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FIXED POINT THEOREMS FOR POINT-TO-SET MAPPINGS AND THE SET OF FIXED POINTS

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# FIXED POINT THEOREMS FOR POINT-TO-SET MAPPINGS AND THE SET OF FIXED POINTS

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Let X be a Banach space and K be a nonempty convex weakly compact subset of X. Belluce and Kirk proved that (1) If  $f: K \to K$  is continuous,  $\inf_{x \in K} ||x-f(x)|| = 0$ and I-f is a convex mapping, then f has a fixed point in K. (2) If  $f: K \to K$  is nonexpansive and I-f is a convex mapping on K, then f has a fixed point in K. In this paper the concept of convex mapping has been extended to pointto-set mappings. Theorems 1 and 2 in § 2 extend the above fixed point theorems by Belluce and Kirk.

Let W stand for the set of fixed points of  $f: K \to cc(K)$ . The set W is called a singleton in a generalized sense if there is  $x_0 \in W$  such that  $W \subset f(x_0)$ . In §3 two examples are given to show that W is not necessarily a singleton in a generalized sense if f is strictly nonexpansive or if I - f is convex. But one can be sure that W is a convex set if I - f is a convex or a semiconvex mapping.

1. Preliminaries.

NOTATIONS AND DEFINITIONS. Let X be a topological space, define

- 1.  $2^{x}$  = the family of all nonempty closed subsets of X.
- 2.  $b(X) = \{A \in 2^x; A \text{ is bounded}\}, \text{ where } X \text{ is a metric space.}$
- 3.  $k(X) = \{A \in 2^X; A \text{ is convex}\}, \text{ where } X \text{ is a linear topological space.}$
- 4.  $cpt(X) = \{A \in 2^X; A \text{ is compact}\}.$
- 5.  $cc(X) = k(X) \cap cpt(X)$ , where X is a linear topological space.

In the remainder of this section we assume X to be a metric space with metric d, unless otherwise stated.

- 6. Let  $x \in X$  and r > 0, define  $S(x, r) = \{y \in X; d(y, x) < r\}$ .
- 7. For  $x \in X$ ,  $A \in 2^x$ , define  $d(x, A) = \inf \{ d(x, y); y \in A \}$ .
- 8. Given  $A \in 2^{x}$  and r > 0, define  $V_{r}(A) = \{x \in X; d(x, A) < r\}$ .

LEMMA 1. Let  $x, y \in X$  and let A be a nonempty subset of X. Then  $d(x, A) \leq d(x, y) + d(y, A)$ .

This is a simple consequence of the triangle inequality.

DEFINITION 1. Let X be a topological space. A mapping

 $f: X \to 2^x$  is said to be upper semicontinuous (abbreviated by u.s.c.) at  $x_0$  if for any open set U containing  $f(x_0)$ , there exists a neighborhood V of  $x_0$  such that  $f(y) \subset U$  for any  $y \in V$ . The mapping f is said to be u.s.c. in X if it is u.s.c. at any x in X.

DEFINITION 2. A map  $f: X \to b(X)$  is continuous if it is continuous from the metric topolgy of X to the Hausdorff metric topology of b(X).

DEFINITION 3. A mapping  $f: X \to b(X)$  is nonexpansive on X if  $D(f(x), f(y)) \leq d(x, y)$  for any x, y in X, where D is the Hausdorff metric on b(X).

DEFINITION 4. A mapping  $f: X \to b(X)$  is a contraction mapping if there is  $0 \le k < 1$ , such that  $D(f(x), f(y)) \le kd(x, y)$  for any  $x, y \in X$ .

It is clear that a nonexpansive mapping  $f: X \rightarrow b(X)$  is continuous. For the relation between a continuous map and an upper semicontinuous map, we have the following:

**PROPOSITION 1.** If  $f: X \rightarrow cpt(X)$  is continuous, then it is upper semicontinuous.

REMARK 1. The condition that the values of f are compact subsets is not removable in the above proposition. As a matter of fact a nonexpansive mapping f on X into  $2^x$  may fail to be upper semicontinuous. Examples like the following seem to be in the folklore.

EXAMPLE 1. Let  $X = [0, 1] \times [0, 1] - \{(0, 1)\}$  with the usual metric. Let  $(x, y) \in X$ , define

 $f((x, y)) = \begin{cases} \text{the segment } \{(x, z); z \in [0, 1]\} \text{ if } x \neq 0 \text{ .} \\ \text{the segment } \{(0, z); z \in [0, 1)\} \text{ if } x = 0 \text{ .} \end{cases}$ 

Then  $f: X \to 2^x$  is nonexpansive on X, but it is not u.s.c. at (0, y) for any  $y \in [0, 1)$ . Because if we take

$$U=\{(x,\,y)\in X;\,x+\,y<1\}$$
 ,

then U is open and contains f((0, y)). However U does not contain f((x, z)) for  $(x, z) \in X$  and  $x \neq 0$ . Therefore no neighborhood of (0, y) exists such that U contains the image of f at every point of the neighborhood. That is, f is not u.s.c. at (0, y).

DEFINITION 5. A real valued function g on X is said to be lower semicontinuous on X if for any real number a, the set

$$\{x \in X; g(x) > a\}$$

is open in X.

**PROPOSITION 2.** If  $f: X \to 2^x$  is upper semicontinuous, then the function g, where g(x) = d(x, f(x)), is lower semicontinuous.

*Proof.* Let a be a real number and  $x_0 \in A = \{x; g(x) > a\}$ . We want to prove that A is an open set. Let  $r = g(x_0) - a$ , then r > 0 and the open set  $V_{r/3}(f(x_0))$  contains  $f(x_0)$ . By the upper semicontinuity of f, there exists a neighborhood V of  $x_0$  such that

$$f(y) \subset V_{r/3}(f(x_0))$$

for any  $y \in V$ . We may assume  $V \subset S(x_0, r/3)$ . Let  $U = V_{r/3}(f(x_0))$ . Then  $z \in U$  implies

$$d(x_0, z) \ge d(x_0, f(x_0)) - d(z, f(x_0)) \quad ext{(by Lemma 1)} \ > r + a - r/3 = a + 2r/3$$
 .

Therefore

$$d(x_{\scriptscriptstyle 0},\ U) = \inf \left\{ d(x_{\scriptscriptstyle 0},\ z);\ z\in U 
ight\} \geqq a \,+\, 2r/3$$
 .

Thus  $y \in V$  implies

$$d(y, f(y)) \ge d(y, U) \ge d(x_0, U) - d(x_0, y)$$
 (by Lemma 1)  
 $\ge a + 2r/3 - r/3 = a + r/3 > a$ .

Hence  $y \in V$  implies  $y \in A$ . Thus A is open. Therefore g is lower semicontinuous.

2. Fixed point theorems. First we state a well known fixed point theorem for a point-to-set contraction mapping (cf. [5] p. 479 for the proof): Let K be a nonempty bounded closed subset of a complete metric space (X, d). If  $f: K \rightarrow b(K)$  is a contraction mapping, then f has a fixed point in K.

The space X in the sequel is assumed to be a Banach space unless otherwise stated.

DEFINITION 6. A mapping f from X into  $2^{x}$  is said to be convex if for any  $x, y \in X$  and  $m = \lambda x + (1 - \lambda)y$  with  $0 \leq \lambda \leq 1$ , and any  $x_1 \in f(x), y_1 \in f(y)$ , there exists  $m_1 \in f(m)$  such that

$$\parallel m_1 \parallel \leq \lambda \parallel x_1 \parallel + (1-\lambda) \parallel y_1 \parallel$$
 .

DEFINITION 7. A mapping  $f: X \to 2^x$  is called semiconvex on X if for any  $x, y \in X$ ,  $m = \lambda x + (1 - \lambda)y$ , where  $0 \le \lambda \le 1$ , and any  $x_1 \in f(x), y_1 \in f(y)$ , there exists  $m_1 \in f(m)$  such that

$$|| m_1 || \le \max \{ || x_1 ||, || y_1 || \}$$
 .

REMARK 2. A convex mapping is semiconvex, but the converse is not true. Take the mapping  $f(x) = \sqrt{x}$ ,  $x \in [0, 1]$ , for instance. The map f is semiconvex because it is strictly increasing. But f is not convex, for example take x = 1 and y = 0,

$$m \, = \, 1/4 \, = \, 1/4 \cdot 1 \, + \, 3/4 \cdot 0$$
 ,

then f(1) = 1, f(0) = 0, but

$$f(m) = \sqrt{1/4} = 1/2 \measuredangle 1/4 f(1) + 3/4 f(0) = 1/4$$
 .

LEMMA 2. Let  $f: X \to 2^x$ , and let  $I: X \to X$  be the identity mapping. If I - f, where  $(I - f)(x) = \{x - y; y \in f(x)\}$ , is convex (semiconvex), then for any  $x, y \in X$  and  $m = \lambda x + (1 - \lambda)y$ ,  $0 \leq \lambda \leq 1$ , we have

$$d(m, f(m)) \leq \lambda d(x, f(x)) + (1 - \lambda) d(y, f(y)) .$$
  
 $(d(m, f(m)) \leq \max \{ d(x, f(x)), d(y, f(y)) \} ).$ 

*Proof.* Let  $x_n \in f(x)$  be such that  $||x_n - x|| \to d(x, f(x))$  and  $y_n \in f(y)$  be such that  $||y_n - y|| \to d(y, f(y))$ . Let I - f be a convex mapping, then there exists  $m_n \in f(m)$  such that

$$\parallel m - m_n \parallel \leq \lambda \parallel x - x_n \parallel + (1 - \lambda) \parallel y - y_n \parallel$$
 .

Now

$$d(m, f(m)) \leq \inf_{n \geq 1} || m - m_n || \leq \lambda || x - x_n || + (1 - \lambda) || y - y_n ||$$

for any  $n \ge 1$ . Thus

$$egin{aligned} d(m,\,f(m)) &\leq \lambda \,||\, x - x_n \,|| + (1 - \lambda) \,||\, y - y_n \,|| \ &\longrightarrow \lambda d(x,\,f(x)) + (1 - \lambda) d(y,\,f(y)) \;. \end{aligned}$$

Similarly one can prove that

$$d(m, f(m)) \leq \max \{ d(x, f(x)), d(y, f(y)) \},\$$

if I - f is semiconvex.

LEMMA 3. Let  $f: X \rightarrow cpt(X)$  be a mapping such that for any  $x, y \in X$  and any  $m = \lambda x + (1 - \lambda)y, 0 \leq \lambda \leq 1$ , we have

$$d(m, f(m)) \leq \lambda d(x, f(x)) + (1 - \lambda) d(y, f(y))$$

 $(d(m, f(m)) \leq \max \{ d(x, f(x)), d(y, f(y)) \}$  respectively).

Then I - f is a convex mapping (semiconvex mapping respectively).

*Proof.* Let  $x_1 \in f(x)$ ,  $y_1 \in f(y)$ ; we have

$$d(x, f(x)) \leq ||x - x_1||$$
 and  $d(y, f(y)) \leq ||y - y_1||$ .

Since f(m) is compact, there is an  $m_1 \in f(m)$  such that

$$||m - m_1|| = d(m, f(m)) \leq \lambda d(x, f(x)) + (1 - \lambda) d(y, f(y))$$
.

Therefore  $||m - m_1|| \leq \lambda ||x - x_1|| + (1 - \lambda) ||y - y_1||$ . Hence I - f is a convex mapping. Similarly one can prove, under the condition that  $d(m, f(m)) \leq \max \{ d(x, f(x)), d(y, f(y)) \}$ , that I - f is a semiconvex mapping.

Lemmas 2 and 3 characterize the convexity (semiconvexity) of I - f in terms of the distance between a point and its image under f, where f is a mapping from X into cpt(X). The following lemma is a simple consequence of Lemma 2.

LEMMA 4. Let  $f: X \rightarrow 2^{X}$ , define

$$H_r = \{x \in X \colon d(x, f(x)) \leq r\}$$
,

where  $r \ge 0$ . If I - f is a semiconvex mapping on X, then  $H_r$  is convex.

THEOREM 1. Let K be a nonempty weakly compact closed convex subset of X. If  $f: K \to 2^{\kappa}$  is upper semicontinuous and

$$\inf \{ d(x, f(x)); x \in K \} = 0$$
,

and I - f is a semiconvex mapping on K, then f has a fixed point in K.

**Proof.** Let r > 0, define  $H_r$  as in Lemma 4. We see that  $H_r \approx \emptyset$  for any r > 0, since  $\inf \{d(x, f(x)); x \in K\} = 0$ . As f is upper semicontinuous,  $H_r$  is closed (by Proposition 2). The map I - f is semiconvex, hence  $H_r$  is convex (by Lemma 4). The set  $H_r$ , being closed and convex, is weakly closed for each r > 0. The family  $\{H_r; r > 0\}$  has the finite intersection property. Therefore, by the weak compactness of K, we have  $\bigcap_{r>0} H_r \approx \emptyset$ . It is clear that any point in  $\bigcap_{r>0} H_r$  is a fixed point of f.

REMARK 3. A convex mapping is semiconvex, therefore Theorem 1 extends Theorem 4.1 of Belluce and Kirk [1]. Example 4.1 and 4.2 in [1], though they are point-to-point mappings, serve the purposes of demonstrating that "inf  $\{d(x, f(x)); x \in K\} = 0$ " or "K is weakly compact" in Theorem 1 is indispensable. The following example, which is a special case of the example given by Kirk [4], shows that the semiconvexity of I - f in Theorem 1 can not be removed.

EXAMPLE 2. Let  $K = \{x \in l_2; ||x|| \leq 1\}$  be the closed unit sphere of the Hilbert space  $l_2$ . Then K is closed, convex and weakly compact. Define f on K as follows: Let  $x = (x_1, x_2, \dots) \in K$ , and let

$$f(x) = (1 - ||x||, x_1, x_2, \cdots)$$
.

Then  $|| f(x) || \le 1$  and  $|| f(x) - f(y) || \le \sqrt{2} || x - y ||$ . i.e., f is a continuous mapping on K into K. We claim that

$$\inf \{ || x - f(x) ||; x \in K \} = 0.$$

Let  $x^{(n)} = (x_1, x_2, \dots) \in l_2$  be such that  $x_1 = x_2 = \dots = x_{n^2} = 1/n$  and  $x_i = 0$  for  $i > n^2$ . Then  $||x^{(n)}|| = 1$  and

$$f(x^{(n)}) = (0, x_1, x_2, \cdots, x_{n^2}, 0, \cdots)$$
.

We see that

$$||x^{(n)} - f(x^{(n)})|| = \sqrt{2}/n \rightarrow 0$$
, as  $n \rightarrow \infty$ .

Hence  $\inf \{ || x - f(x) ||; x \in K \} = 0$ . But I - f is neither convex nor semiconvex. For instance, let  $x = (1/2, 1/2, 0, \cdots), y = (-1/2, -1/2, 0, \cdots)$ . Then  $f(x) = (1 - \sqrt{2}/2, 1/2, 1/2, 0, \cdots), f(y) = (1 - \sqrt{2}/2, -1/2, -1/2, 0, \cdots), || x - f(x) || = (\sqrt{4 - 2\sqrt{2}})/2 < 1, || y - f(y) || = (\sqrt{12 - 6\sqrt{2}}/2 < 1)$ . Take m = 1/2(x + y), then  $m = (0, 0, \cdots)$  and  $f(m) = (1, 0, \cdots)$ . Thus

$$\mid\mid m-f(m)\mid\mid =1>\max\left\{\mid\mid x-f\{x)\mid\mid,\mid\mid y-f(y)\mid\mid
ight\}.$$

Therefore I - f is not semiconvex and hence it is not convex. The map f has no fixed point, for if f(x) = x, where  $x = (x_1, x_2, \dots) \in K$ , then  $x_1 = x_2 = \dots$ , and  $\sum_{i=1}^{\infty} x_i^2 < \infty$ . Thus  $x_i = 0$  for  $i \ge 1$ . But then  $f(x) = (1, 0, \dots) \rightleftharpoons (0, 0, \dots)$ .

DEFINITION 8. A map  $f: X \to 2^x$  is said to be asymptotically regular at  $x_0$  if there exists a sequence of points such that  $x_n \in f(x_{n-1})$ and  $||x_n - x_{n-1}|| \to 0$  as  $n \to \infty$ .

Definition 8 is an extension of the definition of asymtotically

regular point-to-point mapping given by Browder and Petryshyn [2]. One immediate result of Theorem 1 is the following corollary which extends the first part of Theorem 4.3 by Belluce and Kirk [1].

COROLLARY 1. If  $f: K \to 2^{\kappa}$  is asymptotically rgular at some point in K, where K is a nonempty closed convex weakly compact subset of X, and if f is upper semicontinuous in K such that I - fis semiconvex, then f has a fixed point in K.

*Proof.* Assume f is asymptotically regular at  $x_0 \in K$ ; then there exists  $x_n \in K$  such that  $x_n \in f(x_{n-1})$ ,  $n \ge 1$ , and  $||x_n - x_{n-1}|| \to 0$ . Since  $d(x_n, f(x_n)) \le ||x_{n+1} - x_n|| \to 0$ , we have  $\inf \{d(x, f(x)); x \in K\} = 0$ ; hence Corollary 1 follows Theorem 1.

THEOREM 2. Let K be a nonempty weakly compact convex subset of X. If  $f: K \rightarrow cc(K)$  is nonexpansive and if I - f is semiconvex on K, then f has a fixed point in K.

*Proof.* The map f is nonexpansive, so it is upper semi-continuous (by Proposition 1). Theorem 2 follows Theorem 1 provided that the condition "inf  $\{d(x, f(x)); x \in K\} = 0$ " is satisfied. To prove this condition we have the following lemma.

LEMMA 5. Let K be a nonempty bounded closed convex subset of X. If  $f: K \rightarrow b(K)$  is nonexpansive, then  $\inf \{d(x, f(x)); x \in K\} = 0$ .

*Proof.* Let  $x_0 \in K$ . Denote  $K_0 = \{x - x_0; x \in K\}$ , then  $K_0$  is a bounded closed convex subset of X and  $K_0$  contains 0. Let  $0 \leq k < 1$ , define  $f_k$  on  $K_0$  as follows:

$$f_k(x - x_0) = k(f(x) - x_0)$$
.

Then  $f_k(x - x_0) \subset K_0$  for any  $x - x_0 \in K_0$ , since  $K_0$  is convex and contains zero element. As f is nonexpansive,  $f_k$  is contraction. By the fixed point theorem for point-to-set contraction mapping, there exists  $x_k \in K$  such that

$$x_k - x_0 \in f_k(x_k - x_0) = k(f(x_k) - x_0)$$
 .

Thus there is  $y_k \in f(x_k)$  such that  $x_k - x_0 = k(y_k - x_0)$ . Now

$$egin{aligned} d(x_k,\,f(x_k)) &= \inf \left\{ ||\, x_k - y\,||;\, y \in f(x_k) 
ight\} \leq ||\, x_k - y_k\,||| \ &= ||\, x_0 + k(y_k - x) - y_k\,|| = (1-k)\,||\, y_k - x_0\,|| \;. \end{aligned}$$

Therefore

$$egin{aligned} 0 &\leq \inf_{x \, \in \, K} \, d(x, \, f(x)) \, \leq \inf_{0 \leq k < 1} \, d(x_k, \, f(x_k)) \ &\leq \inf_{0 \leq k < 1} \, (1 - k) \, || \, x_0 - y_k \, || = 0 \; , \end{aligned}$$

since the set  $\{||x_0 - y_k||; 0 \le k < 1\}$  is bounded. Hence

$$\inf \{ d(x, f(x)); x \in K \} = 0$$
.

3. The set of fixed points of a point-to-set mapping. Let K be a closed convex subset of a Banach space X. Denote by W the set of fixed points of a mapping  $f: K \to 2^{\kappa}$ . Throught this section we assume W to be nonempty.

DEFINITION 9. A mapping  $f: X \to b(X)$  is strictly nonexpansive if D(f(x), f(y)) < ||x - y|| for any  $x, y \in X$  and  $x \rightleftharpoons y$ .

If f is a point-to-point mapping, then the following properties are true.

(A) If f is strictly nonexpansive, then W is a singleton.

(B) If f is nonexpansive and the norm of the Banach space is strictly convex, then W is convex.

Statement (A) is no longer true for point-to-set mapping. For example, let K be a set containing more than two points, then the set of fixed points of the mapping  $f: K \to 2^{\kappa}$ , such that f(x) = K for any  $x \in K$ , is K itself which is not a singleton.

Statement (B) is obviously not true for a point-to-set mapping. However, as the next example shows, statement (B) is also not true for point-to-set mappings such that the image of each point is a nonempty compact convex set; note that the domain K in our example is also convex.

EXAMPLE 3. Let  $K = [0, 1] \times [0, 1]$  with the usual norm. Define  $f: K \rightarrow cc(K)$  by

$$f((x_1, x_2)) =$$
 the triangle with vertices  
(0, 0),  $(x_1, 0)$  and  $(0, x_2)$ .

Note that  $f((x_1, x_2))$  is a degenerate triangle if  $x_1x_2 = 0$ . We see that f is nonexpansive and the norm in  $R^2$  is strictly convex. But the set W of fixed points of f is

$$W = \{(x_1, x_2); (x_1, x_2) \in K \text{ and } x_1x_2 = 0\}$$

which is not convex.

For a point-to-set mapping f, we have several choices for values of f, e.g.,  $f(x) \in k(X)$ ,  $f(x) \in cpt(X)$  or  $f(x) \in cc(X)$ ; among them,  $f(x) \in cc(X)$  is the strongest assumption. For example, let K be a compact convex subset of X, and let  $g: X \rightarrow cpt(X)$  be an upper semicontinuous mapping such that  $g(x) \subset K$  for any  $x \in K$ , then g does not always have a fixed point (e.g., the map G of Strother [6], p. 990). But if we simply change g as a mapping into cc(X) instead of into cpt(X), then g has a fixed point (see K. Fan [3]). In Example 3, although we have imposed the strongest condition on the values of f, i.e.,  $f(x) \in cc(K)$ , that condition does not force f to satisfy statement (B). However the following proposition shows us a sufficient condition for W to be convex.

**PROPOSITION 3.** Let  $f: K \rightarrow 2^{K}$  be a mapping such that I - f is a semiconvex mapping on K. Then W is convex.

*Proof.* If I - f is semiconvex on K, then Lemma 4 shows that the set  $H_r = \{x \in K; d(x, f(x)) \leq r\}$  is convex. Hence  $W = H_0$  is convex.

Statement (A) can be rephrased as follows:

(A') If f is strictly nonexpansive, then there is  $x_0$  in W such that  $W \subset f(x_0)$ .

For a point-to-point mapping f, statement (A') implicitly shows W to be a singleton. As for a point-to-set mapping f, statement (A') does not require W to be a singleton, and on the other hand it does not rule out the possibility that W is a singleton. Therefore, it is reasonable to define W to be a singleton in a generalized sense if there exists  $x_0 \in W$  such that  $W \subset f(x_0)$ . Unfortunately even for a strictly nonexpansive mapping f on K into cc(K), the set W of fixed points of f is not necessarily a singleton in a generalized sense.

EXAMPLE 4. Let  $K = [0, 1] \times [0, 1]$ , a subset of  $R^2$  with the usual metric. Define  $f; K \rightarrow cc(K)$  as follows:

 $f((x_1, x_2)) =$  the triangle with vertices  $(x_1/2, 0), (x_1/2, 1) \text{ and } (1, 0).$ 

Let  $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in K$ , with  $x \neq y$ , then

$$D(f(x), f(y)) = 1/2 |x_1 - y_1| < d(x, y)$$
.

Hence f is strictly nonexpansive. The set W of fixed points of f is

the set bounded by positive x, y axes and a branch of hyperbola 2x + 2y - xy - 2 = 0. i.e.,

$$W = \{(x, y) \in K; \ 2x + 2y - xy - 2 \leq 0\}$$
.

By an inspection of the shape of the set W, one sees that  $W \subseteq f((x, y))$  for any  $(x, y) \in K$ . Hence W is not a singleton in a generalized sense.

The question arises: Is W a singleton in a generalized sense if f is nonexpansive and I - f is convex? The answer is no. Let us consider the following example.

EXAMPLE 5. Let  $K = [0, 1] \times [0, 1]$  with the usual metric. Let  $(x, y) \in K$ , define

$$f((x, y)) =$$
 the segment  $\{(t, y); 0 \leq t \leq x/2\}$ .

Then  $f: K \to cc(K)$  is nonexpansive. I - f is a convex mapping. To show it, let  $P = (x_1, y_1)$ ,  $Q = (x_2, y_2)$  both in K, and let

$$M = \lambda P + (1 - \lambda)Q$$
 ,

for some  $0 \leq \lambda \leq 1$ . Then

$$egin{aligned} &d(P,\,f(P))=x_1\!/\!2\;,\ &d(Q,\,f(Q))=x_2\!/\!2\;,\ &d(M,\,f(M))=1\!/\!2(\lambda x_1+(1-\lambda)x_2)\ &=\lambda d(P,\,f(P))+(1-\lambda)d(Q,\,f(Q))\;. \end{aligned}$$

By Lemma 3, we see that I - f is convex on K. Now the set of fixed points of f is  $W = \{(0, y); 0 \le y \le 1\}$ . But  $W \subset f((x, y))$  for any  $(x, y) \in K$ . Hence W is not a singleton in the generalized sense.

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