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**INTEGRABILITY OF TRIGONOMETRIC SERIES. II**

MASAKO IZUMI AND SHIN-ICHI IZUMI

## INTEGRABILITY OF TRIGONOMETRIC SERIES II

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**There are proved theorems of the following type. Let  $0 < r < 1$  and  $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$  and suppose  $\int_0^{\pi} g(t) |t-u|^{-r} dt \leq A$  for all  $u > 0$ . Then  $\sum_{n=1}^{\infty} |b_n| n^{r-1} < \infty$ .**

1. Sz.-Nagy [7] (cf. [1]) has proved the following

**THEOREM 1.** *Let  $0 < r < 1$  and  $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$ . If  $g \downarrow$  and is bounded below, then*

$$(1) \quad t^{-r}g(t) \in L(0, \pi) \iff \sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty .$$

*The same holds for even functions.*

In this theorem, the assumption  $g \downarrow$  cannot be replaced by  $g \geq 0$ , that is,

**THEOREM 2.** ([1], [4]) *Let  $0 < r < 1$ , then there is an odd function*

$$g \sim \sum_{n=1}^{\infty} b_n \sin nt$$

*such that*

- (i)  $g \geq 0$  on  $(0, \pi)$ ,
- (ii)  $t^{-r}g(t) \in L(0, \pi)$ ,

*but*

- (iii)  $\sum_{n=1}^{\infty} |b_n|/n^{1-r} = \infty$ .

*The same holds for even functions.*

Sunouchi [6], Edmonds [3] and Boas [2] (cf. [1]) proved that

**THEOREM 3.** *Let  $0 < r < 1$  and  $b_n \downarrow 0$ . Then the function  $g$  defined by  $g(t) = \sum_{n=1}^{\infty} b_n \sin nt$ , satisfies the relation (1). The same holds for even functions.*

The monotonicity of  $(b_n)$  in this theorem cannot be replaced by positivity, that is,

**THEOREM 4.** ([1], [4]) *Let  $0 < r < 1$ , then there is an odd function*

$$g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$$

such that

- (i)  $b_n \geq 0$  (or  $b_n \rightarrow 0$  and  $\sum | \Delta b_n | < \infty$ ),
- (ii)  $\sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty$ ,

but

- (iii)  $t^{-r}g(t) \notin L(0, \pi)$ .

The same holds for even functions.

We shall prove the following theorems.

**THEOREM 5.** Let  $0 < r < 1$  and  $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$ , then

$$(2) \quad \int_0^{\pi} \frac{g(t)}{|t-u|^r} dt \leq A \text{ for all } u > 0 \implies \sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty .$$

The same holds for even functions.

**THEOREM 6.** Let  $0 < r < 1$  and  $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$ . Then

$$(3) \quad \sum_{n=1}^{\infty} n^r | \Delta b_n | < \infty \implies t^{-r}g(t) \in L(0, \pi) .$$

The same holds for even functions.

By Theorem 4, the converse implication of (2) does not hold. We have proved that [5], for  $0 < r < 1$ ,

$$\sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty \implies \exists \int_{+0}^{\pi} g(t)t^{-r}dt .$$

The left side of (2) is satisfied when

- (i)  $g \downarrow$  on  $(0, \delta)$  and bounded on  $(\delta, \pi)$  and  $t^{-r}g(t) \in L(0, \pi)$ , or
- (ii) there is an odd function  $g_1$  such that

$$|g(t)| \leq A g_1(t) \text{ on } (0, \pi)$$

and  $g_1$  satisfies the condition in (i), or

- (iii) there is a finite set of points  $(x_1, x_2, \dots, x_n)$  on  $(0, \pi)$  such that the odd function  $g$  becomes monotonically infinite on one side or both side of each  $x_i$  and is bounded outside of each neighbourhood of  $x_i$  and

$$|g(t)|/|t-x_i|^r \in L(0, \pi) \quad (i = 1, 2, \dots, n) .$$

(i) shows that Theorem 5 contains the  $\int \rightarrow \sum$  part of Theorem 1 as a particular case. (ii) and (iii) are generalizations of (i).

If  $b_n \downarrow 0$ , the right side of (1) is equivalent to the left side of (3), and then the  $\sum \rightarrow \int$  part of Theorem 3 is a particular case of Theorem 6.

2. *Proof of Theorem 5.* We consider the function

$$(3) \quad h(t) = \sum_{n=2}^{\infty} (-1)^n \sin nt/n^{1-r},$$

then

$$k(t) = (g * h)(t) = \frac{2}{\pi} \int_0^{\pi} g(u)h(t + u)du \sim \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1-r}} b_n \cos nt.$$

If  $k \in \text{Lip } 1$ , then its Fourier series converges absolutely by Bernstein's theorem, which gives the conclusion of the theorem. Therefore it is sufficient to prove that

$$(4) \quad |k(t + v) - k(t)| \leq Av \text{ for small } v > 0 \text{ and } 0 < t < \pi.$$

Now,

$$(5) \quad \begin{aligned} k(t + v) - k(t) &= \frac{2}{\pi} \int_0^{\pi} g(u)(h(t + u + v) - h(t + u))du \\ &= \frac{2}{\pi} \int_0^{\pi} g(u - t)(h(u + v) - h(u))du, \end{aligned}$$

where, by (3),

$$(6) \quad \begin{aligned} h(u) &= \sum_{n=1}^{\infty} \left( \frac{\sin 2nu}{(2n)^{1-r}} - \frac{\sin (2n + 1)u}{(2n + 1)^{1-r}} \right) \\ &= \sum_{n=1}^{\infty} \left( \frac{1}{(2n)^{1-r}} - \frac{1}{(2n + 1)^{1-r}} \right) \sin 2nu \\ &\quad + (1 - \cos u) \sum_{n=1}^{\infty} \frac{\sin 2nu}{(2n + 1)^{1-r}} - \sin u \sum_{n=1}^{\infty} \frac{\cos 2nu}{(2n + 1)^{1-r}} \\ &= P(u) + Q(u) - R(u). \end{aligned}$$

We write

$$p(u) = \frac{1}{(2u)^{1-r}} - \frac{1}{(2u + 1)^{1-r}} \text{ for } u > 0$$

and

$$p(n) = p_n (n = 1, 2, \dots),$$

then

$$\begin{aligned} P(u) &= \sum_{n=1}^{\infty} p_n \sin 2nu \\ &= \int_{1/2}^{\infty} p(w) \sin 2uw dw + \int_{1/2}^{\infty} p(w) \sin 2uw dj(w) \end{aligned}$$

where

$$j(w) = -w + [w] + \frac{1}{2} \sim \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2\pi m w}{m}$$

and then

$$\begin{aligned} P(u+v) - P(u) &= \left\{ \int_{1/2}^{\infty} p(w) \sin 2(u+v)w \, dw - \int_{1/2}^{\infty} p(w) \sin 2uw \, dw \right\} \\ &\quad + \left\{ \int_{1/2}^{\infty} p(w) \sin 2(u+v)w \, dj(w) - \int_{1/2}^{\infty} p(w) \sin 2uw \, dj(w) \right\} \\ &= S_1 + S_2. \end{aligned}$$

We shall first show that  $S_1 = O(v/u^r)$ .

$$\begin{aligned} S_1 &= \frac{1}{2(u+v)} \int_{u+v}^{\infty} p\left(\frac{x}{2(u+v)}\right) \sin x \, dx - \frac{1}{2u} \int_u^{\infty} p\left(\frac{x}{2u}\right) \sin x \, dx \\ &= \frac{1}{2} \int_{u+v}^{\infty} \left( \frac{1}{u+v} p\left(\frac{x}{2(u+v)}\right) - \frac{1}{u} p\left(\frac{x}{2u}\right) \right) \sin x \, dx \\ &\quad - \frac{1}{2u} \int_u^{u+v} p\left(\frac{x}{2u}\right) \sin x \, dx \\ &= \frac{1}{2} \int_{u+v}^{\infty} \left\{ \left( \frac{1}{(u+v)^r} - \frac{1}{u^r} \right) \frac{1}{x^{1-r}} \right. \\ (7) \quad &\quad \left. - \left( \frac{1}{(u+v)^r (x+u+v)^{1-r}} - \frac{1}{u^r (x+u)^{1-r}} \right) \right\} \sin x \, dx \\ &\quad - \frac{1}{2u^r} \int_u^{u+v} \left( \frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x \, dx \\ &= \frac{1}{2} \left( \frac{1}{(u+v)^r} - \frac{1}{u^r} \right) \int_{u+v}^{\infty} \left( \frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x \, dx \\ &\quad + \frac{1}{2(u+v)^r} \int_{u+v}^{\infty} \left( \frac{1}{(x+u)^{1-r}} - \frac{1}{(x+u+v)^{1-r}} \right) \sin x \, dx \\ &\quad - \frac{1}{2u^r} \int_u^{u+v} \left( \frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x \, dx \\ &= O(v/u^r), \end{aligned}$$

since

$$\begin{aligned} &\int_{u+v}^{\infty} \left( \frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \sin x \, dx \\ &= (1 - \cos u) \int_{2u+v}^{\infty} \frac{\sin x}{x^{1-r}} \, dx + \int_{2u+v}^{\infty} \frac{\sin u \cdot \cos x}{x^{1-r}} \, dx \\ &\quad + \int_{u+v}^{2u+v} \frac{\sin x}{x^{1-r}} \, dx \end{aligned}$$

and similarly for the integral of the second term of the right side of (7). The estimation of  $S_2$  is a little more complicated. By integration by parts,

$$\begin{aligned}
 S_2 &= \int_{1/2}^{\infty} p(w) \sin 2(u + v)w \, dj(w) - \int_{1/2}^{\infty} p(w) \sin 2uw \, dj(w) \\
 &= - \int_{1/2}^{\infty} p'(w) \sin 2(u + v)w \, j(w) \, dw \\
 &\quad - 2(u + v) \int_{1/2}^{\infty} p(w) \cos 2(u + v)w \, j(w) \, dw \\
 &\quad + \int_{1/2}^{\infty} p'(w) \sin 2uw \, j(w) \, dw \\
 &\quad + 2u \int_{1/2}^{\infty} p(w) \cos 2uw \, j(w) \, dw,
 \end{aligned}$$

and using the Fourier expansion of  $j(w)$ , we get

$$\begin{aligned}
 S_2 &= - \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \left\{ \int_{1/2}^{\infty} p'(w) (\sin 2(u + v)w \cdot \sin 2\pi m w \right. \\
 &\quad \left. - \sin 2uw \cdot \sin 2\pi m w) \, dw \right. \\
 &\quad \left. - 2 \int_{1/2}^{\infty} p(w) ((u + v) \cos 2(u + v)w \cdot \sin 2\pi m w \right. \\
 &\quad \left. - u \cos 2uw \sin 2\pi m w) \, dw \right\} \\
 &= \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \{2(1 - r)T_1 + 2T_2\}.
 \end{aligned}$$

In order to get  $S_2 = O(v)$ , it is sufficient to prove that  $T_1$  and  $T_2$  are of order  $O(v/m)$ . Now, for  $m \geq 3$ , we have

$$\begin{aligned}
 T_1 &= \int_{1/2}^{\infty} \left( \frac{1}{(2w)^{2-r}} - \frac{1}{(2w + 1)^{2-r}} \right) \\
 &\quad \cdot \{(\cos 2(m\pi - u - v)w - \cos 2(m\pi - u)w) \\
 &\quad - (\cos 2(m\pi + u + v)w - \cos (m\pi + u)w)\} \, dw \\
 &= \frac{1}{2} \left\{ \int_{m\pi - u - v}^{\infty} \left( \left( \frac{m\pi - u - v}{x} \right)^{2-r} - \left( \frac{m\pi - u - v}{x + m\pi - u - v} \right)^{2-r} \right) \frac{\cos x}{m\pi - u - v} \, dx \right. \\
 &\quad \left. - \int_{m\pi - u}^{\infty} \left( \left( \frac{m\pi - u}{x} \right)^{2-r} - \left( \frac{m\pi - u}{x + m\pi - u} \right)^{2-r} \right) \frac{\cos x}{m\pi - u} \, dx \right\} \\
 &\quad - \frac{1}{2} \left\{ \int_{m\pi + u + v}^{\infty} \left( \left( \frac{m\pi + u + v}{x} \right)^{2-r} - \left( \frac{m\pi + u + v}{x + m\pi + u + v} \right)^{2-r} \right) \frac{\cos x}{m\pi + u + v} \, dx \right. \\
 &\quad \left. - \int_{m\pi + u}^{\infty} \left( \left( \frac{m\pi + u}{x} \right)^{2-r} - \left( \frac{m\pi + u}{x + m\pi + u} \right)^{2-r} \right) \frac{\cos x}{m\pi + u} \, dx \right\} \\
 &= \frac{1}{2} \left\{ \int_{m\pi - u}^{\infty} \left( \frac{(m\pi - u - v)^{1-r}}{x^{2-r}} - \frac{(m\pi - u)^{1-r}}{x^{2-r}} \right. \right. \\
 &\quad \left. \left. - \frac{(m\pi - u - v)^{1-r}}{(x + m\pi - u - v)^{2-r}} + \frac{(m\pi - u)^{1-r}}{(x + m\pi - u)^{2-r}} \right) \cos x \, dx + O(v/m) \right\} \\
 &\quad - \frac{1}{2} \{\text{similar terms}\} \\
 &= O(v/m)
 \end{aligned}$$

$$\begin{aligned}
 T_2 &= \int_{1/2}^{\infty} p(w) \{ (u+v)(\sin 2(m\pi + u + v)w + \sin 2(m\pi - u - v)w) \\
 &\quad - u(\sin 2(m\pi + u)w + \sin 2(m\pi - u)w) \} dw \\
 &= \frac{1}{2} \left\{ (u+v) \int_{m\pi+u+v}^{\infty} \left( \left( \frac{m\pi + u + v}{x} \right)^{1-r} \right. \right. \\
 &\quad \left. \left. - \left( \frac{m\pi + u + v}{x + m\pi + u + v} \right)^{1-r} \right) \frac{\sin x}{m\pi + u + v} dx \right. \\
 &\quad \left. - u \int_{m\pi+u}^{\infty} \left( \left( \frac{m\pi + u}{x} \right)^{1-r} - \left( \frac{m\pi + u}{x + m\pi + u} \right)^{1-r} \right) \frac{\sin x}{m\pi + u} dx \right\} \\
 &\quad + \frac{1}{2} \{ \text{similar terms} \} \\
 &= O(v/m).
 \end{aligned}$$

When  $m = 1$  or  $2$ , we get easily

$$|T_1| \leq A \int_{1/2}^{\infty} \frac{v}{w^{2-r}} dw \leq Av, \quad |T_2| \leq Av.$$

Thus we have proved  $T_1 + T_2 = O(v/m)$  for all  $m$ , and then  $S_2 = O(v)$ . Collecting the above estimates, we get

$$(8) \quad \int_0^{\pi} |g(u-t)| |P(u+v) - P(u)| du \leq Av \int_0^{\pi} |g(u-t)| u^{-r} du \leq Av.$$

Now, by (6),

$$\begin{aligned}
 Q(u+v) - Q(u) &= (1 - \cos(u+v)) \sum_{n=1}^{\infty} \frac{\sin 2n(u+v)}{(2n+1)^{1-r}} \\
 &\quad - (1 - \cos u) \sum_{n=1}^{\infty} \frac{\sin 2nu}{(2n+1)^{1-r}} \\
 &= (\cos u - \cos(u+v)) \sum_{n=1}^{\infty} \frac{\sin 2n(u+v)}{(2n+1)^{1-r}} \\
 &\quad + (1 - \cos u) \left( \sum_{n=1}^{\infty} \frac{\sin 2n(u+v)}{(2n+1)^{1-r}} - \sum_{n=1}^{\infty} \frac{\sin 2nu}{(2n+1)^{1-r}} \right) \\
 &= U + (1 - \cos u) V
 \end{aligned}$$

where

$$|U| \leq v(u+v) \left| \sum_{n=1}^{\infty} \Delta \left( \frac{1}{(2n+1)^{1-r}} \right) \tilde{D}_n(2u+v) \right| \leq Av,$$

$\tilde{D}_n$  being the  $n$ th conjugate Dirichlet kernel, and

$$\begin{aligned}
 V &= \left\{ \int_{1/2}^{\infty} \frac{\sin 2(u+v)w}{(2w+1)^{1-r}} dw - \int_{1/2}^{\infty} \frac{\sin 2uw}{(2w+1)^{1-r}} dw \right\} \\
 &\quad + \left\{ \int_{1/2}^{\infty} \frac{\sin 2(u+v)w}{(2w+1)^{1-r}} dj(w) - \int_{1/2}^{\infty} \frac{\sin 2uw}{(2w+1)^{1-r}} dj(w) \right\} \\
 &= V_1 + V_2.
 \end{aligned}$$

It is sufficient to prove that  $(1 - \cos u)V = O(v)$ . Proceeding as for  $S_1$  and  $S_2$ , we get

$$V_1 = \frac{1}{2(u+v)^r} \int_{u+v}^{\infty} \frac{\sin x}{(x+u+v)^{1-r}} dx - \frac{1}{2u^r} \int_u^{\infty} \frac{\sin x}{(x+u)^{1-r}} dx = O(v/u)$$

and

$$V_2 = O(v) .$$

Therefore

$$Q(u+v) - Q(u) = O(v) ,$$

and

$$(9) \quad \int_0^{\pi} |g(u-t)| |Q(u+v) - Q(u)| du \leq Av \int_0^{\pi} |g(u)| du \leq Av .$$

Finally we shall estimate  $R(u+v) - R(u)$ . By the definition (6),

$$R(u) = \sin u \sum_{n=1}^{\infty} \frac{\cos 2nu}{(2n+1)^{1-r}} = \sin u \int_{1/2}^{\infty} \frac{\cos 2uw}{(2w+1)^{1-r}} dw + \sin u \int_{1/2}^{\infty} \frac{\cos 2uw}{(2w+1)^{1-r}} dj(w)$$

and

$$\begin{aligned} R(u+v) - R(u) &= (\sin(u+v) - \sin u) \int_{1/2}^{\infty} \frac{\cos 2uw}{(2w+1)^{1-r}} dm \\ &\quad + \sin(u+v) \int_{1/2}^{\infty} \frac{\cos 2(u+v)w - \cos 2uw}{(2w+1)^{1-r}} dw \\ &\quad + \left\{ \sin(u+v) \int_{1/2}^{\infty} \frac{\cos 2(u+v)w}{(2w+1)^{1-r}} dj(w) \right. \\ &\quad \left. - \sin u \int_{1/2}^{\infty} \frac{(2w+1)^{1-r}}{\cos 2uw} dj(w) \right\} \\ &= W_1 + W_2 + W_3 \end{aligned}$$

where

$$\begin{aligned} |W_1| &\leq A \frac{v}{u^r} \left| \int_u^{\infty} \frac{\cos x}{(x+u)^{1-r}} dx \right| \leq A \frac{v}{u^r} , \\ |W_2| &= \left| \frac{\sin(u+v)}{(u+v)^r} \int_{u+v}^{\infty} \frac{\cos x}{(x+u+v)^{1-r}} dx - \frac{\sin(u+v)}{u^r} \int_u^{\infty} \frac{\cos x}{(x+u)^{1-r}} dx \right| \\ &= \left| \frac{\sin(u+v)}{(u+v)^r} \int_{u+v}^{\infty} \left( \frac{1}{(x+u+v)^{1-r}} - \frac{1}{(x+u)^{1-r}} \right) \cos x dx \right. \\ &\quad \left. + \left( \frac{1}{u^r} - \frac{1}{(u+v)^r} \right) \sin(u+v) \int_{u+v}^{\infty} \frac{\cos x}{(x+u)^{1-r}} dx \right| \end{aligned}$$



$$\begin{aligned} & \left| -\frac{\sin(u+v)}{u^r} \int_u^{u+v} \frac{\cos x}{(x+u)^{1-r}} dx \right| \\ \leq & A(u+v)^{1-r} \int_{u+v}^{5\pi/2} \frac{v}{(x+u)^{2-r}} dx + A \frac{v(u+v)}{u^{1+r}} \left| \int_{u+v}^{\infty} \frac{\cos x}{(x+u)^{1-r}} dx \right| \\ & + A \frac{u+v}{u} \int_u^{u+v} dx \\ \leq & Av + Av/u^r + Av \end{aligned}$$

and  $W_3 = O(v)$  by an estimation similar to that used for  $S_2$ . Therefore

$$(10) \quad \int_0^\pi |g(u-t)| |R(u+v) - R(u)| du \leq Av.$$

Combining (8), (9) and (10), we get the required result (4). Thus we have completed the proof of Theorem 5.

3. *Proof of Theorem 6.* We write  $s = [\pi/t]$  and

$$\begin{aligned} \int_0^\pi t^{-r} |g(t)| dt & \leq \int_0^\pi t^{-r} \left| \sum_{n=1}^s b_n \sin nt \right| dt \\ & \quad + \int_0^\pi t^{-r} \left| \sum_{n=s+1}^\infty b_n \sin nt \right| dt = U + V \end{aligned}$$

where

$$\begin{aligned} U & \leq \int_0^\pi t^{1-r} \left| \sum_{n=1}^s nb_n \right| = \sum_{m=1}^\infty \int_{\pi/(m+1)}^{\pi/m} t^{1-r} \left| \sum_{n=1}^m nb_n \right| dt \\ & \leq A \sum_{m=1}^\infty \frac{1}{m^{3-r}} \sum_{n=1}^m n |B_n| \leq A \sum_{n=1}^\infty |b_n|/n^{1-r} \\ & \leq A \sum_{n=1}^\infty n^r |\Delta b_n| < \infty \end{aligned}$$

and

$$\begin{aligned} V & = \int_0^\pi t^{-r} \left| -b_{s+1} \tilde{D}_s(t) + \sum_{n=s+1}^\infty \Delta b_n \tilde{D}_n(t) \right| dt \\ & \leq A \sum_{m=1}^\infty \frac{|b_m|}{m^{1-r}} + A \sum_{m=1}^\infty \frac{1}{m^{1-r}} \sum_{n=m+1}^\infty |\Delta b_n| \\ & \leq A \sum_{n=1}^\infty n^r |\Delta b_n| < \infty. \end{aligned}$$

Thus we have proved  $t^{-r}g(t) \in L$  under the assumption

$$\sum_{n=1}^\infty n^r |\Delta b_n| < \infty.$$

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