{n} \varDelta_{ij}$ *and* $Q_{kl} = \sum_{i=k+1}^{m} \varDelta_{il}$. *If for all* $k = 1, 2, \cdots, m$ *and* $l = 0, 1, \cdots, n$

$$P\left[|R_l| > \frac{\varepsilon}{2}\right] \leqq 1 - \sqrt{\frac{1}{2}}$$

and

$$P\left[|Q_{kl}| > \frac{\varepsilon}{2}\right] \leqq 1 - \sqrt{\frac{1}{2}} \,,$$

then

$$P\left[\max_{\substack{k=1,2,\cdots,m \\ l=1,2,\cdots,n}} |S_{kl}| > 2\varepsilon\right] \leqq 2P[|S_{mn}| > \varepsilon] \,.$$

Proof. Let A_{ij} be defined for $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$ by

$$A_{ij} \equiv [|S_{kl}| \leqq 2\varepsilon \quad \text{for} \quad l < j \quad \text{and} \quad k \leqq m, \ |S_{kj}| \leqq 2\varepsilon \quad \text{for} \quad k < i, \ |S_{ij}| > 2\varepsilon]$$
$$A_{11} \equiv [|S_{11}| > 2\varepsilon] \,.$$

Let $T \equiv \{(i, j): 1 = 1, 2, \cdots, m \text{ and } j = 1, 2, \cdots, n\}$. It is clear that

$$\left[\max_{(i,j)\in T}|S_{ij}|>2\varepsilon\right]=\bigcup_{i=1}^{m}\bigcup_{j=1}^{n}A_{ij}$$

and the A_{ij}'s are disjoint. Now let

$$B_{kl}\equiv\left[|R_l|<\frac{\varepsilon}{2},\ |Q_{kl}|<\frac{\varepsilon}{2}\right].$$

Then,

$$A_{kl}\cap B_{kl}\subset[|S_{mn}|>\varepsilon]$$

and so,

$$\bigcup_{l=1}^{n}\bigcup_{k=1}^{m}(A_{kl}\cap B_{kl})\subset[|S_{mn}|>\varepsilon].$$

Since $X(s, t)$ is biadditive, A_{kl} and B_{kl} are independent events, and R_l and Q_{kl} are independent random variables. It follows that

$$P(B_{kl})=P\left[|R_l|<\frac{\varepsilon}{2}\right]\cdot P\left[|Q_{kl}|<\frac{\varepsilon}{2}\right]\geqq\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}=\frac{1}{2}.$$

Hence,

$$\begin{aligned}\frac{1}{2}P\left[\max|S_{ij}|>2\varepsilon\right]&=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}P(A_{ij})\leqq\sum_{i=1}^{m}\sum_{j=1}^{n}P(A_{ij}\cap B_{ij})\\&=P\left(\bigcup_{i=1}^{m}\bigcup_{j=1}^{n}A_{ij}\cap B_{ij})\right)\\&\leqq P[|S_{mn}|>\varepsilon].\end{aligned}$$

LEMMA 4.4. *Let $X(s, t)$ be a biadditive process on a probability space $(\Omega, \mathfrak{B}, P)$ with $(s, t)\in D=[0, 1]\times[0, 1]$. Let $m(s, t)\equiv EX(s, t)$ and $v(s, t)\equiv \operatorname{var}(X(s, t))$ be continuous on D. Then for any point $(s_0, t_0)\in D$ and for any sequence of points $\{(s_n, t_n)\}\subset D$ such that*

$$\lim_{n\to\infty}(s_n, t_n)=(s_0, t_0)$$

$$P\left[\lim_{n\to\infty}X(s_n, t_n)=X(s_0, t_0)\right]=1.$$

Proof. Let $\varepsilon>0$ be chosen arbitrarily except for the condition $\varepsilon<1-\sqrt{1/2}<1/2$. Chebychef's Inequality and the uniform continuity of $m(s, t)$ and $v(s, t)$ imply that there is a $\delta>0$ such that for (s', t') and $(s, t)\in[s_0-\delta, s_0+\delta]\times[t_0-\delta, t_0+\delta]$

$$P\left[|X(s, t)-X(s', t')|\geqq\frac{\varepsilon}{2}\right]<\frac{\varepsilon}{4}. \tag{1}$$

Now let S be a countable dense set in D and let S_1, S_2, S_3, and S_4

denote the sets

$$S_1 \equiv S \cap ([s_0, s_0 + \delta] \times [t_0, t_0 + \delta])$$
$$S_2 \equiv S \cap ([s_0, s_0 + \delta] \times [t_0 - \delta, t_0])$$
$$S_3 \equiv S \cap ([s_0 - \delta, s_0] \times [t_0, t_0 + \delta])$$
$$S_4 \equiv S \cap ([s_0 - \delta, s_0] \times [t_0 - \delta, t_0]) .$$

The first part of the proof will show that

$$(2) \qquad P\Big[\sup_{(s,t)\in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon .$$

The same kind of argument can be used to show that for $i = 2, 3,$ and 4

$$(3) \qquad P\Big[\sup_{(s,t)\in S_i} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon$$

and so only the case for S_1 will be done here.

Let the elements of S_1 be numbered in an arbitrary manner so that $S_1 = \{(s_i, t_i): i = 1, 2, \cdots\}$. Then

$$(4) \qquad \begin{aligned} &P\Big[\sup_{(s,t)\in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \\ &\qquad = \lim_{n\to\infty} P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] . \end{aligned}$$

Thus it suffices to show that

$$(5) \qquad P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon$$

in order to prove (2). Now clearly

$$(6) \qquad \begin{aligned} &P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] \\ &\quad \leqq P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_i) - X(s_i, t_0) + X(s_0, t_0)| > 2\varepsilon\Big] \\ &\quad + P\Big[\max_{i=1,\cdots,n} |X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon\Big] \\ &\quad + P\Big[\max_{i=1,\cdots,n} |X(s_0, t_i) - X(s_0, t_0)| > 2\varepsilon\Big] . \end{aligned}$$

Consider the first n points $(s_1, t_1), \cdots, (s_n, t_n)$ in S_1. Let $\sigma_1, \cdots, \sigma_n$ and $\tau_1, \cdots, \tau_n$ be rearrangements of $s_1, \cdots, s_n$ and $t_1, \cdots, t_n$ respectively so that $s_0 \leqq \sigma_1 \leqq \sigma_2 \leqq, \cdots, \leqq \sigma_n \leqq s_0 + \delta$ and $t_0 \leqq \tau_1 \leqq \tau_2 \leqq, \cdots, \leqq \tau_n \leqq t_0 + \delta$. Since $X(s, t)$ is biadditive,

$$X(\sigma_i, \tau_j) - X(\sigma_i, t_0) - X(s_0, \tau_j) + X(s_0, t_0)$$
$$= \sum_{m=1}^{i} \sum_{l=1}^{j} \{X(\sigma_m, \tau_l) - X(\sigma_{m-1}, \tau_l) - X(\sigma_m, \tau_{l-1}) + X(\sigma_{m-1}, \tau_{l-1})\}$$

$$X(\sigma_i, t_0) - X(s_0, t_0) = \sum_{m=1}^{i} \{X(\sigma_m, t_0) - X(\sigma_{m-1}, t_0)\}$$

$$X(s_0, t_j) - X(s_0, t_0) = \sum_{l=1}^{j} \{X(s_0, \tau_l) - X(s_0, \tau_{l-1})\}$$

are sums of independent random variables. Now if (s', t') and (s'', t'') are any two points in $[s_0 - \delta, s_0 + \delta] \times [t_0 + \delta, t_0 + \delta]$, then using (1) we may verify that the hypotheses of the Ottaviani inequalities, Lemmas 4.2 and 4.3, are satisfied. Thus

$$(7) \quad P\Big[\max_{i=1,\cdots,n} |X(\sigma_i, t_0) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2P[|X(\sigma_n, t_0) - X(s_0, t_0)| > \varepsilon]$$

$$(8) \quad P\Big[\max_{j=1,\cdots,n} |X(s_0, \tau_j) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2P[|X(s_0, \tau_n) - X(s_0, t_0)| > \varepsilon]$$

and

$$(9) \quad \begin{aligned} &P\Big[\max_{\substack{i=1,\cdots,n \\ j=1,\cdots,n}} |X(\sigma_i, \tau_j) - X(s_0, \tau_j) - X(\sigma_i, t_0) + X(s_0, t_0)| > 2\varepsilon\Big] \\ &\leqq 2P[|X(\sigma_n, \tau_n) - X(s_0, \tau_n) - X(\sigma_n, t_0) + X(s_0, t_0)| > \varepsilon] \,. \end{aligned}$$

From the choice of δ we see that the right sides of inequalities (7), (8), and (9) are each not greater than 2ε. Since the σ_i's are s_i's and τ_i's are t_i's, we have

$$(10) \qquad P\Big[\max_{i=1,\cdots,n} |X(s_i, t_0) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2\varepsilon$$

$$(11) \qquad P\Big[\max_{j=1,\cdots,n} |X(s_0, t_j) - X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2\varepsilon$$

and

$$(12) \quad P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_i) - X(s_i, t_0) + X(s_0, t_0)| > 2\varepsilon\Big] \leqq 2\,\varepsilon \,.$$

Substituting (10), (11), and (12) into (6) we get (5), i.e.

$$P\Big[\max_{i=1,\cdots,n} |X(s_i, t_i) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon \,.$$

Then

$$P\Big[\sup_{(s,t) \in S_1} |X(s, t) - X(s_0, t_0)| > 6\varepsilon\Big] \leqq 6\varepsilon \,.$$

Since the proof of (2) is similar, it is omitted.

Now let $V = S_1 \cup S_2 \cup S_3 \cup S_4$. Then

$$P\Big[\sup_{(s,t)\in V} |X(s,t) - X(s_0,t_0)| > 6\varepsilon\Big] \leqq \sum_{i=1}^{4} P\Big[\sup_{(s,t)\in S_i} |X(s,t) - X(s_0,t_0)| > 6\varepsilon\Big] \tag{13}$$

and hence

$$P\Big[\sup_{(s,t)\in V} |X(s,t) - X(s_0,t_0)| > 6\varepsilon\Big] \leqq 24\varepsilon\,. \tag{14}$$

Taking limits as $\delta \downarrow 0$, we obtain

$$P\Big[\limsup_{\delta\downarrow 0} \sup_{V} |X(s,t) - X(s_0,t_0)| > 6\varepsilon\Big] \leqq 24\varepsilon\,. \tag{15}$$

Now let $\varepsilon \downarrow 0$ and take complements to get

$$P\Big[\lim_{\delta\downarrow 0} \sup_{V} |X(s,t) - X(s_0,t_0)| = 0\Big] = 1\,. \tag{16}$$

If an arbitrary sequence (s_n, t_n) with $\lim_{n\to\infty}(s_n, t_n) = (s_0, t_0)$ is given, we extend the point set $\{s_n, t_n\}$ to a countable dense set S in D. Then

$$\Big[\lim_{n\to\infty} X(s_n, t_n) = X(s_0, t_0)\Big] \supset \Big[\lim_{\delta\downarrow 0} \sup_{V} |X(s,t) - X(s_0,t_0)| = 0\Big]$$

and by (16)

$$P\Big[\lim_{n\to\infty} X(s_n, t_n) = X(s_0, t_0)\Big] = 1\,.$$

LEMMA 4.5. *Let* $X(s, t)$ *be a biadditive process on a probability space* $(\Omega, \mathfrak{B}, P)$ *with* $(s, t) \in D = [0, 1] \times [0, 1]$. *Suppose that* $v(s, t) \equiv$ var $(X(s, t))$ *is continuous over* D. *Furthermore, suppose that for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} P\Big[\Big|X\Big(\frac{k}{n}, \frac{j}{n}\Big) - X\Big(\frac{k-1}{n}, \frac{j}{n}\Big) - X\Big(\frac{k}{n}, \frac{j-1}{n}\Big) + X\Big(\frac{k-1}{n}, \frac{j-1}{n}\Big)\Big| > \varepsilon\Big] = 0 \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} P\Big[\Big|X\Big(1, \frac{k}{n}\Big) - X\Big(1, \frac{k-1}{n}\Big)\Big| > \varepsilon\Big] = 0 \tag{2}$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{n} P\Big[\Big|X\Big(\frac{j}{n}, 1\Big) - X\Big(\frac{j-1}{n}, 1\Big)\Big| > \varepsilon\Big] = 0\,. \tag{3}$$

Then there is a process $Y(s, t)$ *equivalent to* $X(s, t)$ *such that almost* *every sample function of* $Y(s, t)$ *is continuous on* D.

Proof. Let S be the set of all rational numbers in $[0, 1]$ and let $D' = S \times S$. Define Ω' by $\Omega' = \{\omega \in \Omega: X(s, t)$ is uniformly continuous on $D'\}$. In the first part of the proof, we show that $P(\Omega') = 1$.

Let Z_n be defined on $(\Omega, \mathfrak{B}, P)$ by

$$Z_n = \sup\left\{\left|X(s'', t'') - X(s', t')\right|: (s'', t'') \in D',\ (s', t') \in D' \quad \text{and} \quad |s'' - s'| < \frac{1}{n}, |t'' - t'| < \frac{1}{n}\right\}.$$

Then $X(s, t)$ is uniformly continuous on D' if and only if $\lim_{n\to\infty} Z_n = 0$. Hence,

$$P(\Omega') = P\left[\lim_{n\to\infty} Z_n = 0\right]. \tag{4}$$

Let $S_j \equiv S \cap [(j-1)/n, j/n]$ $j = 1, \cdots, n$, and fix n. We number the elements of S_j in an arbitrary manner for each $j = 1, \cdots, n$. Let j and k be now fixed and let $s_1, \cdots, s_{m-1}$ and $t_1, \cdots, t_{m-1}$ denote the first $m-1$ elements of S_j and S_k respectively. Let $\sigma_1, \cdots, \sigma_{m-1}$ and $\tau_1, \cdots, \tau_{m-1}$ be the arrangements of $\{s_1, \cdots, s_{m-1}\}$ and $\{t_1, \cdots, t_{m-1}\}$ respectively in ascending order so that $\sigma_1 < \sigma_2 < \cdots < \sigma_{m-1}$ and $\tau_1 < \tau_2 < \cdots < \tau_{m-1}$. Choose $\sigma_0 = (j-1)/n$, $\sigma_m = j/n$, $\tau_0 = (k-1)/n$, and $\tau_m = k/n$, and define $S_{jm} \equiv \{\tau_0, \tau_1, \cdots, \tau_m\}$. We will use the notation:

$$\Delta(s, t, s', t') \equiv X(s', t') - X(s, t') - X(s', t) + X(s, t).$$

Since $X(s, t)$ is biadditive, the three collections of random variables below are systems of independent random variables:

$$\{\Delta(\sigma_{\mu-1}, \tau_{\gamma-1}, \sigma_\mu, \tau_\gamma): \mu, \gamma = 1, \cdots, m\}$$

$$\left\{\Delta\left(\frac{j-1}{n}, \tau_{\gamma-1}, \frac{j}{n}, \tau_\gamma\right): \gamma = 1, \cdots, m \quad \text{and} \quad j = 1, \cdots, n\right\}$$

$$\left\{\Delta\left(\sigma_{\mu-1}, \frac{k-1}{n}, \sigma_\mu, \frac{k}{n}\right): \mu = 1, \cdots, m \quad \text{and} \quad k = 1, \cdots, n\right\}.$$

Let $\varepsilon > 0$ be chosen arbitrarily. Since $v(s, t)$ and $m(s, t)$ are continuous on D, they are uniformly continuous and if n is sufficiently large and if $0 < s'' - s' < 1/n$ or $0 < t'' - t' < 1/n$, then from Chebychef's inequality it follows that

$$P\left[|\Delta(s', t', s'', t'')| > \frac{\varepsilon}{2}\right] \leqq 1 - \sqrt{\frac{1}{2}}. \tag{5}$$

Let $Y_{n,j,k} \equiv \sup_{S_j \times S_k} |X(s, t) - X((j-1)/n, (k-1)/n))|$. Then from the triangle inequality we get

$$
\begin{aligned}
Y_n &\equiv \max_{j,k=1,\cdots,n} Y_{n,j,k} \\
&\leqq \max_{j,k,1,\cdots,n} \sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| \\
&\quad + \max_{j,k=1,\cdots,n} \sup_{s \in S_j} \left| X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| \\
&\quad + \max_{j,k=1,\cdots,n} \sup_{t \in S_k} \left| X\left(\frac{j-1}{n}, t\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| .
\end{aligned} \tag{6}
$$

Consequently,

$$
\begin{aligned}
&P[Y_n > 6\varepsilon] \\
&\quad \leqq P\left[\max_{j,k=1,\cdots,n} \sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon\right] \\
&\qquad + P\left[\max_{j,k=1,\cdots,n} \sup_{s \in S_j} \left| X\left(s, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) \right| > 2\varepsilon\right] \\
&\qquad + P\left[\max_{j,k=1,\cdots,n} \sup_{t \in S_k} \left| X\left(\frac{j-1}{n}, t\right) - X\left(\frac{j-1}{n}, \frac{k-1}{m}\right) \right| > 2\varepsilon\right] .
\end{aligned} \tag{7}
$$

For $(\sigma_\mu, \tau_\gamma) \in S_{jm} \times S_{km}$, we see that

$$
\Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \sigma_\mu, \tau_\gamma\right) = \sum_{q=1}^{\gamma} \sum_{p=1}^{\mu} \Delta(\sigma_{p-1}, \tau_{q-1}, \sigma_p, \tau_q)
$$

a sum of independent random variables. Now (5) implies that the hypotheses of the extended Ottaviani's Inequality (Lemma 4.3) are satisfied and consequently

$$
P\left[\max_{\mu,\gamma=1,\cdots,m} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \sigma_\mu, \tau_\gamma\right) \right| > 2\varepsilon\right] \leqq 2P\left[\left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon\right] .
$$

Letting $m \to \infty$, it follows that

$$
P\left[\sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon\right] \leqq 2P\left[\left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon\right]
$$

and hence

$$
\begin{aligned}
&P\left[\max_{j,k=1,\cdots,n} \sup_{S_j \times S_k} \left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, s, t\right) \right| > 2\varepsilon\right] \\
&\qquad \leqq 2 \sum_{j=1}^{n} \sum_{k=1}^{n} P\left[\left| \Delta\left(\frac{j-1}{n}, \frac{k-1}{n}, \frac{j}{n}, \frac{k}{n}\right) \right| > \varepsilon\right] .
\end{aligned} \tag{9}
$$

Now if $\sigma_\mu \in S_{im}$, since $X(\sigma_\mu, 0) = X(0, (k-1)/n) = 0$, we have

$$
X\left(\sigma_\mu, \frac{k-1}{n}\right) - X\left(\frac{j-1}{n}, \frac{k-1}{n}\right) = \sum_{p=1}^{\mu} \sum_{q=1}^{k-1} \Delta\left(\sigma_{p-1}, \frac{q-1}{n}, \sigma_p, \frac{q}{n}\right) ,
$$

as before, a sum of independent random variables. Again, (5) allows us to use the extended Ottaviani's Inequality to obtain

$$P\Big[\max_{k=1,\cdots,n}\max_{t\in S_{jm}}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$\leqq 2P\Big[\Big|X\Big(\frac{j}{n},1\Big)-X\Big(\frac{j-1}{n},1\Big)\Big|>\varepsilon\Big].$$

Letting $m\to\infty$, we get

$$P\Big[\max_{k=1,\cdots,n}\sup_{s\in S_j}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$\leqq 2P\Big[\Big|X\Big(\frac{j}{n},1\Big)-X\Big(\frac{j-1}{n},1\Big)\Big|>\varepsilon\Big]$$

and

$$P\Big[\max_{j,k=1,\cdots,n}\sup_{s\in S_j}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$(10)\qquad = P\Big\{\bigcup_{j=1}^{n}\Big[\max_{k=1,\cdots,n}\sup_{s\in S_j}\Big|X\Big(s,\frac{k-1}{n}\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]\Big\}$$
$$\leqq 2\sum_{j=1}^{n}P\Big[\Big|X\Big(\frac{j}{n},1\Big)-X\Big(\frac{j-1}{n},1\Big)\Big|>\varepsilon\Big].$$

Similarly for $\tau_r\in S_{k,m}$

$$X\Big(\frac{j-1}{n},\tau_r\Big)-X\Big(\frac{j-1}{n},\ \frac{k-1}{n}\Big)=\sum_{p=1}^{j-1}\sum_{q=1}^{r}\varDelta\Big(\frac{p-1}{n},\tau_{q-1},\frac{p}{n},\tau_q\Big),$$

a sum of independent random variables, and so by (5) we may again apply the extended Ottaviani's Inequality and take limits as $m\to\infty$. We get

$$(11)\qquad P\Big[\max_{j,k=1,\cdots,n}\sup_{t\in S_k}\Big|X\Big(\frac{j-1}{n},t\Big)-X\Big(\frac{j-1}{n},\frac{k-1}{n}\Big)\Big|>2\varepsilon\Big]$$
$$\leqq 2\sum_{k=1}^{n}P\Big[\Big|X\Big(1,\frac{k}{n}\Big)-X\Big(1,\frac{k-1}{n}\Big)\Big|>\varepsilon\Big].$$

Inserting (9), (10), and (11) into (7) and letting $n\to\infty$, we see from the hypotheses (1), (2), and (3) that

$$(12)\qquad \lim_{n\to\infty}P[Y_n>6\varepsilon]=0.$$

The inequality $Z_n\leqq 4Y_n$ can be checked by succesive applications of the triangle inequality. (If $|s'-s''|<1/n$ and $|t'-t''|<1/n$, $(s',t')\in[(j-1)/n,j/n]\times[(k-1)/n,k/n]$ implies that $(s'',t'')\in[(j-2)/n,(j+1)/n]\times[(k-2)/n,(k+1)n]$ and it suffices to check each possibility.) It follows that

$$P[Z_n > 24\varepsilon] \leqq P[Y_n > 6\varepsilon] .$$

Since $0 \leqq Z_n$ and $Z_{n+1} \leqq Z_n$ for all n,

$$\lim_{n\to\infty} P[Z_n > 24\varepsilon] = P\left[\lim_{n\to\infty} Z_n > 24\varepsilon\right] = 0$$

by (12). Letting $\varepsilon \downarrow 0$, we obtain

$$P\left[\lim_{n\to\infty} Z_n > 0\right] = 0 ,$$

and since $Z_n \geqq 0$, we get

$$P(\Omega') = P\left[\lim_{n\to 0} Z_n = 0\right] = 1 ,$$

which finishes the first part of the proof.

Now if $x(s, t)$ is any real-valued function uniformly continuous on a set D, it has a unique continuous extension to the closure of D. Let $Y(s, t, \omega)$ be defined for $\omega \in \Omega'$ by $Y(s, t, \omega) = X(s, t, \omega)$ if $(s, t) \in D'$.

If $(s, t) \notin D'$, choose a sequence of points (s_n, t_n) in D' such that $\lim_{n\to\infty} (s_n, t_n) = (s, t)$ and define $Y(s, t, \omega) \equiv \lim_{n\to\infty} Y(s_n, t_n, \omega)$ for $\omega \in \Omega'$. Since for $\omega \in \Omega'$ $Y(s, t, \omega)$ is uniformly continuous on D' which is dense in D, $Y(s, t, \omega)$ is well-defined for $\omega \in \Omega'$. If $\omega \notin \Omega'$, let $Y(s, t, \omega) \equiv 0$. Then for $(s, t) \in D'$,

$$P[Y(s, t) = X(s, t)] \geqq P(\Omega') = 1$$

and if $(s, t) \in D$ but $(s, t) \notin D'$,

$$P\left[Y(s, t) = \lim_{n\to\infty} X(s_n, t_n)\right] \geqq P(\Omega') = 1$$

for some sequence $\{(s_n, t_n)\}$ in D' such that $\lim_{n\to\infty} (s_n, t_n) = (s, t)$. But by Lemma 2.6,

$$P\left[X(s, t) = \lim_{n\to\infty} X(s_n, t_n)\right] = 1$$

and hence for any $(s, t) \in D$,

$$P[Y(s, t) = X(s, t)] = 1 .$$

That is, $Y(s, t)$ is a process which is equivalent to $X(s, t)$. It follows from the definition of $Y(s, t)$, that its sample functions are continuous on Ω', a set of probability one.

5. Proof of Theorem 2.

Proof. Let $\Phi(a, b, c, d)$ denote the normal probability distribution with mean zero and variance $v(c, d) - v(a, d) - v(c, b) + v(a, b)$ where $0 \leqq a < c \leqq 1$ and $0 \leqq b < d \leqq 1$. Then since the convolution of normal distributions is a normal distribution whose mean and variance are the respective sums of the means and variances of the original distributions, for any $\alpha > 0$ we have

$$\Phi(a, b, c + \alpha, d) = \Phi(a, b, c, d) * \Phi(c, b, c + \alpha, d)$$
$$\Phi(a, b, c, d + \alpha) = \Phi(a, b, c, d) * \Phi(a, d, c, d + \alpha)$$

where "$*$" denotes the operation of convolution.

By Lemma 4.1, there is a biadditive process $Y(s, t)$ such that for $s' < s''$ and $t' < t''$, $Y(s'', t'') - Y(s', t'') - Y(s'', t') + Y(s', t')$ is normally distributed with mean zero and variance $v(s'', t'') - v(s', t'') - v(s'', t') + v(s', t')$. If $Y(s, t)$ satisfies conditions (1), (2), and (3) of Lemma 4.5 there is a process $Y_0(s, t)$ equivalent to $Y(s, t)$ such that almost every sample function of $Y_0(s, t)$ is continuous over D. Define $X(s, t) = Y_0(s, t) + m(s, t)$. Then $X(s, t)$ satisfies (i) and (ii) and is biadditive since $Y_0(s, t)$ is. Furthermore almost every sample function of $X(s, t)$ is continuous over D.

Let Δ_{jk} denote the random variable

$$\Delta_{jk} \equiv Y\left(\frac{j}{n}, \frac{k}{n}\right) - Y\left(\frac{j-1}{n}, \frac{k}{n}\right) - Y\left(\frac{j}{n}, \frac{k-1}{n}\right) + Y\left(\frac{j-1}{n}, \frac{k-1}{n}\right)$$

where n is a positive integer. Conditions (1), (2), and (3) of Lemma 4.5 are

$$(1) \qquad \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} P[|\Delta_{jk}| > \varepsilon] = 0$$

$$(2) \qquad \lim_{n\to\infty} \sum_{k=1}^{n} P\left[\left| Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] = 0$$

$$(3) \qquad \lim_{n\to\infty} \sum_{j=1}^{n} P\left[\left| Y\left(\frac{j}{n}, 1\right) - Y\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] = 0$$

where $\varepsilon > 0$ is chosen in an arbitrary manner. We will use the following inequality which is valid for $\lambda > 0$.

$$\int_\lambda^\infty e^{-t^2/2} dt \leqq \frac{1}{\lambda}\int_\lambda^\infty t e^{-t^2/2} dt = \frac{1}{\lambda} e^{-\lambda^2/2} .$$

For $\varepsilon > 0$ since Δ_{jk} is normally distributed,

$$P[|\Delta_{jk}| > \varepsilon] = \frac{2}{\sqrt{2\pi v_{jk}}} \int_\varepsilon^\infty e^{-(t^2/2v_{jk})} dt$$
$$= \frac{2}{\sqrt{2\pi}} \int_\lambda^\infty e^{-t^2/2} dt$$

or

$$P[|\Delta_{jk}| > \varepsilon] \leqq \frac{2}{\lambda\sqrt{2\pi}} \exp\left\{-\frac{\lambda^2}{2}\right\} = \frac{2}{\varepsilon}\sqrt{\frac{v_{jk}}{2\pi}} \exp\left\{-\frac{\varepsilon^2}{2v_{jk}}\right\}$$

where

$$v_{jk} \equiv v\left(\frac{j}{n}, \frac{k}{n}\right) - v\left(\frac{j-1}{n}, \frac{k}{n}\right) - v\left(\frac{j}{n}, \frac{k-1}{n}\right) + v\left(\frac{j-1}{n}, \frac{k-1}{n}\right)$$

and $\lambda = \varepsilon(v_{jk})^{-(1/2)}$. Since $v(s, t)$ is uniformly continuous over D, we can choose N independently of j and k such that $n \geqq N$ implies $v_{jk}/\varepsilon^2 < 1/M_\delta^2$ where M_δ is determined as follows. Since $(1/x)\exp\{-(x^2/2)\} = o(x^{-2})$ as $x \to \infty$, we have for every positive integer δ, a number M_δ such that $x > M_\delta$ implies $x \exp\{-(x^2/2)\} < 1/\delta$, that is, for $x > M_\delta$,

$$\frac{1}{x} \exp\left\{-\frac{x^2}{2}\right\} < \frac{1}{\delta x^2} .$$

Now $v_{jk}/\varepsilon^2 < 1/M_\delta^2$ entails $\varepsilon/\sqrt{v_{jk}} > M_\delta$ and with $x = \varepsilon/\sqrt{v_{jk}}$ we get

$$\frac{\sqrt{v_{jk}}}{\varepsilon} \exp\left\{-\frac{\varepsilon^2}{2v_{jk}}\right\} \leqq \frac{1}{\delta}\frac{v_{jk}}{\varepsilon^2} .$$

Then for $n \geqq N$

$$P[|\Delta_{jk}| > \varepsilon] \leqq \frac{2}{\sqrt{2\pi}} \cdot \frac{v_{jk}}{\varepsilon^2} .$$

But $v(1, 1) - v(1, 0) - v(0, 1) + v(0, 0) = v(1, 1) = \sum_{k=1}^{n} \sum_{j=1}^{n} v_{jk}$, and so

$$\sum_{j=1}^{n} \sum_{k=1}^{n} P[|\Delta_{jk}| < \varepsilon] \leqq \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\delta\varepsilon^2} v(1, 1) .$$

Since we may take δ arbitrarily large, choosing N sufficiently large for each δ,

$$\lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} P[|\Delta_{ik}| > \varepsilon] = 0$$

and (1) holds for $Y(s, t)$. A similar argument proves (2) and (3). Since $Y(s, 0) = Y(0, t) = 0$ for all (s, t) in D, $Y(1, k/n) - Y(1, (k-1)/n)$ is normally distributed with mean zero and variance $v(1, k/n) - v(1, (k-1)/n)$, and $Y(j/n, 1) - Y((j-1)/n, 1)$ is normally distributed with mean 0 and variance $v(j/n, 1) - v((j-1)/n, 1)$. Thus

$$P\left[\left| Y\left(1, \frac{k}{n}\right) - Y\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] = \frac{2}{\sqrt{2\pi v_k}} \int_\varepsilon^\infty \exp\{-t^2/2v_k\} dt$$

$$\leqq \frac{2\sqrt{v_k}}{\sqrt{2\pi}\varepsilon} \exp\{-\varepsilon^2/2v_k\}$$

and

$$P\left[\left| Y\left(\frac{j}{n}, 1 - Y\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] = \frac{2}{\sqrt{2\pi\nu_j}}\int_\varepsilon^\infty \exp\{-t^2/2v\}dt$$
$$\leqq \frac{2\sqrt{v_j}}{\sqrt{2\pi}\varepsilon}\exp\{-\varepsilon^2/2v_j\}$$

where $v_j \equiv v(j/n, 1 - v((j-1)/n, 1)$ and $v_k \equiv v(1, k/n) - v(1, (k-1)/n)$. Again we may choose δ, M_δ, N', and N'' so that when $n \geqq N'$ or $n \geqq N''$, the respective inequalities

$$\frac{v_j}{\varepsilon^2} < \frac{1}{M_\delta^2} \quad \text{or} \quad \frac{v_k}{\varepsilon^2} < \frac{1}{M_\delta^2}$$

hold. Since $v(1, 1) = \sum_{j=1}^n v_j = \sum_{k=1}^n v_k$,

$$\sum_{k=1}^n P\left[\left|X\left(1, \frac{k}{n}\right) - X\left(1, \frac{k-1}{n}\right)\right| > \varepsilon\right] \leqq \frac{2}{\sqrt{2\pi}\delta\varepsilon^2}v(1, 1)$$

and

$$\sum_{j=1}^n P\left[\left|X\left(\frac{j}{n}, 1\right) - X\left(\frac{j-1}{n}, 1\right)\right| > \varepsilon\right] \leqq \frac{2}{\sqrt{2\pi}\delta\varepsilon^2}v(1, 1)$$

when $n > N''$ or $n > N'$ respectively. Thus there is a process $Y_0(s, t)$ equivalent to $Y(s, t)$ such that almost every sample function of Y_0 is continuous over D. Setting $X(s, t) = Y_0(s, t) + m(s, t)$ we obtain a biadditive process satisfying (i), (ii), and (iii).

References

1. H. G. Tucker, *A Graduate Course in Probability*, Academic Press, New York and London (1967).
2. J. Yeh, *Wiener measure in the space of functions of two variables*, Trans. Amer. Math. Soc., **95** (1960), 433-450.
3. ———, Unpublished Lecture Notes.

Received April 29, 1971. The author wishes to thank Professor J. Yeh for suggesting this problem. This paper forms part of the author's dissertation and the research for it was sponsored in part by the National Science Foundation (Grant No. GP-13288) and in part by the Air Force (Grant No. 1321-67).

UNIVERSITY OF CALIFORNIA,
SANTA BARBARA, CALIFORNIA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 42, No. 2 February, 1972