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INTEGRABILITY OF TRIGONOMETRIC SERIES. II

MASAKO IZUMI AND SHIN-ICHI IZUMI

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There are proved theorems of the following type. Let $0<r<1$ and $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$ and suppose $\int_0^u g(t)\,|t-u|^{-r}dt \leqq A$ for all $u>0$. Then $\sum_{n=1}^{\infty} |b_n| n^{r-1} < \infty$.

1. Sz.-Nagy [7] (cf. [1]) has proved the following

THEOREM 1. *Let $0 < r < 1$ and $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$. If $g \downarrow$ and is bounded below, then*

$$(1) \qquad t^{-r}g(t) \in L(0, \pi) \Longleftrightarrow \sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty .$$

The same holds for even functions.

In this theorem, the assumption $g \downarrow$ cannot be replaced by $g \geqq 0$, that is,

THEOREM 2. ([1]), [4]) *Let $0 < r < 1$, then there is an odd function*

$$g \sim \sum_{n=1}^{\infty} b_n \sin nt$$

such that

(i) $g \geqq 0$ *on* $(0, \pi)$,
(ii) $t^{-r}g(t) \in L(0, \pi)$,

but

(iii) $\sum_{n=1}^{\infty} |b_n|/n^{1-r} = \infty$.

The same holds for even functions.

Sunouchi [6], Edmonds [3] and Boas [2] (cf. [1]) proved that

THEOREM 3. *Let $0 < r < 1$ and $b_n \downarrow 0$. Then the function g defined by $g(t) = \sum_{n=1}^{\infty} b_n \sin nt$, satisfies the relation* (1). *The same holds for even functions.*

The monotonicity of (b_n) in this theorem cannot be replaced by positivity, that is,

THEOREM 4. ([1], [4]) *Let $0 < r < 1$, then there is an odd function*

$$g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$$

such that

(i) $b_n \geqq 0$ (*or* $b_n \to 0$ *and* $\sum |\Delta b_n| < \infty$),

(ii) $\sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty$,

but

(iii) $t^{-r}g(t) \notin L(0, \pi)$.

The same holds for even functions.

We shall prove the following theorems.

THEOREM 5. *Let* $0 < r < 1$ *and* $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$, *then*

$$(2) \qquad \int_0^{\pi} \frac{g(t)}{|t-u|^r} dt \leqq A \text{ for all } u > 0 \Longrightarrow \sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty .$$

The same holds for even functions.

THEOREM 6. *Let* $0 < r < 1$ *and* $g(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$. *Then*

$$(3) \qquad \sum_{n=1}^{\infty} n^r |\Delta b_n| < \infty \Longrightarrow t^{-r}g(t) \in L(0, \pi) .$$

The same holds for even functions.

By Theorem 4, the converse implication of (2) does not hold. We have proved that [5], for $0 < r < 1$,

$$\sum_{n=1}^{\infty} |b_n|/n^{1-r} < \infty \Longrightarrow \exists \int_{+0}^{\pi} g(t)t^{-r}dt .$$

The left side of (2) is satisfied when

(i) $g \downarrow$ on $(0, \delta)$ and bounded on (δ, π) and $t^{-r}g(t) \in L(0, \pi)$, or

(ii) there is an odd function g_1 such that

$$|g(t)| \leqq A\, g_1(t) \text{ on } (0, \pi)$$

and g_1 satisfies the condition in (i), or

(iii) there is a finite set of points $(x_1, x_2, \cdots, x_n)$ on $(0, \pi)$ such that the odd function g becomes monotonically infinite on one side or both side of each x_i and is bounded outside of each neighbourhood of x_i and

$$|g(t)|/|t - x_i|^r \in L(0, \pi) \qquad (i = 1, 2, \cdots, n) .$$

(i) shows that Theorem 5 contains the $\int \to \sum$ part of Theorem 1 as a particular case. (ii) and (iii) are generalizations of (i).

If $b_n \downarrow 0$, the right side of (1) is equivalent to the left side of (3), and then the $\sum \to \int$ part of Theorem 3 is a particular case of Theorem 6.

2. *Proof of Theorem* 5. We consider the function

$$h(t) = \sum_{n=2}^{\infty} (-1)^n \sin nt/n^{1-r}, \tag{3}$$

then

$$k(t) = (g*h)(t) = \frac{2}{\pi}\int_0^{\pi} g(u)h(t+u)du \sim \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1-r}} b_n \cos nt .$$

If $k \in \text{Lip } 1$, then its Fourier series converges absolutely by Bernstein's theorem, which gives the conclusion of the theorem. Therefore it is sufficient to prove that

$$|k(t+v) - k(t)| \leqq Av \text{ for small } v > 0 \text{ and } 0 < t < \pi . \tag{4}$$

Now,

$$\begin{aligned} k(t+v) - k(t) &= \frac{2}{\pi}\int_0^{\pi} g(u)(h(t+u+v) - h(t+u))du \\ &= \frac{2}{\pi}\int_0^{\pi} g(u-t)(h(u+v) - h(u))du , \end{aligned} \tag{5}$$

where, by (3),

$$\begin{aligned} h(u) &= \sum_{n=1}^{\infty} \left(\frac{\sin 2nu}{(2n)^{1-r}} - \frac{\sin (2n+1)u}{(2n+1)^{1-r}} \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{(2n)^{1-r}} - \frac{1}{(2n+1)^{1-r}} \right) \sin 2nu \\ &\quad + (1 - \cos u) \sum_{n=1}^{\infty} \frac{\sin 2nu}{(2n+1)^{1-r}} - \sin u \sum_{n=1}^{\infty} \frac{\cos 2nu}{(2n+1)^{1-r}} \\ &= P(u) + Q(u) - R(u) . \end{aligned} \tag{6}$$

We write

$$p(u) = \frac{1}{(2u)^{1-r}} - \frac{1}{(2u+1)^{1-r}} \text{ for } u > 0$$

and

$$p(n) = p_n (n = 1, 2, \cdots) ,$$

then

$$\begin{aligned} P(u) &= \sum_{n=1}^{\infty} p_n \sin 2nu \\ &= \int_{1/2}^{\infty} p(w) \sin 2uw \, dw + \int_{1/2}^{\infty} p(w) \sin 2uw \, dj(w) \end{aligned}$$

where

$$j(w) = -w + [w] + \frac{1}{2} \sim \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin 2\pi m w}{m}$$

and then

$$\begin{aligned} P(u+v) - P(u) &= \left\{\int_{1/2}^{\infty} p(w)\sin 2(u+v)w\,dw - \int_{1/2}^{\infty} p(w)\sin 2uw\,dw\right\} \\ &\quad + \left\{\int_{1/2}^{\infty} p(w)\sin 2(u+v)w\,dj(w) - \int_{1/2}^{\infty} p(w)\sin 2uw\,dj(w)\right\} \\ &= S_1 + S_2 . \end{aligned}$$

We shall first show that $S_1 = 0(v/u^r)$.

$$\begin{aligned} S_1 &= \frac{1}{2(u+v)}\int_{u+v}^{\infty} p\Big(\frac{x}{2(u+v)}\Big)\sin x\,dx - \frac{1}{2u}\int_{u}^{\infty} p\Big(\frac{x}{2u}\Big)\sin x\,dx \\ &= \frac{1}{2}\int_{u+v}^{\infty}\Big(\frac{1}{u+v}p\Big(\frac{x}{2(u+v)}\Big) - \frac{1}{u}p\Big(\frac{x}{2u}\Big)\Big)\sin x\,dx \\ &\qquad - \frac{1}{2u}\int_{u}^{u+v} p\Big(\frac{x}{2u}\Big)\sin x\,dx \\ &= \frac{1}{2}\int_{u+v}^{\infty}\Big\{\Big(\frac{1}{(u+v)^r} - \frac{1}{u^r}\Big)\frac{1}{x^{1-r}} \\ &\qquad - \Big(\frac{1}{(u+v)^r(x+u+v)^{1-r}} - \frac{1}{u^r(x+u)^{1-r}}\Big)\Big\}\sin x\,dx \qquad (7) \\ &\quad - \frac{1}{2u^r}\int_{u}^{u+v}\Big(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}}\Big)\sin x\,dx \\ &= \frac{1}{2}\Big(\frac{1}{(u+v)^r} - \frac{1}{u^r}\Big)\int_{u+v}^{\infty}\Big(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}}\Big)\sin x\,dx \\ &\quad + \frac{1}{2(u+v)^r}\int_{u+v}^{\infty}\Big(\frac{1}{(x+u)^{1-r}} - \frac{1}{(x+u+v)^{1-r}}\Big)\sin x\,dx \\ &\quad - \frac{1}{2u^r}\int_{u}^{u+v}\Big(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}}\Big)\sin x\,dx \\ &= O(v/u^r) , \end{aligned}$$

since

$$\begin{aligned} &\int_{u+v}^{\infty}\Big(\frac{1}{x^{1-r}} - \frac{1}{(x+u)^{1-r}}\Big)\sin x\,dx \\ &= (1-\cos u)\int_{2u+v}^{\infty}\frac{\sin x}{x^{1-r}}dx + \int_{2u+v}^{\infty}\frac{\sin u\cdot\cos x}{x^{1-r}}dx \\ &\qquad + \int_{u+v}^{2u+v}\frac{\sin x}{x^{1-r}}dx \end{aligned}$$

and similarly for the integral of the second term of the right side of (7). The estimation of S_2 is a little more complicated. By integration by parts,

$$
\begin{aligned}
S_2 &= \int_{1/2}^{\infty} p(w) \sin 2(u+v)w \, dj(w) - \int_{1/2}^{\infty} p(w) \sin 2uw \, dj(w) \\
&= -\int_{1/2}^{\infty} p'(w) \sin 2(u+v)w \, j(w) \, dw \\
&\quad - 2(u+v)\int_{1/2}^{\infty} p(w) \cos 2(u+v)w \, j(w) \, dw \\
&\quad + \int_{1/2}^{\infty} p'(w) \sin 2uw \, j(w) \, dw \\
&\quad + 2u \int_{1/2}^{\infty} p(w) \cos 2uw \, j(w) \, dw ,
\end{aligned}
$$

and using the Fourier expansion of $j(w)$, we get

$$
\begin{aligned}
S_2 = -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\Big\{\int_{1/2}^{\infty} p'(w)&(\sin 2(u+v)w \cdot \sin 2\pi m w \\
&- \sin 2uw \cdot \sin 2\pi m w) dw \\
&- 2\int_{1/2}^{\infty} p(w)((u+v)\cos 2(u+v)w \cdot \sin 2\pi m w \\
&- u \cos 2uw \sin 2\pi m w) dw\Big\} \\
= \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\{2(1-r)T_1 + 2T_2\} &.
\end{aligned}
$$

In order to get $S_2 = O(v)$, it is sufficient to prove that T_1 and T_2 are of order $O(v/m)$. Now, for $m \geqq 3$, we have

$$
\begin{aligned}
T_1 = \int_{1/2}^{\infty}&\Big(\frac{1}{(2w)^{2-r}} - \frac{1}{(2w+1)^{2-r}}\Big) \\
&\cdot\{(\cos 2(m\pi - u - v)w - \cos 2(m\pi - u)w) \\
&- (\cos 2(m\pi + u + v)w - \cos (m\pi + u)w)\}dw \\
= \frac{1}{2}\Big\{&\int_{m\pi-u-v}^{\infty}\Big(\Big(\frac{m\pi-u-v}{x}\Big)^{2-r} - \Big(\frac{m\pi-u-v}{x+m\pi-u-v}\Big)^{2-r}\Big)\frac{\cos x}{m\pi-u-v}dx \\
&- \int_{m\pi-u}^{\infty}\Big(\Big(\frac{m\pi-u}{x}\Big)^{2-r} - \Big(\frac{m\pi-u}{x+m\pi-u}\Big)^{2-r}\Big)\frac{\cos x}{m\pi-u}dx\Big\} \\
- \frac{1}{2}\Big\{&\int_{m\pi+u+v}^{\infty}\Big(\Big(\frac{m\pi+u+v}{x}\Big)^{2-r} - \Big(\frac{m\pi+u+v}{x+m\pi+u+v}\Big)^{2-r}\Big)\frac{\cos x}{m\pi+u+v}dx \\
&- \int_{m\pi+u}^{\infty}\Big(\Big(\frac{m\pi+u}{x}\Big)^{2-r} - \Big(\frac{m\pi+u}{x+m\pi+u}\Big)^{2-r}\Big)\frac{\cos x}{m\pi+u}dx\Big\} \\
= \frac{1}{2}\Big\{&\int_{m\pi-u}^{\infty}\Big(\frac{(m\pi-u-v)^{1-r}}{x^{2-r}} - \frac{(m\pi-u)^{1-r}}{x^{2-r}} \\
&- \frac{(m\pi-u-v)^{1-r}}{(x+m\pi-u-v)^{2-r}} + \frac{(m\pi-u)^{1-r}}{(x+m\pi-u)^{2-r}}\Big)\cos x \, dx + O(v/m)\Big\} \\
- \frac{1}{2}&\{\text{similar terms}\} \\
= O(v/m)&
\end{aligned}
$$

$$
\begin{aligned}
T_2 &= \int_{1/2}^{\infty} p(w)\{(u+v)(\sin 2(m\pi+u+v)w + \sin 2(m\pi-u-v)w) \\
&\qquad - u(\sin 2(m\pi+u)w + \sin 2(m\pi-u)w)\}\,dw \\
&= \frac{1}{2}\Big\{(u+v)\int_{m\pi+u+v}^{\infty}\Big(\Big(\frac{m\pi+u+v}{x}\Big)\Big)^{1-r} \\
&\qquad\qquad -\Big(\frac{m\pi+u+v}{x+m\pi+u+v}\Big)^{1-r}\Big)\frac{\sin x}{m\pi+u+v}dx \\
&\qquad - u\int_{m\pi+u}^{\infty}\Big(\Big(\frac{m\pi+u}{x}\Big)^{1-r} - \Big(\frac{m\pi+u}{x+m\pi+u}\Big)^{1/r}\Big)\frac{\sin x}{m\pi+u}dx\Big\} \\
&\quad + \frac{1}{2}\{\text{similar terms}\} \\
&= O(v/m).
\end{aligned}
$$

When $m = 1$ or 2, we get easily

$$
|T_1| \leqq A\int_{1/2}^{\infty}\frac{v}{w^{2-r}}dw \leqq Av,\ |T_2| \leqq Av\ .
$$

Thus we have proved $T_1 + T_2 = O(v/m)$ for all m, and then $S_2 = O(v)$. Collecting the above estimates, we get

$$
(8)\qquad \int_0^{\pi}|g(u-t)|\,|P(u+v)-P(u)|\,du \leqq Av\int_0^{\pi}|g(u-t)|\,u^{-r}du \leqq Av\ .
$$

Now, by (6),

$$
\begin{aligned}
Q(u+v)-Q(u) &= (1-\cos(u+v))\sum_{n=1}^{\infty}\frac{\sin 2n(u+v)}{(2n+1)^{1-r}} \\
&\qquad - (1-\cos u)\sum_{n=1}^{\infty}\frac{\sin 2nu}{(2n+1)^{1-r}} \\
&= (\cos u - \cos(u+v))\sum_{n=1}^{\infty}\frac{\sin 2n(u+v)}{(2n+1)^{1-r}} \\
&\quad + (1-\cos u)\Big(\sum_{n=1}^{\infty}\frac{\sin 2n(u+v)}{(2n+1)^{1-r}} - \sum_{n=1}^{\infty}\frac{\sin 2nu}{(2n+1)^{1-r}}\Big) \\
&= U + (1-\cos u)\,V
\end{aligned}
$$

where

$$
|U| \leqq v(u+v)\,|\sum_{n=1}^{\infty}\Delta\Big(\frac{1}{(2n+1)^{1-r}}\Big)\tilde{D}_n(2u+v))| \leqq Av\ ,
$$

$\tilde{D}_n$ being the nth conjugate Dirichlet kernel, and

$$
\begin{aligned}
V &= \Big\{\int_{1/2}^{\infty}\frac{\sin 2(u+v)w}{(2w+1)^{1-r}}dw - \int_{1/2}^{\infty}\frac{\sin 2uw}{(2w+1)^{1-r}}dw\Big\} \\
&\quad + \Big\{\int_{1/2}^{\infty}\frac{\sin 2(u+v)w}{(2w+1)^{1-r}}dj(w) - \int_{1/2}^{\infty}\frac{\sin 2uw}{(2w+1)^{1-r}}dj(w)\Big\} \\
&= V_1 + V_2.
\end{aligned}
$$

It is sufficient to prove that $(1 - \cos u)V = 0(v)$. Proceeding as for S_1 and S_2, we get

$$V_1 = \frac{1}{2(u+v)^r}\int_{u+v}^{\infty} \frac{\sin x}{(x+u+v)^{1-r}}dx - \frac{1}{2u^r}\int_u^{\infty}\frac{\sin x}{(x+u)^{1-r}}dx$$
$$= O(v/u)$$

and

$$V_2 = O(v) .$$

Therefore

$$Q(u+v) - Q(u) = O(v) ,$$

and

$$(9) \qquad \int_0^{\pi} |g(u-t)|\,|Q(u+v) - Q(u)|\,du \leqq Av\int_0^{\pi}|g(u)|\,du \leqq Av .$$

Finally we shall estimate $R(u+v) - R(u)$. By the definition (6),

$$R(u) = \sin u \sum_{n=1}^{\infty}\frac{\cos 2nu}{(2n+1)^{1-r}}$$
$$= \sin u \int_{1/2}^{\infty}\frac{\cos 2uw}{(2w+1)^{1-r}}dw + \sin u\int_{1/2}^{\infty}\frac{\cos 2uw}{(2w+1)^{1-r}}dj(w)$$

and

$$R(u+v) - R(u) = (\sin(u+v) - \sin u)\int_{1/2}^{\infty}\frac{\cos 2uw}{(2w+1)^{1-r}}dm$$
$$+ \sin(u+v)\int_{1/2}^{\infty}\frac{\cos 2(u+v)w - \cos 2uw}{(2w+1)^{1-r}}dw$$
$$+ \Big\{\sin(u+v)\int_{1/2}^{\infty}\frac{\cos 2(u+v)w}{(2w+1)^{1-r}}dj(w)$$
$$- \sin u\int_{1/2}^{\infty}\frac{(2w+1)^{1-r}}{\cos 2uw}dj(w)\Big\}$$
$$= W_1 + W_2 + W_3$$

where

$$|W_1| \leqq A\frac{v}{u^r}\left|\int_u^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx\right| \leqq A\frac{v}{u^r} ,$$

$$|W_2| = \left|\frac{\sin(u+v)}{(u+v)^r}\int_{u+v}^{\infty}\frac{\cos x}{(x+u+v)^{1-r}}dx - \frac{\sin(u+v)}{u^r}\int_u^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx\right|$$
$$= \left|\frac{\sin(u+v)}{(u+v)^r}\int_{u+v}^{\infty}\left(\frac{1}{(x+u+v)^{1-r}} - \frac{1}{(x+u)^{1-r}}\right)\cos x\,dx\right.$$
$$\left. + \left(\frac{1}{u^r} - \frac{1}{(u+v)^r}\right)\sin(u+v)\int_{u+v}^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx\right.$$

$$
\begin{aligned}
&- \frac{\sin(u+v)}{u^r}\int_u^{u+v}\frac{\cos x}{(x+u)^{1-r}}dx\Big| \\
&\leqq A(u+v)^{1-r}\int_{u+v}^{5\pi/2}\frac{v}{(x+u)^{2-r}}dx + A\frac{v(u+v)}{u^{1+r}}\left|\int_{u+v}^{\infty}\frac{\cos x}{(x+u)^{1-r}}dx\right| \\
&\quad + A\,\frac{u+v}{u}\int_u^{u+v}dx \\
&\leqq Av + Av/u^r + Av
\end{aligned}
$$

and $W_3 = O(v)$ by an estimation similar to that used for S_2. Therefore

$$
\int_0^{\pi}|g(u-t)|\,|R(u+v)-R(u)|\,du \leqq Av\,. \tag{10}
$$

Combining (8), (9) and (10), we get the required result (4). Thus we have completed the proof of Theorem 5.

3. *Proof of Theorem* 6. We write $s = [\pi/t]$ and

$$
\begin{aligned}
\int_0^{\pi}t^{-r}|g(t)|\,dt &\leqq \int_0^{\pi}t^{-r}\left|\sum_{n=1}^{s}b_n\sin nt\right|dt \\
&\quad + \int_0^{\pi}t^{-r}\left|\sum_{n=s+1}^{\infty}b_n\sin nt\right|dt = U + V
\end{aligned}
$$

where

$$
\begin{aligned}
U &\leqq \int_0^{\pi}t^{1-r}\left|\sum_{n=1}^{s}nb_n\right| = \sum_{m=1}^{\infty}\int_{\pi/(m+1)}^{\pi/m}t^{1-r}\left|\sum_{n=1}^{m}nb_n\right|dt \\
&\leqq A\sum_{m=1}^{\infty}\frac{1}{m^{3-r}}\sum_{n=1}^{m}n|B_n| \leqq A\sum_{n=1}^{\infty}|b_n|/n^{1-r} \\
&\leqq A\sum_{n=1}^{\infty}n^r|\Delta b_n| < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
V &= \int_0^{\pi}t^{-r}\Big|-b_{s+1}\tilde{D}_s(t) + \sum_{n=s+1}^{\infty}\Delta b_n\tilde{D}_n(t)\Big|dt \\
&\leqq A\sum_{m=1}^{\infty}\frac{|b_m|}{m^{1-r}} + A\sum_{m=1}^{\infty}\frac{1}{m^{1-r}}\sum_{n=m+1}^{\infty}|\Delta b_n| \\
&\leqq A\sum_{n=1}^{\infty}n^r|\Delta b_n| < \infty\,.
\end{aligned}
$$

Thus we have proved $t^{-r}g(t) \in L$ under the assumption

$$
\sum_{n=1}^{\infty}n^r|\Delta b_n| < \infty\,.
$$

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