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GREGERS LOUIS KRABBE

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Let (a, b) be any open sub-interval of the real line, such that $-\infty \leqq a < 0 < b \leqq \infty$. Let $L^{\mathrm{loc}}(a, b)$ be the space of all the functions which are integrable on each interval (a', b') with $a < a' < b' < b$. There is a one-to-one linear transformation $\mathfrak{T}$ which maps $L^{\mathrm{loc}}(a, b)$ into a commutative algebra $\mathscr{A}$ of (linear) operators. This transformation $\mathfrak{T}$ maps convolution into operator-multiplication; therefore, this transformation $\mathfrak{T}$ is a useful substitute for the two-sided Laplace transformation; it can be used to solve problems that are not solvable by the distributional transformations (Fourier or bi-lateral Laplace).

In essence, the theme of this paper is a commutative algebra $\mathscr{A}$ of generalized functions on the interval (a, b); besides containing the function space $L^{\mathrm{loc}}(a, b)$, the algebra $\mathscr{A}$ contains every element of the distribution space $\mathscr{D}'(a, b)$ which is regular on the interval $(a, 0)$. The algebra $\mathscr{A}$ is the direct sum $\mathscr{A}_- \oplus \mathscr{A}_+$, where $\mathscr{A}_-$ (respectively, $\mathscr{A}_+$) $(a, 0)$ (respectively, to the interval $(0, b)$). There is a subspace $\mathscr{Y}$ of $\mathscr{A}$ such that, if $y \in \mathscr{Y}$, then y has an "initial value" $\langle y, 0- \rangle$ and a "derivative" $\partial_t y$ (which corresponds to the usual distributional derivative). If y is a function $f(\ )$ which is locally absolutely continuous on (a, b), then y belongs to $\mathscr{Y}$, the initial value $\langle y, 0- \rangle$ equals $f(0)$, and $\partial_t y$ corresponds to the usual derivative $f'(\ )$. If y is a distribution (such as the Dirac distribution) whose support is a locally finite subset of the interval (a, b), then both y and $\partial_t y$ belong to the subspace $\mathscr{Y}$. In case $a = -\infty$ and $b = \infty$, the subspace $\mathscr{Y}$ contains the distribution space $\mathscr{D}'_+$.

The resulting operational calculus takes into account the behavior of functions to the left of the origin (in case $a = -\infty$ and $b = \infty$, the whole real line is accounted for—whereas Mikusiński's operational calculus only accounts for the positive axis). Since the functions are not subjected to growth restrictions, the transformation $\mathfrak{T}$ is a useful substitute for the two-sided Laplace transformation (no strips of convergence need to be considered: see Examples 2.21 and the four problems 6.3–6.7). Problems such as

$$\frac{d^2}{dt^2} y + y = \sec \frac{\pi t}{2\alpha} \qquad (-\alpha < t < \alpha)$$

can be solved by calculations which duplicate the ones that would arise if the Laplace transformation could be applied to such problems.

The differential equation

$$\partial_t^2 y + y = \sum_{k=-\infty}^{\infty} \delta(t - 2k\pi) \tag{1}$$

is solved in 6.7 in order to illustrate our operational calculus; the right-hand side of this equation represents a series of unit impulses starting at $t = -\infty$. The differential equation (1) cannot be solved by the distributional Fourier transformation nor by the distributional two-sided Laplace transformation. When $-\infty = a < t < b = \infty$ the equation

$$y(t) = c_0 \cos t + c_1 \sin t + \left(1 + \left[\frac{t}{2\pi}\right]\right) \sin t$$

defines the general solution of the equation (1).

The paper is subdivided as follows. § 1: the space of generalized functions, § 2: two-sided operational calculus, § 3: translation properties, § 4: the topological space $\mathscr{A}_\varpi$, § 5: derivative of an operator, § 6: four problems.

The concepts introduced in § 5 (initial value, derivative, antiderivative of an operator) are more general and more appropriate than the corresponding ones in my textbook [5].

0. Preliminaries. Henceforth, ω is an open sub-interval (ω_-, ω_+) of the real line $\mathbf{R}$; we suppose that $\omega_- < 0 < \omega_+$. If $h(\ )$ is a function on ω, we denote by $h_+(\ )$ the function defined by

$$h_+(t) = \begin{cases} 0 & \text{for } t < 0 \\ h(t) & \text{for } t \geqq 0 ; \end{cases} \tag{0.1}$$

we set

$$h_{\sqcup}(\ ) = h(\ ) - h_+(\ ) . \tag{0.2}$$

As usual, the support of a function $f(\ )$ (denoted Supp f) is the complement of the largest open subset of $\mathbf{R}$ on which $f(\ )$ vanishes. Let $e_t(\ )$ be the function defined by

$$e_t(u) = \begin{cases} 1 & \text{for } 0 \leqq u < t \\ -1 & \text{for } t < u < 0 , \end{cases} \tag{0.3}$$

and by $e_t(u) = 0$ for all other values of u. It will be convenient to denote by e_t the support of the function $e_t(\ )$; thus, e_t is the interval with end-points 0 and t:

$$e_t = (t, 0) \cup [0, t] = \begin{cases} [0, t) & \text{for } t \geqq 0 \\ (t, 0) & \text{for } t < 0 . \end{cases} \tag{0.4}$$

Unless otherwise specified, suppose that $f(\ )$ and $g(\ )$ belong to $L^{\text{loc}}(\omega)$ (this is the space of all the complex-valued functions which are Lebesgue integrable on each interval (a, b) with $\omega_- < a < 0 < b < \omega_+$). We denote by $f \wedge g(\ )$ the function defined by

$$f \wedge g(t) = \int_0^t f(t-u)g(u)du \qquad \text{(all } t \text{ in } \omega); \tag{0.5}$$

that is,

$$f \wedge g(t) = \int_{e_t} f(t-u)e_t(u)g(u)du . \tag{0.6}$$

REMARK 0.7. Suppose that $\omega_- \leqq a \leqq 0 \leqq b < \omega_+$:

$$\text{if } a < t < b \text{ and } u \in e_t \text{ then } (t-u) \in e_t \subset (a, b) . \tag{0.8}$$

This is easily verified.

REMARKS 0.9. The following properties are direct consequences of (0.1)–(0.8):

$$f \wedge g(t) = f_+ \wedge g(t) = f_+ \wedge g_+(t) \qquad (\text{for } t > 0), \tag{0.10}$$

and

$$f \wedge g(t) = f_{\text{Ц}} \wedge g(t) = f_{\text{Ц}} \wedge g_{\text{Ц}}(t) \qquad (\text{for } t < 0). \tag{0.11}$$

FINAL REMARK 0.12. If $f_1(\ ) = f(\ )$ and $g_1(\ ) = g(\ )$ almost-everywhere on ω, then $f_1 \wedge g_1(\ ) = f \wedge g(\ )$ almost-everywhere on ω. This is another easy consequence of (0.5)–(0.8).

LEMMA 0.13. *If* $a \leqq 0 \leqq b$ *and if* $f(\ ) = 0$ *almost-everywhere on the interval* (a, b), *then* $f \wedge g(\ ) = 0$ *on* (a, b).

Proof. If $t \in (a, b)$ it follows from (0.8) that

$$u \in e_t \quad \text{implies} \quad (t-u) \in e_t \subset (a, b) ;$$

therefore, $(t-u) \in (a, b)$, whence our hypothesis ($f(\ ) = 0$ almost-everywhere on (a, b)) gives $f(t-u) = 0$ for u almost-everywhere on the interval e_t: the conclusion $f \wedge g(t) = 0$ now follows directly from (0.6).

LEMMA 0.14. *Suppose that* $a < 0 < b$. *If* $f(\ ) = 0$ *on the interval* (ω_-, b), *then*

(0.15) $$f \wedge g(t) = \int_0^{t-b} f(t-\tau)g(\tau)d\tau \qquad (\textit{for } b < t < \omega_+).$$

If $h(\ ) \in L^{\text{loc}}(\omega)$ *and if* $h(\ ) = 0$ *on the interval* (a, ω_+), *then*

(0.16) $$h \wedge g(t) = -\int_{t-a}^{0} h(t-\tau)g(\tau)d\tau \qquad (\textit{for } \omega_- < t < a).$$

Proof. First, the case $b < t < \omega_+$. From (0.5) we have

(1) $$f \wedge g(t) = \int_0^{t-b} f(t-\tau)g(\tau)d\tau + \int_{t-b}^{t} f(t-u)g(u)du .$$

From (0.8) we see that

$$u \in [0, t) \quad \text{implies} \quad (t-u) \in e_t \subset \omega ,$$

so that $(t-u) \in \omega$. If $u > t-b$, then $b > t-u$, whence $(t-u) \in (\omega_-, b)$; consequently, our hypothesis ($f(\ ) = 0$ on (ω_-, b)) gives $f(t-u) = 0$ whenever $u > t-b$: Conclusion (0.15) is now immediate from (1).

Next, the case $\omega_- < t < a$. From (0.5) we have

(2) $$h \wedge g(t) = -\int_{t}^{t-a} h(t-u)g(u)du - \int_{t-a}^{0} h(t-\tau)g(\tau)d\tau .$$

From (0.8) we again see that

$$u \in (t, 0) \quad \text{implies} \quad (t-u) \in e_t \subset \omega ,$$

so that $(t-u) \in \omega$. If $u < t-a$ then $t-u > a$, whence $(t-u) \in (a, \omega_+)$; consequently, our hypothesis ($h(\ ) = 0$ on (a, ω_+)) gives $h(t-u) = 0$ whenever $u < t-a$: Conclusion (0.16) is now immediate from (2).

0.17. *Convolution.* If $F(\ )$ and $G(\ )$ belong to $L^1(\mathbf{R})$, then $F * G(\ )$ is the function defined by

$$F * G(x) = \int_{\mathbf{R}} F(x-u)G(u)du \qquad (\text{all } x \text{ in } \mathbf{R});$$

it is well-known that $F * G(\ ) \in L^1(\mathbf{R})$ (see [1], p. 634). Further,

(0.18) $$\operatorname{Supp} F * G \subset (\operatorname{Supp} F) + (\operatorname{Supp} G):$$

see p. 385 in [2].

THEOREM 0.19. *If* $f(\ )$ *and* $g(\ )$ *belong to* $L^{\text{loc}}(\omega)$, *then* $f \wedge g(\ )$ *belongs to* $L^{\text{loc}}(\omega)$, *and*

(0.20) $$f \wedge g(\ ) = g \wedge f(\ ) \quad \textit{almost-everywhere on } \omega.$$

Proof. Suppose that $\omega_- < a < 0 < b < \omega_+$. If $h(\ ) \in L^{\mathrm{loc}}(\omega)$, we can define the function $h_b(\ )$ by

$$h_b(t) = \begin{cases} h(t) & \text{for } 0 < t < b \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Similarly, $h_a(\ )$ is defined by

$$h_a(t) = \begin{cases} h(t) & \text{for } a < t < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Note that both $h_b(\ )$ and $h_a(\ )$ belong to $L^1(\boldsymbol{R})$. Set

$$F(\ ) = -f_a * g_a(\ ) + f_b * g_b(\ ) \,. \tag{3}$$

The four functions on the right-hand side of (3) are all integrable on $\boldsymbol{R}$; consequently, both $f_a * g_a(\ )$ and $f_b * g_b(\ )$ are integrable on $\boldsymbol{R}$; from (3) it now follows that $F(\ )$ is integrable on $\boldsymbol{R}$. In consequence, if we can prove that

$$F(t) = f \wedge g(t) \qquad \text{for } a < t \neq 0 < b \,, \tag{4}$$

then $f \wedge g(\ )$ is integrable on the arbitrary sub-interval (a, b) of the interval ω; our conclusion $f \wedge g \in L^{\mathrm{loc}}(\omega)$ is at hand; moreover, Conclusion (0.20) comes from (4)–(3) and the property $F_1 * F_2(\ ) = F_2 * F_1(\ )$ (see [1], p. 635). Accordingly, the proof will be accomplished by proving (4).

The proof of (4) is divided into two cases. *First case*: $a < t < 0$. Since $\mathrm{Supp} f_b$ and $\mathrm{Supp}\, g_b$ are subsets of the interval $[0, \infty)$, we see from (0.18) that

$$\mathrm{Supp} f_b * g_b \subset [0, \infty) \,;$$

consequently, $f_b * g_b(\ )$ vanishes for $t < 0$; therefore, (3) gives

$$F(t) = -f_a * g_a(t) = -\int_a^0 f_a(t-u) g(u) du \tag{5}$$

(for $a < t < 0$); the second equation comes from (2) and the fact that $g_a(u) = 0$ when $u < a$ and when $u > 0$. From (5) it follows that

$$F(t) = -\int_a^t f_a(t-u) g(u) du - \int_t^0 f_a(t-\tau) g(\tau) d\tau \,;$$

but $a < u < t$ implies $t - u > 0$, so that $f_a(t-u) = 0$; therefore,

$$F(t) = -\int_t^0 f_a(t-\tau) g(\tau) d\tau \,; \tag{6}$$

but $0 > \tau > t$ implies $t < t - \tau < 0$; in consequence, since $a < t$, we have $a < t - \tau < 0$, so that (2) gives $f_a(t-\tau) = f(t-\tau)$: Equation (6) becomes

$$F(t) = \int_{e_t} f(t-u)e_t(u)g(u)du \,.$$

In view of (0.6), this concludes the proof of (4) in case $a < t < 0$.

Second case. $0 < t < b$. As in the first case, we observe that $f_a * g_a(t) = 0$; it is a question of proving that $F(t) = f_b * g_b(t)$: the reasoning is entirely analogous to the one used in the first case.

THEOREM 0.21[1]. *Suppose that the functions $f(\ )$, $g(\ )$, and $h(\ )$ all belong to $L^{loc}(\omega)$. If the function $|f| \wedge (|g| \wedge |h|)(\ )$ is continuous on ω then*

(0.22) $$f \wedge (g \wedge h)(x) = (f \wedge g) \wedge h(x) \qquad \text{for every } x \text{ in } \omega.$$

Proof. From (0.6) it follows that

(1) $$F \wedge (G \wedge H)(x) = \int_{e_x} \int_{e_t} F(x-t)G(t-u)H(u)dudt \,.$$

Since $|f| \wedge (|g| \wedge |h|)(\ )$ is continuous on ω (by hypothesis), we therefore have $|f| \wedge (|g| \wedge |h|)(x) < \infty$, so that (1) gives

$$\int_{e_x} \int_{e_t} |f(x-t)g(t-u)h(u)| dudt < \infty \,;$$

we may therefore apply Tonelli's Theorem [3, p. 131] to write

(2) $$f \wedge (g \wedge h)(x) = \int_{e_x} \int_{x_u} f(x-t)g(t-u)h(u)dtdu \,,$$

where x_u is the appropriate interval. Let us prove that

(3) $$f \wedge (g \wedge h)(x) = \int_0^x h(u) \int_u^x f(x-t)g(t-u)dtdu \,.$$

In case $x > 0$ the double integral is taken over the interior of the triangle

$$\{(u, t) : 0 < t < x \text{ and } 0 < u < t\} \,;$$

consequently, the range of t (in the integral (2)) is the interval $x_u = [u, x]$: this establishes (3). In case $x < 0$ the double integral is taken over the triangle

$$\{(u, t) : x < t < 0 \text{ and } t < u < 0\} \,;$$

[1] The principle of this proof is due to R. B. Darst.

consequently, the range of t (in the integral (2)) is the interval $x_u = [x, u]$; the integral (2) becomes

$$f \wedge (g \wedge h)(x) = \int_x^0 \int_x^u f(x - t)g(t - u)h(u)dtdu ,$$

which again establishes the equation (3). The change of variable $\tau = t - u$ brings (3) into the form

$$f \wedge (g \wedge h)(x) = \int_0^x h(u) \int_0^{x-u} f(x - u - \tau)g(\tau)d\tau du ;$$

consequently, (0.5) gives

$$f \wedge (g \wedge h)(x) = \int_0^x h(u)[f \wedge g(x - u)]du :$$

Conclusion (0.22) is now immediate from (0.5).

DEFINITION 0.23. *For any integer $n \geqq 1$ we denote by $q_n(\ )$ the function defined by the equation $q_n(0) = 0$ and*

$$q_n(t) = \exp\left(\frac{-1}{|nt|}\right) \qquad (for\ t \neq 0).$$

THEOREM 0.24. *Suppose that $f(\ )$ belongs to $L^{\mathrm{loc}}(\omega)$. If $\omega_- \leqq a \leqq 0 \leqq b \leqq \omega_+$ and if*

(4) $\quad f \wedge q_n(t) = 0$ *for $a < t < b$ and every integer $n \geqq 1$,*

then $f(\ )$ vanishes almost-everywhere on the interval (a, b).

Proof. From (4) and (0.20) it follows that

$$0 = \lim_{n\to\infty} q_n \wedge f(t) = \lim_{n\to\infty} \int_{e_t} q_n(t - u)e_t(u)f(u)du ;$$

since $|q_n(\ )| \leqq 1$ we may apply the Lebesgue Dominated Convergence Theorem:

(5) $$0 = \int_{e_t} \lim_{n\to\infty} \left[\exp \frac{-1}{n(t - u)}\right] e_t(u)f(u)du = \int_{e_t} e_t(u)f(u)du .$$

From (5) and (0.3)–(0.4) we see that

$$0 = \int_0^t f \text{ for } 0 < t < b, \text{ and } 0 = -\int_t^0 f \text{ for } a < t < 0 ,$$

which implies our conclusion: $f(\ )$ vanishes almost-everywhere on the interval (a, b).

1. The space $\mathscr{A}_\omega$ of generalized functions. As before, ω is an arbitrary sub-interval of $\boldsymbol{R} = (-\infty, \infty)$ such that $\omega \ni 0$. If $f(\ )$ and $g(\ )$ are functions, the equation $f(\ ) = g(\ )$ will mean that the functions are equal almost-everywhere on the interval ω.

NOTATION 1.0. Let $\mathscr{C}_0(\omega)$ be the space of all the functions which are continuous on ω and which vanish at the origin.

NOTATION 1.1. We denote by $1(\ )$ the constant function defined by $1(t) = 1$ for all t in $\boldsymbol{R}$.

LEMMA 1.2. *If $g(\ ) \in L^{\mathrm{loc}}(\omega)$ then $1 \wedge g(\ ) \in \mathscr{C}_0(\omega)$.*

Proof. From (0.5) we see that

$$1 \wedge g(t) = \int_0^t 1(t-u)g(u)du = \int_0^t g(u)du\ . \tag{1.3}$$

On the other hand, $g(\ ) \in L^1(a, b)$ whenever (a, b) is a compact sub-interval of the open set ω: the conclusion is now at hand.

LEMMA 1.4. *If $\Psi(\ )$ is continuous on ω, then $(1 \wedge \Psi)' = \Psi(\ )$.*

Proof. The equations

$$(1 \wedge \Psi)'(t) = \frac{d}{dt}(1 \wedge \Psi)(t) = \Psi(t)$$

are immediate from (1.3) and the Fundamental Theorem of Calculus.

LEMMA 1.5. *Suppose that $v(\ ) \in \mathscr{C}_0(\omega)$. If $v'(\ )$ has only countably many discontinuities and is integrable in each compact sub-interval of the open interval ω, then $v(\ ) = 1 \wedge v'(\ )$.*

Proof. Take t in ω. If $t > 0$ the equations

$$v(t) = v(t) - v(0) = \int_0^t v'(u)du = 1 \wedge v(t)$$

are from $v(0) = 0$, [4, p. 143], and (1.3). If $t < 0$, the same reasoning yields

$$v(t) = -[v(0) - v(t)] = -\int_t^0 v'(u)du = 1 \wedge v(t)\ .$$

THEOREM 1.6. *Let $G(\ )$ be a function whose derivative is continuous on the interval ω. If $f(\ ) \in L^{\mathrm{loc}}(\omega)$, then $G \wedge f(\ ) \in \mathscr{C}_0(\omega)$ and*

(1.7) $$G \wedge f(\ ) = G(0)(1 \wedge f)(\ ) + 1 \wedge (G' \wedge f)(\ ) .$$

Proof. Clearly, the function $v(\ ) = G(\ ) - G(0)1(\ )$ belongs to $\mathscr{C}_0(\omega)$; consequently, 1.5 gives

$$G(\ ) - G(0)1(\ ) = 1 \wedge G'(\ ) ,$$

so that 0.12 implies

(1) $$G \wedge f(\ ) - G(0)(1 \wedge f)(\ ) = (1 \wedge G') \wedge f(\ ) .$$

From 0.19 it follows that $(|G'| \wedge |f|)(\ ) \in L^{\text{loc}}(\omega)$; we can therefore conclude from 1.2 that the function $|1| \wedge (|G'| \wedge |f|)(\ )$ is continuous on ω, whence the equation

(2) $$(1 \wedge G') \wedge f(\ ) = 1 \wedge (G' \wedge f)(\ )$$

now comes from 0.21. Conclusion (1.7) is immediate from (1)–(2). It still remains to prove that $G \wedge f(\ ) \in \mathscr{C}_0(\omega)$.

Set $g_1(\ ) = G' \wedge f(\ )$; Equation (1.7) becomes

(3) $$G \wedge f(\ ) = G(0)(1 \wedge f)(\ ) + 1 \wedge g_1(\ ) .$$

From 0.19 we see that $g_1(\ ) \in L^{\text{loc}}(\omega)$; the conclusion $G \wedge f(\ ) \in \mathscr{C}_0(\omega)$ is obtained from (3) by setting $g = f$ and then $g = g_1$ in 1.2.

1.8. The space of test-functions. Let W_ω be the linear space of all the complex-valued functions which are infinitely differentiable on ω and whose every derivative vanishes at the origin. Thus, $w(\ ) \in W_\omega$ if $w(\ ) \in \mathscr{C}_0(\omega)$ and $w^{(k)} \in \mathscr{C}_0(\omega)$ for every integer $k \geqq 1$.

EXAMPLE 1.9. Let $q_n(\ )$ be the function defined in 0.23; it is easily verified that $q_n^{(k)}(0) = 0$ for every integer $k \geqq 1$; therefore, $q_n(\ ) \in W_\omega$.

LEMMA 1.10. *If $f(\ ) \in L^{\text{loc}}(\omega)$ and $q(\ ) \in W_\omega$ then*

(1.11) $$q \wedge f(\ ) \in \mathscr{C}_0(\omega)$$

and

(1.12) $$(q \wedge f)'(\ ) = q' \wedge f(\ ) .$$

Proof. Since $q'(\ ) \in \mathscr{C}_0(\omega)$, we can set $G = q$ in 1.6 to obtain (1.11) and the equations

(4) $$q \wedge f(\ ) = q(0)(1 \wedge f)(\ ) + 1 \wedge (q' \wedge f)(\ ) = 1 \wedge (q' \wedge f)(\ )$$

now come from (1.7) and $q(0) = 0$ (since $q(\ ) \in \mathscr{C}_0(\omega)$). Next, set

(5) $$\Psi(\ ) = q' \wedge f(\ ):$$

Equation (4) becomes

(6) $$q \wedge f(\ ) = 1 \wedge \Psi(\ ) .$$

Setting $G = q'$ in 1.6, we see from (5) that $\Psi(\ ) \in \mathscr{C}_0(\omega)$; the equations

(7) $$(1 \wedge \Psi)'(\ ) = \Psi(\ ) = q' \wedge f(\ )$$

therefore follow from 1.4 and (5). Conclusion (1.12) is immediate from (6)-(7).

LEMMA 1.13. *If* $f(\ ) \in L^{\text{loc}}(\omega)$ *and* $w(\ ) \in W_\omega$, *then* $w \wedge f(\ ) \in W_\omega$, *and*

(1.14) $$(f \wedge w)'(\ ) = w' \wedge f(\ ) = f \wedge w'(\ ) .$$

Proof. If the equation

(8) $$(w \wedge f)^{(k)}(\ ) = w^{(k)} \wedge f(\ )$$

holds for $k = n$, then it holds for $k = n + 1$: this is easily seen by observing that the equations

$$[(w \wedge f)^{(n)}]'(\ ) = (w^{(n)} \wedge f)'(\ ) = w^{(n+1)} \wedge f(\ )$$

come from (8) and (1.12). Since (8) holds for $k = 0$, it holds for any integer $k \geqq 0$. From (8) and (1.11) (with $q = w^{(k)}$) it follows that

$$(w \wedge f)^{(k)}(\ ) \in \mathscr{C}_0(\omega) \qquad \text{for any integer } k \geqq 0 ;$$

therefore, $w \wedge f(\ ) \in W_\omega$. Conclusion (1.14) comes from (1.12) and (0.20).

DEFINITIONS 1.15. An **operator** is a linear mapping of W_ω into W_ω. If A is an operator and $w(\ ) \in W_\omega$, we denote by $.Aw(\ )$ the function that the operator A assigns to $w(\ )$.

As usual, the product A_1A_2 of two operators is defined by

(1.16) $$.A_1A_2w(\ ) = .A_1(.A_2w)(\ ) \qquad (\text{every } w(\ ) \text{ in } W_\omega).$$

1.17. The space of generalized functions. Let $\mathscr{A}_\omega$ be the set of all the operators A such that the equation

(1.18) $$.A(w_1 \wedge w_2)(\ ) = (.Aw_1) \wedge w_2(\ )$$

holds whenever $w_1(\ )$ and $w_2(\ )$ belong to W_ω.

DEFINITION 1.19. If $f(\ ) \in L^{\text{loc}}(\omega)$ we denote by f^* the operator which assigns to each $w(\ )$ in W_ω the function $f \wedge w(\ )$:

$$.f^*w(\ ) = f \wedge w(\ ) \qquad \text{(for each } w(\ ) \text{ in } W_\omega). \tag{1.20}$$

THEOREM 1.21. *If $f_1(\ )$ and $f_2(\ )$ belong to $L^{\text{loc}}(\omega)$, then*

$$f_1^* f_2^* = (f_1 \wedge f_2)^* . \tag{1.22}$$

Proof. Take any $w_2(\ )$ *in* W_ω. From 1.13 and (0.20) we see that $|f_2| \wedge |w_2|(\ ) \in W_\omega$; consequently, we can set $w = |f_2| \wedge |w_2|$ and $f = |f_1|$ in 1.13 to obtain

$$|f_1| \wedge (|f_2| \wedge |w_2|)(\ ) \in W_\omega :$$

from 0.21 it therefore follows that

$$f_1 \wedge (f_2 \wedge w_2)(\ ) = (f_1 \wedge f_2) \wedge w_2(\ ) , \tag{1.23}$$

which, in view of 1.19, means that

$$.f_1^*(.f_2^*w_2)(\ ) = .(f_1 \wedge f_2)^*w_2(\ ) .$$

Since $w_2(\ )$ is an arbitrary element of W_ω, Conclusion (1.22) is immediate from (1.16).

REMARK 1.24. If $f(\ ) \in L^{\text{loc}}(\omega)$ then $f^* \in \mathscr{A}_\omega$. Indeed, f^* is an operator (by (1.20), (0.20), and 1.13): it only remains to prove that the equation (1.18) holds for $A = f^*$. Setting $f_1 = f$ and $f_2 = w_1$ in (1.23), we obtain

$$f \wedge (w_1 \wedge w_2)(\ ) = (f \wedge w_1) \wedge w_2(\ ) ;$$

in view of (1.20), this becomes

$$.f^*(w_1 \wedge w_2)(\ ) = (.f^*w_1) \wedge w_2(\ ) :$$

therefore, (1.18) holds when $A = f^*$.

DEFINITIONS 1.25. We denote by D the differentiation operator:

$$.Dw(\ ) = w'(\ ) \qquad \text{(all } w(\ ) \text{ in } W_\omega). \tag{1.26}$$

Let I be the identity-operator:

$$.Iw(\ ) = w(\ ) \qquad \text{(all } w(\ ) \text{ in } W_\omega). \tag{1.27}$$

If $f(\ ) \in L^{\text{loc}}(\omega)$, we denote by $\{f(t)\}$ the operator defined by

$$.\{f(t)\}w(\ ) = f \wedge w'(\ ) \qquad \text{(all } w(\ ) \text{ in } W_\omega) ; \tag{1.28}$$

the operator $\{f(t)\}$ will be called **the operator** of the function $f(\ )$.

REMARK 1.29. $\{1(t)\} = I$. Indeed, the equations

$$.\{1(t)\}w(\ ) = 1 \wedge w'(\ ) = w(\ )$$

are from (1.28) and 1.5.

REMARK 1.30. $D \in \mathscr{A}_\omega$. Indeed, D is clearly an operator, and the equations

$$.D(w_1 \wedge w_2)(\ ) = (w_1 \wedge w_2)'(\ ) = w_1' \wedge w_2(\ ) = (.Dw_1) \wedge w_2(\ )$$

are from (1.26), (1.14), and (1.26).

DEFINITION 1.31. *Let (a, b) be a sub-interval of ω such that $a \leqq 0 \leqq b$; if $A \in \mathscr{A}_\omega$ and $B \in \mathscr{A}_\omega$, we say that A* **agrees with** *B* **on** *(a, b) if*

$$.Aw(t) = .Bw(t) \quad \textit{for } a < t < b \textit{ and for every } w(\ ) \textit{ in } W_\omega.$$

THEOREM 1.32. *Suppose that $f_k(\ ) \in L^{\text{loc}}(\omega)$ for $k = 1, 2$. If $\{f_1(t)\}$ agrees with $\{f_2(t)\}$ on $(a, b\}$, then $f_1(\ ) = f_2(\ )$ almost-everywhere on the interval (a, b). Conversely, if the functions are equal almost-everywhere on (a, b), then their operators agree on (a, b).*

Proof. Set $h(\ ) = f_1(\ ) - f_2(\ )$. By hypothesis, the relation

$$(1) \qquad .\{h(t)\}w(t) = 0 \qquad (\text{for } a < t < b)$$

holds for every $w(\ )$ in W_ω: it will suffice to show that $h(\ ) = 0$ almost-everywhere on (a, b). Take any integer $n \geqq 1$, and let $q_n(\ )$ be the function that was defined in 0.23; since $q_n(\ ) \in W_\omega$ (see 1.9), it follows from 1.13 (with $f = 1$) that $q_n \wedge 1(\ ) \in W_\omega$; in view of (0.20) we may therefore set $w(\ ) = 1 \wedge q_n(\ )$ in (1) to obtain

$$(2) \qquad .\{h(t)\}(1 \wedge q_n)(t) = 0 \qquad (\text{for } a < t < b).$$

The equations

$$(3) \qquad .\{h(t)\}(1 \wedge q_n)(\ ) = h \wedge (1 \wedge q_n)'(\ ) = h \wedge q_n(\ )$$

are from (1.28) and 1.4. Combining (2) and (3), we see that $h \wedge q_n(t) = 0$ for $a < t < b$ and for every integer $n \geqq 1$; the conclusion $h(\ ) = 0$ (almost-everywhere on (a, b)) now comes from 0.24.

Conversely, suppose that $f_1(\ ) = f_2(\ )$ almost-everywhere; this means that $h(\ ) = 0$ almost-everywhere on (a, b); we may therefore apply 0.13 to conclude that

$$h \wedge w'(\ ) = 0 \qquad \text{for } a < t < b \text{ and every } w(\ ) \text{ in } W_\omega ;$$

consequently, (1.28) gives $.\{h(t)\}w(t) = 0$, so that

$$.\{f_1(t)\}w(t) = .\{f_2(t)\}w(t) \qquad \text{for } a < t < b \text{ and } w(\ ) \in W_\omega :$$

this proves that $\{f_1(t)\}$ agrees with $\{f_2(t)\}$ on (a, b).

COROLLARY 1.33. *Suppose that $f_1(\ )$ and $f_2(\ )$ belong to $L^{\text{loc}}(\omega)$:*

$$f_1(\ ) = f_2(\ ) \textit{ if (and only if) } \{f_1(t)\} = \{f_2(t)\} .$$

Proof. Set $a = \omega_-$ and $b = \omega_+$ in 1.32: by definition, two operators are equal if they agree on (a, b); moreover, we agree that the equation $f_1(\ ) = f_2(\ )$ means that these functions are equal almost-everywhere on (a, b). The conclusion is now immediate from 1.32.

THEOREM 1.34. *The mapping $f(\ ) \mapsto \{f(t)\}$ is an injective linear transformation of $L^{\text{loc}}(\omega)$ into $\mathscr{A}_\omega$ such that*

$$\{f(t)\} = f^* D . \tag{1.35}$$

Proof. The equation (1.35) is immediate from (1.28), (1.16), and (1.26). On the other hand, it is easily verified that $\mathscr{A}_\omega$ is an algebra (if $A_k \in \mathscr{A}_\omega$ for $k = 1, 2$, then $A_1 A_2 \in \mathscr{A}_\omega$): since $f^* \in \mathscr{A}_\omega$ (by 1.24), and since $D \in \mathscr{A}_\omega$ (by 1.30), the conclusion $\{f(t)\} \in \mathscr{A}_\omega$ comes from (1.35). From 1.33 we may now conclude that $f(\ ) \mapsto \{f(t)\}$ is an injective transformation of $L^{\text{loc}}(\omega)$ into $\mathscr{A}_\omega$: the linearity is clear from (1.28).

LEMMA 1.36. *If $B \in \mathscr{A}_\omega$ then the equation*

$$.B(p_1 \wedge p_2)(\ ) = p_1 \wedge (.Bp_2)(\ ) \tag{1.37}$$

holds for every $p_1(\ )$ and $p_2(\ )$ in W_ω.

Proof. The equations

$$.B(p_1 \wedge p_2)(\ ) = .B(p_2 \wedge p_1)(\ ) = (.Bp_2) \wedge p_1(\ )$$

are from (0.20), (0.12), and (1.18); conclusion (1.37) is now immediate from (0.20).

THEOREM 1.38. *$\mathscr{A}_\omega$ is a commutative algebra.*

Proof. The multiplication of the algebra $\mathscr{A}_\omega$ is the usual operator-multiplication (defined in (1.16)); it is easily verified that $\mathscr{A}_\omega$ is an algebra. Take A_1 and A_2 in $\mathscr{A}_\omega$; to prove the commutativity, it will suffice to demonstrate that $A_1A_2 - A_2A_1 = 0$. Let $q_1(\ )$ and $q_2(\ )$ be any two elements of W_ω; we begin by observing that

$$(1)\qquad .A_1A_2(q_1 \wedge q_2')(\ ) = .A_1[(.A_2q_1) \wedge q_2'](\ ) = (.A_2q_1) \wedge (.A_1q_2')(\ ):$$

these equations are from (1.16), (1.18), and (1.37) (with $p_1 = .A_2q_1'$ and $p_2 = q_2'$). On the other hand, the equations

$$(2)\qquad .A_2A_1(q_1 \wedge q_2')(\ ) = .A_2(q_1 \wedge (.A_1q_2')) = (.A_2q_1) \wedge (.A_1q_2')(\ )$$

are from (1.16), (1.37), and (1.18). We now subtract (2) from (1) to obtain

$$(3)\qquad .A(q_1 \wedge q_2')(\ ) = 0, \text{ where } A = A_1A_2 - A_2A_1.$$

From (3) and (1.18) it results that

$$0 = (.Aq_1) \wedge q_2'(\ ) = \{.Aq_1(t)\}q_2(\ ) \qquad (\text{all } q_2(\ ) \text{ in } W_\omega);$$

the last equation is from (1.28). Consequently, $0 = \{.Aq_1(t)\}$; we may now infer from 1.33 that $0 = .Aq_1(\ )$ for each $q_1(\ )$ in W_ω: the desired conclusion $A = 0$ is at hand.

THEOREM 1.39. *If* $A \in \mathscr{A}_\omega$ *and* $w(\ ) \in W_\omega$, *then* $\{.Aw(t)\} = A\{w(t)\}$.

Proof. Let $w_2(\ )$ be an arbitrary element of W_ω; the equations

$$(4)\qquad .\{.Aw(t)\}w_2(\ ) = (.Aw) \wedge w_2'(\ ) = .A(w \wedge w_2')(\ )$$

are from (1.28) and (1.18). On the other hand, the equations

$$(5)\qquad .A\{w(t)\}w_2(\ ) = .A(.\{w(t)\}w_2)(\ ) = .A(w \wedge w_2')(\ )$$

come from (1.16) and (1.28). Comparing (4) and (5):

$$(6)\qquad .\{.Aw(t)\}w_2(\ ) = .(A\{w(t)\})w_2(\ ) .$$

Since (6) holds for every $w_2(\ )$ in W_ω, the proof is complete.

THEOREM 1.40. *If* $f_1(\ )$ *and* $f_2(\ )$ *both belong to* $L^{\text{loc}}(\omega)$, *then*

$$(7)\qquad D\{f_1 \wedge f_2(t)\} = \{f_1(t)\}\{f_2(t)\} .$$

Proof. The equations

$$(8)\qquad D\{f_1 \wedge f_2(t)\} = D(f_1 \wedge f_2)^*D = Df_1^*f_2^*D = (f_1^*D)(f_2^*D)$$

are obtained by using (1.35) (with $f = f_1 \wedge f_2$), by using (1.22), and by utilizing the commutativity and the associativity of the multiplication in $\mathscr{A}_\omega$. Conclusion (7) comes directly from (8) and two more applications of 1.35.

2. Two-sided operational calculus. If c is a scalar (that is, a complex number), the equation $\{c1(t)\} = cI$ comes from 1.29 and the linearity of the transformation $f(\ ) \mapsto \{f(t)\}$; consequently, $cI \in \mathscr{A}_\omega$ (recall that I is the identity: (1.27)). Since the correspondence $c \mapsto cI$ is an algebraic isomorphism of the field of scalars into the algebra $\mathscr{A}_\omega$, there is no reason to distinguish between the scalar c and the operator cI:

$$c = cI = \{c1(t)\} \qquad \text{for any scalar } c\,. \tag{2.0}$$

Since $ct^n1(t) = ct^n$ for all t in $\mathbf{R}$, it is natural to write $\{ct^n\}$ instead of $\{ct^n1(t)\}$; in particular,

$$c = cI = \{c\} \text{ and } 1 = I = \{1\}\,. \tag{2.1}$$

Substituting $f_1 = 1$ into 1.40:

$$D\{1 \wedge f_2(t)\} = \{f_2(t)\}\,. \tag{2.2}$$

We can also combine the linearity property with (2.1) to obtain

$$\{c_1f_1(t) + c_2f_2(t) + c_3\} = c_1\{f_1(t)\} + c_2\{f_2(t)\} + c_3\,; \tag{2.3}$$

of course, we suppose throughout that c_k ($k = 1, 2, 3$) are scalars, and $f_k(\ )$ ($k = 1, 2$) belong to $L^{\mathrm{loc}}(\omega)$.

THEOREM 2.4. *Suppose that $f(\ )$ is a function which is continuous on the interval ω. If $f'(\ )$ has at most countably-many discontinuities and is integrable in each compact sub-interval of ω, then*

$$\{f'(t)\} = D\{f(t)\} - f(0)D\,. \tag{2.5}$$

Proof. If $v(\ ) = f(\ ) - f(0)1$, then $v'(\ ) = f'(\ )$ and we may apply 1.5:

$$f(\ ) - f(0)1 = v(\ ) = 1 \wedge f'(\ )\,. \tag{1}$$

From (1) and (2.3) it follows that

$$\{f(t)\} - f(0) = \{1 \wedge f'(t)\}\,. \tag{2}$$

Multiplying by D both sides of (2), we obtain

$$D\{f(t)\} - f(0)D = D\{1 \wedge f'(t)\} = \{f'(t)\}:$$

the last equation is from (2.2).

2.6. Invertibility. As usual, an operator A is called **invertible**

if $A \in \mathscr{A}_\omega$ and there exists an operator X in $\mathscr{A}_\omega$ such that $AX = 1$. Suppose that A is an invertible operator; since $\mathscr{A}_\omega$ is a commutative algebra, it is easily verified that there exists exactly one operator A^{-1} such that $A^{-1} \in \mathscr{A}_\omega$ and $AA^{-1} = 1$. Setting $f(t) = t$ in 2.4, we obtain

$$\{1\} = D\{t\}\,; \tag{2.7}$$

consequently, D is an invertible operator, and $D^{-1} = \{t\}$.

THEOREM 2.8. *Suppose that $Y \in \mathscr{A}_\omega$ and $V \in \mathscr{A}_\omega$. If the equation $VY = R$ holds for some invertible R in $\mathscr{A}_\omega$, then V is invertible, and $Y = R/V$, where R/V denotes RV^{-1}.*

Proof. Easy; see 1.76 in [5].

REMARKS 2.9. From (2.5) we see that

$$D\{\sin t\} = \{\cos t\}\,, \tag{2.10}$$

whence $D^2\{\sin t\} = D\{\cos t\} = -\{\sin t\} + D$ (this last equation also comes from (2.5)); we may therefore use 2.8 to obtain

$$\{\sin t\} = \frac{D}{D^2 + 1}\,. \tag{2.11}$$

The equation

$$D^{-k} = \left\{\frac{t^k}{k!}\right\} \qquad \text{(for any integer } k \geqq 0) \tag{2.12}$$

is an easy consequence of (2.7) and (2.5).

2.13. NOTATION. *We shall often write f instead of $\{f(t)\}$.* Consequently, (2.3) can be re-written in the form

$$\{c_1 f_1(t) + c_2 f_2(t) + c_3\} = c_1 f_1 + c_2 f_2 + c_3\,, \tag{2.14}$$

and 1.33 becomes

$$f_1 = f_2 \text{ if (and only if) } f_1(\ ) = f_2(\ ). \tag{2.15}$$

Combining 1.40 with (0.5):

$$f_1 \wedge f_2 = f_1 D^{-1} f_2 = \left\{\int_0^t f_1(t-u) f_2(u)\,du\right\}. \tag{2.16}$$

Also, note that (2.2) gives

$$f_2 = D(1 \wedge f_2)\,; \tag{2.17}$$

that is,

$$D^{-1}f_2 = 1 \wedge f_2 \,; \tag{2.18}$$

combining with (1.3):

$$\left\{\int_0^t f_2\right\} = D^{-1}f_2 \,. \tag{2.19}$$

Finally, note that Theorem 1.39 becomes

$$.Aw = Aw \qquad (\text{for } A \in \mathscr{A}_\omega \text{ and } w(\ ) \in W_\omega). \tag{2.20}$$

APPLICATION 2.21. Given a function $f(\ )$ in $L^{\mathrm{loc}}(-\alpha, \alpha)$, let us solve the differential equation

$$y''(t) + y(t) = f(t) \qquad (-\alpha < t < \alpha)\,; \tag{1}$$

for example, we could have $f(t) = \sec(\pi t/2\alpha)$. To solve (1), set $\omega = (-\alpha, \alpha)$, $c_0 = y(0)$, $c_1 = y'(0)$, and inject both sides of (1) into $\mathscr{A}_\omega$; this gives $D^2y + y = c_1D + c_0D^2 + f$; solving for y:

$$y = c_1 \frac{D}{D^2+1} + c_0D\frac{D}{D^2+1} + \frac{D}{D^2+1}D^{-1}f :$$

we can now use (2.11), (2.10), and (2.16) to write

$$y = c_1 \sin + c_0 \cos + \left\{\int_0^t (\sin(t-u))f(u)du\right\} .$$

3. Translation properties. In this section we shall describe some two-sided analogues of the translation properties described in [5].

If $b \geqq 0$ we define the function $\mathsf{T}_b(\ )$ by

$$\mathsf{T}_b(t) = \begin{cases} 0 & \text{for } t < b \\ 1 & \text{for } t \geqq b\,. \end{cases} \tag{3.0}$$

If $a < 0$ we set

$$\mathsf{T}_a(t) = \begin{cases} -1 & \text{for } t < a \\ 0 & \text{for } t \geqq a\,. \end{cases} \tag{3.1}$$

Observe that

$$\mathsf{T}_x(\ ) = 0 \quad \text{on} \quad (-|x|, |x|) \qquad (\text{for any } x \text{ in } \mathbf{R}). \tag{3.2}$$

Until further notice, let $g(\ )$ be a function in $L^{\mathrm{loc}}(\omega)$, and let $g_x(\ )$ be the function defined by

$$g_x(u) = \mathsf{T}_x(u)g(u-x) \qquad (\text{for } u \in \omega)\,; \tag{3.3}$$

note that $g_x(\ ) \in L^{\mathrm{loc}}(\omega)$.

LEMMA 3.4. *If* $b \geqq 0$ *then* $1 \wedge g_b(\ ) = \mathsf{T}_b \wedge g(\ )$.

Proof. Observe that $g_b(\ ) = 0 = \mathsf{T}_b(\ )$ on the interval (ω_-, b); from 0.13 it therefore follows that

$$g_b \wedge 1(t) = 0 = \mathsf{T}_b \wedge g(t) \qquad (\text{for } t \in (\omega_-, b)). \tag{1}$$

Next, suppose that $t > b$ and $t \in \omega$: the equation

$$1 \wedge g_b(t) = \int_0^t 1(t-u)\mathsf{T}_b(u)g(u-x)du$$

comes from (0.5) and (3.3); in view of (3.0), we see that

$$1 \wedge g_b(t) = \int_b^t g(u-x)du = \int_0^{t-b} g(\tau)d\tau = \mathsf{T}_b \wedge g(t): \tag{2}$$

the second equation is obtained by the change of variable $\tau = u - b$; the last equation comes from (0.15) by setting $f = \mathsf{T}_b$ in 0.14. The conclusion is immediate from (1)–(2).

THEOREM 3.5. *If* $x \in \mathbf{R}$ *then* $1 \wedge g_x(\ ) = \mathsf{T}_x \wedge g(\ )$ *and*

$$g_x = g\mathsf{T}_x\,. \tag{3.6}$$

Proof. In view of 3.4, it only remains to consider the case $x = a < 0$. Observe that $g_a(\ ) = 0 = \mathsf{T}_a(\ )$ on the interval (a, ω_+); from 0.13 it therefore follows that

$$g_a \wedge 1(t) = 0 = \mathsf{T}_a \wedge g(t) \qquad (\text{for } t \in (a, \omega_+)). \tag{3}$$

Next, suppose that $t < a$ and $t \in \omega$: as in the proof of 3.4, we see that

$$1 \wedge g_a(t) = -\int_t^a g(u-x)du = -\int_{t-a}^0 g(\tau)d\tau: \tag{4}$$

the second equation is obtained by the change of variable $\tau = u - a$. Note that $\mathsf{T}_a(\ ) = 0$ on the interval (a, ω_+): we can therefore set $h = \mathsf{T}_a$ in 0.14 and use (0.16) to obtain

$$\mathsf{T}_a \wedge g(t) = -\int_{t-a}^0 \mathsf{T}_a(t-\tau)g(\tau)d\tau = -\int_{t-a}^0 g(\tau)d\tau\,. \tag{5}$$

From (4)–(5) it results that $1 \wedge g_a(t) = \mathsf{T}_a \wedge g(t)$ for $\omega_- < t < a$; the conclusion $1 \wedge g_a(\ ) = \mathsf{T}_a \wedge g(\ )$ is now immediate from (3). The equations

$$g_x = D(1 \wedge g_x) = D(\mathsf{T}_x \wedge g) = \mathsf{T}_x g$$

are from (2.17), from our conclusion $(1 \wedge g_x(\ ) = \mathsf{T}_x \wedge g)$, and from (2.17) : this proves (3.6).

3.7. *Particular cases.* In view of (3.3), we can write (3.6) in the form

(3.8) $$\{\mathsf{T}_x(t)g(t-x)\} = \mathsf{T}_x g \qquad (\text{for } x \in \mathbf{R} \text{ and } g(\ ) \in L^{\mathrm{loc}}(\omega)).$$

This equation is a useful substitute for the Laplace-transform identity

$$\mathfrak{L}[\mathsf{T}_x(t)g(t-x)] = e^{-xs}\mathfrak{L}[g(t)] .$$

Let $\text{Ц}(\ )$ be the function $1(\ ) - 1_+(\ )$; that is,

(3.9) $$\text{Ц}(\ ) = 1(\ ) - \mathsf{T}_0(\ ) .$$

From (0.1) and (3.0) it follows that $g_+(\ ) = T_0(\ )g(\ )$; but (3.8) then gives $\{g_+(t)\} = \mathsf{T}_0 g$, so that

(3.9.1) $$\{g_{\text{Ц}}(t)\} = g - \mathsf{T}_0 g = \text{Ц}g \qquad (\text{by (0.2) and (3.9)}).$$

Setting $g(\ ) = T_0(\ )$ in (3.8) we see that $\mathsf{T}_0 = \{\mathsf{T}_0(t)\mathsf{T}_0(t)\} = \mathsf{T}_0\mathsf{T}_0$, whence it results that

(3.10) $$\mathsf{T}_0\text{Ц} = 0, \quad \mathsf{T}_0^2 = \mathsf{T}_0, \text{ and } \text{Ц}^2 = \text{Ц}.$$

If $A \in \mathscr{A}_\omega$ we set $A_+ = \mathsf{T}_0 A$ and $A_{\text{Ц}} = \text{Ц}A$; clearly, $A = A_{\text{Ц}} + A_+$ and $A_{\text{Ц}}A_+ = 0$. If $B \in \mathscr{A}_\omega$ then

(3.11) $$A_{\text{Ц}}B = A_{\text{Ц}}B_{\text{Ц}} = \text{Ц}(AB)$$

and

(3.12) $$A_+B = AB_+ = A_+B_+ = (AB)_+ .$$

Let $(B\mathscr{A})$ denote the set $\{BA : A \in \mathscr{A}\}$; it is easily seen that $(\text{Ц}\mathscr{A})$ and $(\mathsf{T}_0\mathscr{A})$ are ideals in the algebra $\mathscr{A}_\omega$, and $\mathscr{A}_\omega$ is the direct sum of these ideals:

(3.13) $$\mathscr{A} = (\text{Ц}\mathscr{A}) \oplus (T_0\mathscr{A}) .$$

Note that $\operatorname{sgn} t = -\text{Ц}(t) + \mathsf{T}_0(t)$, so that $\operatorname{sgn} = -\text{Ц} + \mathsf{T}_0$. It is easily verified that $\{|t|\} = D^{-1}\operatorname{sgn}$, and

(3.14) $$\{e^{a|t|}\} = \frac{D^2 + aD\operatorname{sgn}}{D^2 - a^2} .$$

If $\alpha > 0$ we set

$$1^\alpha(\ ) = -\mathsf{T}_{-\alpha}(\ ) + \mathsf{T}_\alpha(\ ) ;$$

from (3.8) it follows readily that

$$1^\alpha g = \{-\mathsf{T}_{-\alpha}(t)g(t+\alpha) + \mathsf{T}_\alpha(t)g(t-\alpha)\} .$$

If $h(\ )$ is a periodic function of period α, then

$$h = \frac{\{[1 - 1^\alpha(t)]h(t)\}}{1 - 1^\alpha} .$$

Finally, if $\alpha \geqq 0$ and $\beta \geqq 0$ then $1^\alpha 1^\beta = 1^{\alpha+\beta}$ and

(3.15) $$\mathsf{T}_\alpha \mathsf{T}_\beta = \mathsf{T}_{\alpha+\beta} :$$

we define 1^α to be 1 in case $\alpha = 0$.

3.16. *Other operational calculi.* Mikusiński's injection (of $L^{\mathrm{loc}}(0, \infty)$ into the Mikusiński field) is an extension of the Laplace transformation; analogously, our injection $f(\ ) \mapsto \{f(t)\}$ is comparable to the two-sided Laplace transformation. However, if $\mathfrak{L}\{f(t)\}$ denotes the Laplace transform of the function $f(\ )$, then

$$\mathfrak{L}\{e^{-t} - e^t\}(s) = \frac{2}{1 - s^2} = \mathfrak{L}\{e^{-|t|}\}(s) ;$$

the first equation holds for $s > 1$, the second for $0 < s < 1$. This contrasts with

$$\{e^{-t} - e^t\} = \frac{2D}{1 - D^2} \neq \{e^{-|t|}\} \qquad \text{(see (3.14)).}$$

A problem which is not Laplace-transformable is discussed in 6.7.

THEOREM 3.17. *If $\alpha > 0$ and $h(\ ) \in L^{\mathrm{loc}}(\omega)$, then the equation*

(3.18) $$\left\{\sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t) g(t - k\alpha)\right\} = g\left\{\sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t)\right\}$$

holds for any scalar-valued sequence c_k ($k = 0, \pm 1, \pm 2, \pm 3, \cdots$).

Proof. Set

(1) $$g(\mathsf{T}_\alpha)(\ ) = \sum_{k=-\infty}^{\infty} c_k g_{k\alpha}(\ ) .$$

Take any t in ω: there exists an integer $m > 0$ such that $|t| < m\alpha$. Clearly,

(2) $$g(\mathsf{T}_\alpha)(t) = \sum_{|k|<m} c_k g_{k\alpha}(t) + \sum_{|i|\geqq m} c_i g_{i\alpha}(t) .$$

Since $t \in (-m\alpha, m\alpha) \subset (-|i|\alpha, |i|\alpha)$ and since $g_{i\alpha}(\ ) = 0$ on the interval

$(-|i|\alpha, |i|\alpha)$ (by (3.2) and (3.3)), we have $g_{i\alpha}(t) = 0$: consequently, the series (1) converges, and (3.3) gives

$$(3) \qquad g(\mathsf{T}_\alpha)(t) = \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t) g(t - k\alpha) .$$

The equations

$$g(\mathsf{T}_\alpha) = D\{1 \wedge g(\mathsf{T}_\alpha)\} = D\Big\{\sum_{k=-\infty}^{\infty} c_k(1 \wedge g_{k\alpha})(t)\Big\}$$

are from (2.17) and (1); from 3.5 it therefore follows that

$$(4) \qquad g(\mathsf{T}_\alpha) = D\Big\{\sum_{k=-\infty}^{\infty} c_k(\mathsf{T}_{k\alpha} \wedge g)(t)\Big\} .$$

Equation (4) gives

$$(5) \qquad g(\mathsf{T}_\alpha) = D\Big\{g \wedge \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t)\Big\} = g\Big\{\sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t)\Big\} :$$

the second equation is from 1.40. Conclusion (3.18) now comes from (3) and (5).

REMARK 3.19. If c is a scalar and if $\lambda \geqq 0$, the equation

$$\frac{1^\lambda h}{1 - c1^\alpha} = \Big\{\sum_{k=0}^{\infty} c^k(h_{\amalg}(t + k\alpha + \lambda) + h_{+}(t - k\alpha - \lambda))\Big\}$$

is not hard to verify; it is the two-sided analogue of Theorem 5.29 in [5].

THEOREM 3.20. *If $x \in \mathbf{R}$ and $w(\ ) \in W_\omega$ then*

$$(3.21) \qquad .\mathsf{T}_x w(t) = \mathsf{T}_x(t) w(t - x) \qquad (\textit{for } t \in \omega).$$

Proof. The equations

$$\{\mathsf{T}_x(t) w(t - x)\} = \mathsf{T}_x w = .\mathsf{T}_x w$$

come from (3.8) and (2.20): Conclusion (3.21) now follows from (2.15).

LEMMA 3.22. *If $R \in \mathscr{A}_\omega$ and $w(\ ) \in W_\omega$ then*

$$(3.23) \qquad .R_{\amalg} w(\ ) = [.Rw]_{\amalg}(\ ) .$$

Proof. Setting $g = .Rw$ in (3.9.1), we obtain

$$(1) \qquad \{[.Rw]_{\amalg}(t)\} = \amalg\{.Rw(t)\} = \amalg R\{w(t)\} :$$

the last equation is from 1.39. Since $B_{\amalg} = \amalg B$ (by definition), Equation (1) becomes

$$\{[.Rw]_{\amalg}(t)\} = R_{\amalg}\{w(t)\} = \{.R_{\amalg}w(t)\}: \tag{2}$$

the second equation is from 1.39. Conclusion (3.23) is immediate from (2) and 1.33.

THEOREM 3.24. *If* $A \in \mathscr{A}_\omega$ *and* $B \in \mathscr{A}_\omega$, *then*

$$A_{\amalg} = B_{\amalg} \text{ if (and only if) } A \text{ agrees with } B \text{ on } (\omega_-, 0).$$

Proof. Recall that $(\omega_-, 0) = \omega \cap (-\infty, 0)$. Let $w(\ )$ be any element of W_ω; the equations

$$[.Aw]_{\amalg}(\ ) = .A_{\amalg}w(\ ) = .B_{\amalg}w(\ ) = [.Bw]_{\amalg}(\ ) \tag{3}$$

are from (3.23), our hypothesis $A_{\amalg} = B_{\amalg}$, and (3.23). Since $h_{\amalg}(t) = h(t)$ for $t < 0$ (see (0.1)–(0.2)), Equation (3) implies

$$.Aw(t) = .Bw(t) \qquad (\text{for } \omega_- < t < 0). \tag{4}$$

From (4) and 1.31 we see that A agrees with B on $(\omega_-, 0)$. Conversely, if A agrees with B on $(\omega_-, 0)$, then (4) holds, whence the equation $[.Aw]_{\amalg}(\ ) = [.Bw]_{\amalg}(\ )$: combining this with (3.23), we obtain

$$.A_{\amalg}w(\ ) = .B_{\amalg}w(\ ) \qquad (\text{for every } w(\ ) \text{ in } W_\omega),$$

which gives $A_{\amalg} = B_{\amalg}$.

THEOREM 3.25. *The space* $(\mathsf{T}_0\mathscr{A})$ *consists of all the elements of* $\mathscr{A}_\omega$ *which agree with* 0 *on* $(\omega_-, 0)$. *Moreover,*

$$B \in (\mathsf{T}_0\mathscr{A}) \Longleftrightarrow B_{\amalg} = 0 \Longleftrightarrow B = B_+ . \tag{3.26}$$

Proof. We begin with (3.26). If $B \in (\mathsf{T}_0\mathscr{A})$ then $B = \mathsf{T}_0A$ for some A in $\mathscr{A}_\omega$; therefore, $\amalg B = 0$ (by (3.10)); this gives $B_{\amalg} = 0$; since $B = B_{\amalg} + B_+$, the equation $B_{\amalg} = 0$ implies $B = B_+$; if $B = B_+$ then $B = \mathsf{T}_0B$, whence $B \in (\mathsf{T}_0\mathscr{A})$. This proves (3.26).

If $B \in (\mathsf{T}_0\mathscr{A})$ then $B_{\amalg} = 0$ (by (3.26)), which implies that B agrees with 0 on the interval $(\omega_-, 0)$ (by 3.24). Conversely, if B agrees with 0 on the interval $(\omega_-, 0)$, then $B_{\amalg} = 0$ (by (3.24)): the conclusion $B \in (\mathsf{T}_0\mathscr{A})$ now comes from (3.26).

THEOREM 3.27. *If* $B \in \mathscr{A}_\omega$ *is such that the equation* $f = B_{\amalg}$ *holds for some* $f(\ )$ *in* $L^{\mathrm{loc}}(\omega)$, *then* f *agrees with* B *on the interval* $(\omega_-, 0)$.

Proof. The equations

$$f_{\text{Ц}} = \text{Ц}f = \text{Ц}B_{\text{Ц}} = \text{Ц}^2B = \text{Ц}B = B_{\text{Ц}} \tag{3.28}$$

are from the definition ($f_{\text{Ц}} = \text{Ц}f$), from our hypothesis, from the definition ($B_{\text{Ц}} = \text{Ц}B$), from (3.10), and again from the definition ($B_{\text{Ц}} = \text{Ц}B$). From (3.28) and 3.24 we see that f agrees with B on the interval $(\omega_-, 0)$.

4. **The topological space $\mathscr{A}_\omega$.** Let the function space W_ω be endowed with the topology of pointwise convergence on the interval ω: this enables us to topologize $\mathscr{A}_\omega$ by endowing it with the product topology (recall that $\mathscr{A}_\omega$ consists of mappings of W_ω into the topological space W_ω). Consequently, the equation

$$B = \lim_{\lambda \to \mu} A_\lambda \qquad (\text{for } B \text{ and } A_\lambda \text{ in } \mathscr{A}_\omega)$$

means that

$$.Bw(t) = \lim_{\lambda \leftarrow \mu} .A_\lambda w(t) \qquad (\text{for } t \in \omega \text{ and } w(\ ) \in \omega_\omega). \tag{1}$$

It is immediately clear that $\mathscr{A}_\omega$ is a locally convex Hausdorff vector space: in fact, H. Shultz has proved that it is sequentially complete and that the multiplication of the algebra $\mathscr{A}_\omega$ is sequentially continuous.

We denote by $\lim A_\lambda$ the mapping that assigns to each $w(\ )$ in W_ω the function $.Bw(\ )$ defined by (1):

$$.\Big(\lim_{\lambda \to \mu} A_\lambda\Big) w(\ ) = \lim_{x \to \mu} .A_\lambda w(\ ) \qquad (\text{every } w(\ ) \text{ in } W_\omega). \tag{4.1}$$

If $x \mapsto F(x)$ is a mapping into $\mathscr{A}_\omega$, we set

$$\frac{d}{dx} F(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [F(x + \varepsilon) - F(x)]\,; \tag{4.2}$$

in view of (4.1), this means that $dF(x)/dx$ is the operator defined for any $w(\ )$ in W_ω by

$$.\Big(\frac{d}{dx} F(x)\Big) w(\ ) = \frac{\partial}{\partial x} (.F(x)w(\ ))\,. \tag{4.3}$$

THEOREM 4.4. *If $x \in \mathbf{R}$, then* $\Big(\dfrac{d}{dx}\Big)\mathsf{T}_x = -\mathsf{T}_x D$.

Proof. Take any $w(\ )$ in W_ω, take any $t \neq x$ in ω; from (4.3) we see that

$$.\Big(\frac{d}{dx} \mathsf{T}_x\Big) w(t) = \frac{\partial}{\partial x} (.\mathsf{T}_x w(t)) = \frac{\partial}{\partial x} \mathsf{T}_x(t) w(t - x)\,: \tag{2}$$

the second equation is from (3.21). Set $E_1 = \{x: x > t\}$ and $E_2 = \{x: x < t\}$: note that the function $x \mapsto \mathsf{T}_x(t)$ is constant on E_k when $k = 1, 2$; consequently, since $x \neq t$ then $x \in E_k$ for some k, whence $\partial \mathsf{T}_x(t)/\partial x = 0$; we can use this to infer from (2) that

$$.\left(\frac{d}{dx}\,\mathsf{T}_x\right)w(t) = \mathsf{T}_x(t)\,\frac{\partial}{\partial x}\,w(t-x) = -\mathsf{T}_x(t)w'(t-x) \qquad (\text{all } t \neq x).$$

Consequently, we may use (3.21) to write

$$.\left(\frac{d}{dx}\,\mathsf{T}_x\right)w(\ ) = -.\mathsf{T}_x w'(\ ) \qquad (\text{all } w(\ ) \text{ in } W_\omega).$$

Calling $B = dT_x/dx$, this gives $.Bw(\ ) = -.\mathsf{T}_x Dw(\ )$, whence the conclusion $B = -\mathsf{T}_x D$.

COROLLARY 4.5. *if* $x \in \mathbf{R}$ *then* $D\mathsf{T}_x = \lim_{\varepsilon \to 0+} (1/\varepsilon)(\mathsf{T}_x - \mathsf{T}_{x+\varepsilon})$.

Proof. From 4.4 and (4.2) it follows that

$$-\mathsf{T}_x D = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\,(\mathsf{T}_{x+\varepsilon} - \mathsf{T}_x)\ ,$$

which implies directly our conclusion.

REMARK 4.6. Corollary 4.5 indicates that DT_x corresponds to the Dirac delta distribution δ_x concentrated at the point x.

THEOREM 4.7. *If* $F_k(\ )$ $(k = 0, \pm 1, \pm 2, \pm 3, \cdots)$ *is a sequence in* $L^{\mathrm{loc}}(\omega)$, *then*

$$\sum_{k=-\infty}^{\infty} \mathsf{T}_{k\alpha} F_k = \left\{\sum_{k=-\infty}^{\infty} \mathsf{T}_{k\alpha}(t) F_k(t - k\alpha)\right\}. \tag{4.8}$$

Proof. Let $\mathsf{T}_{k\alpha} F_k(\ )$ be the function defined by

$$\mathsf{T}_{k\alpha} F_k(t) = \mathsf{T}_{k\alpha}(t) F_k(t - k\alpha)\ . \tag{1}$$

Set

$$f_s(\ ) = \sum_{k=-s}^{s} \mathsf{T}_{k\alpha} F_k(\ )\ . \tag{2}$$

For any integer $n \geqq 1$, observe that

$$f_\infty(\ ) = f_n(\ ) + \sum_{|i|>n} \mathsf{T}_{i\alpha} F_i(\ )\ ; \tag{3}$$

since $(-n\alpha, n\alpha) \subset (-|i|\alpha, |i|\alpha)$ and since $\mathsf{T}_{i\alpha} F_i(\ ) = 0$ on the interval $(-|i|\alpha, |i|\alpha)$ (because of (3.2) and (1)), we may conclude that $\mathsf{T}_{i\alpha} F_i(\ ) =$

0 on the interval $(-n\alpha, n\alpha)$: consequently, (3) becomes

$$(4) \qquad f_\infty(\ ) = f_n(\ ) \text{ on } (-n\alpha, n\alpha) \text{ for any integer } n \geqq 1.$$

If $t \in \omega$ there exists an integer $m \geqq 1$ such that $t \in (-m\alpha, m\alpha)$: from (4), (2), and (1) we see that

$$(5) \qquad \sum_{k=-\infty}^{\infty} \mathsf{T}_{k\alpha}(t) F_k(t - k\alpha) = f_\infty(t) = \sum_{k=-m}^{\infty} \mathsf{T}_{k\alpha} F_k(t) .$$

On the other hand,

$$(6) \qquad f_n = \left\{ \sum_{k=-n}^{n} \mathsf{T}_{k\alpha} F_k(t) \right\} = \sum_{k=-k}^{n} \mathsf{T}_{k\alpha} F_k ;$$

the second equation is from (3.8) and (1).

In view of (5)-(6), the proof of (4.8) will be accomplished by showing that

$$(7) \qquad \lim_{n\to\infty} f_n = f_\infty .$$

To that effect, take any $w(\ )$ in W_ω, and any t in the interval ω; we must prove that

$$(8) \qquad \lim_{n\to\infty} .f_n w(t) = .f_\infty w(t) .$$

Observe that there exists an integer $m \geqq 1$ such that $|t| < m\alpha$; suppose that $n \geqq m$; from (4) and 1.32 it follows that the operators f_n and f_∞ agree on $(-n\alpha, n\alpha)$: therefore, 1.31 gives

$$(9) \qquad .f_n w(t) = .f_\infty w(t) \qquad \text{(for all } n \geqq m);$$

this is because $w(\ ) \in W_\omega$ and $-m\alpha < t < m\alpha$. Conclusion (8) is immediate from (9).

REMARK 4.9. Let c_k $(k = 0, \pm 1, \pm 2, \pm 3, \cdots)$ be a scalar-valued sequence. Setting $F_k(\ ) = c_k$ in (4.8), we obtain

$$(4.10) \qquad \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha} = \left\{ \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t) \right\} ;$$

combining with (3.18):

$$(4.11) \qquad \left\{ \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t) g(t - k\alpha) \right\} = g \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha} .$$

Obviously, if $g(\ )$ is a periodic function of period $\alpha > 0$, then (4.11) becomes

$$(4.12) \qquad g \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha} = \left\{ g(t) \sum_{k=-\infty}^{\infty} c_k \mathsf{T}_{k\alpha}(t) \right\} .$$

5. Derivative of an operator. Given $A \in \mathscr{A}_\omega$ and $B \in \mathscr{A}_\omega$, let us indicate by $A \subset B$ *the existence of a number* $a < 0$ *such that* A *agrees with* B *on the interval* $(a, 0)$. The notion of "agreeing with" has been defined in 1.31. Recall that $F = \{F(t)\}$ (see 2.13); as usual, $F(0-)$ denotes the limit of $F(t)$ as t approaches zero through negative values.

THEOREM 5.0. *Suppose that* $B \in \mathscr{A}_\omega$. *There is at most one scalar* c_1 *such that the equation* $c_1 = f_1(0-)$ *holds for some function* $f_1(\ )$ *in* $L^{\text{loc}}(\omega)$ *with* $f_1 \subset B$.

Proof. Suppose that the equation $c_2 = f_2(0-)$ holds for some function $f_2(\ )$ in $L^{\text{loc}}(\omega)$ with $f_2 \subset B$: we must prove that $c_1 = c_2$. By definition, there exists an interval $(a_k, 0)$ such that f_k agrees with B on the interval $(a_k, 0)$ (for $k = 1, 2$); from 1.31 we now see that f_1 agrees with f_2 on $(a, 0)$, where a is the largest of the two negative numbers a_1 and a_2; from 1.32 it follows that $f_1(\ ) = f_2(\ )$ on $(a, 0)$, whence $f_1(0-) = f_2(0-)$: this proves that $c_1 = c_2$.

5.1. Derivable operators. An operator B is said to be **derivable** *if* $B \in \mathscr{A}_\omega$ *and if there exists a function* $f_1(\ )$ *in* $L^{\text{loc}}(\omega)$ *such that* $|f_1(0-)| < \infty$ *and* $f_1 \subset B$.

5.2. Initial value of an operator. If B is derivable, we denote by $\langle B, 0-\rangle$ the unique scalar c_1 such that the equation $c_1 = f_1(0-)$ holds for some function $f_1(\ )$ in $L^{\text{loc}}(\omega)$ such that $f \subset B$; we also set

$$\partial_t B = DB - \langle B, 0-\rangle D . \tag{5.3}$$

The uniqueness of c_1 comes from 5.0, while the existence of c_1 can be verified by setting $c_1 = f_1(0-)$ in 5.1.

REMARKS 5.4. If $f(\ )$ is a function in $L^{\text{loc}}(\omega)$ such that $|f(0-)| < \infty$, then the operator f is derivable, and $\langle f, 0-\rangle = f(0-)$ (this is immediate from 5.1); from (5.3) we see that

$$\partial_t f = Df - f(0-)D .$$

5.5. Suppose that $f(\ )$ is continuous on ω; if $f'(\ )$ has at most countably-many discontinuities and is integrable an each compact sub-interval of the open interval ω, then

$$\partial_t f = \{f'(t)\} \quad \text{and} \quad \langle f, 0-\rangle = f(0):$$

this follows immediately from 2.4, 2.13, and 5.4.

5.6. Suppose that $B \in \mathscr{S}_\omega$. If $f(\ ) \in L^{\text{loc}}(\omega)$ is such that $|f(0-)| < \infty$ and $f \subset B$, then B is derivable and $\langle B, 0- \rangle = f(0-)$: this follows directly from 5.0–5.2.

5.7. If $B \in \mathscr{S}_\omega$ is such that the equation $B_{\amalg} = f$ holds for some function $f(\ )$ in $L^{\text{loc}}(\omega)$ such that $|f(0-)| < \infty$, then B is derivable and $\langle B, 0- \rangle = f(0-)$. This is immediate from 3.27 and 5.6.

THEOREM 5.8. *Suppose that $\alpha > 0$. If A_k ($k = 0, \pm 1, \pm 2, \pm 3, \cdots$) is a sequence in $\mathscr{S}_\omega$ such that the equation*

$$B = \sum_{k=-\infty}^{\infty} \mathsf{T}_{k\alpha} A_k \tag{1}$$

defines an element B of $\mathscr{S}_\omega$, then B is derivable, $\langle B, 0- \rangle = 0$, and $\partial_t B = DB$.

Proof. Take any $w(\ )$ in W_ω. From (1) and (3.21) it follows that

$$.Bw(t) = \mathsf{T}_0(t).A_0 w(t) + \sum_{k \neq 0} \mathsf{T}_{k\alpha}(t).A_k w(t - k\alpha) \qquad (\text{for } t \in \omega). \tag{2}$$

If $k \neq 0$ we see from (3.2) that $\mathsf{T}_{k\alpha}(\ ) = 0$ on $(-\alpha, \alpha)$: consequently, the equation (2) implies that

$$.Bw(t) = \mathsf{T}_0(t).A_0 w(t) \qquad (\text{for } |t| < \alpha). \tag{3}$$

Since $\mathsf{T}_0(\ ) = 0$ on $(-\alpha, 0)$, it now follows from (3) that $.Bw(t) = 0$ for $-\alpha < t < 0$ and for any $w(\ )$ in W_ω: therefore, the operator 0 agrees with B on $(-\alpha, 0)$, whence $0 \subset B$; the conclusion $\langle B, 0- \rangle = 0$ now follows from 5.6; in view of (5.3), the proof is concluded.

THEOREM 5.9. *Suppose that $x \in \mathbf{R}$. Each element of $(\mathsf{T}_x.\mathscr{S})$ is infinitely derivable; in fact,*

$$\langle B, 0- \rangle = 0 \quad \text{and} \quad \partial_t^k B = D^k B \qquad (\text{for each integer } k \geqq 1) \tag{5.10}$$

whenever $B \in (\mathsf{T}_x.\mathscr{S})$.

Proof. Note that $(\mathsf{T}_x.\mathscr{S})$ is the set $\{\mathsf{T}_x A \colon A \in \mathscr{S}_\omega\}$. If B is an element of $(\mathsf{T}_x.\mathscr{S})$, then $B = \mathsf{T}_x A$ for some A in $\mathscr{S}_\omega$: clearly, B can be written in the form (1) (set $\alpha = |x|$ and $A_k = A$ for $k = \operatorname{sgn} x$ and $A_k = 0$ for other values of k): the conclusion $\langle B, 0- \rangle = 0$ now comes from 5.8. Since $\partial_t^k B = B$ (by definition) for $k = 0$, we proceed by induction on $k \geqq 1$. To that effect, we assume that $\partial_t^n B = D^n B$: clearly,

$$(4) \qquad \partial_t^{n+1}B = \partial_t(D^nB) = D^{n+1}B + \langle D^nB, 0-\rangle D .$$

On the other hand, $D^nB = D^n\mathsf{T}_xA = \mathsf{T}_xD^nA$; consequently, D^nB belongs to $(\mathsf{T}_x\mathscr{A})$, whence $\langle D^nB, 0-\rangle = 0$ (by what we established at the beginning of this proof); therefore (4) gives $\partial_t^{n+1}B = D^{n+1}B$. The induction proof is completed.

Note 5.11. Both T_x and the Dirac delta distribution $D\mathsf{T}_x$ belong to the space $(\mathsf{T}_x\mathscr{A})$. If $B = B_+$ or if $B_{\sqcup} = 0$ then B belongs to $(\mathsf{T}_0\mathscr{A})$: see 3.25.

THEOREM 5.12. *Set $a = \omega_-$ and suppose that $B \in \mathscr{A}_\omega$. If the equation $B_{\sqcup} = f$ holds for some function $f(\ )$ in $L^1(a, 0)$, there exists a unique scalar c_1 such that the equation*

$$(5) \qquad c_1 = \int_a^0 f_1(u)du$$

holds for some $f_1(\ )$ in $L^1(a, 0)$ with $f_1 = B_{\sqcup}$.

Proof. Clearly, such a scalar exists. If

$$(6) \qquad c_2 = \int_a^0 f_2(u)du$$

for $f_2(\ )$ in $L^1(a, 0)$ and $f_2 = B_{\sqcup}$, then both f_1 and f_2 agree with B on $(a, 0)$ (by 3.27): therefore, $f_1(\ )$ equals $f_2(\ )$ almost-everywhere on $(a, 0)$ (by 1.32); the conclusion $c_1 = c_2$ now comes from (5)-(6).

5.13. The anti-derivative. Let B be as in 5.12. We set

$$(7) \qquad \int_a^t B = D^{-1}B + c_1 .$$

In a subsequent paper we shall prove that

$$\left\langle \int_a^t B, 0- \right\rangle = c_1 \quad \text{and} \quad \partial_t \int_a^t B = B .$$

In case $B = f$ with $f(\ ) \in L^1(a, 0)$ and $f(\ ) \in L^{\mathrm{loc}}(\omega)$, it follows immediately from (2.19) and (3) (7) that

$$\int_a^t f = \left\{ \int_a^t f(u)du \right\} .$$

6. Four problems. Recall that $D\mathsf{T}_x$ corresponds to the Dirac delta distribution concentrated at the point x (see 4.6), it is infinitely derivable (see 5.11). If an operator A is twice derivable, it follows directly from (5.3) that

(6.0) $$\partial_t^2 A = D^2 A - \langle A, 0-\rangle D^2 - \langle \partial_t A, 0-\rangle D\ .$$

We shall need two more facts. Each operator A in $\mathscr{A}_\omega$ can be written as a sum

(6.1) $$A = A_{\text{II}} + A_+,\ \text{where}\ A_+ = A\mathsf{T}_0 \qquad \text{(see 3.7)};$$

moreover, if $g(\ ) \in L^{\text{loc}}(\omega)$ then

(6.2) $$g\mathsf{T}_0 = \{\mathsf{T}_0(t)g(t)\} \qquad \text{(see (3.8))}\ .$$

6.3. *First problem.* Given two scalars m and a, to find an operator y such that

(6.4) $$m\partial_t y = D\mathsf{T}_0 \quad \text{and} \quad \langle y, 0-\rangle = a:$$

Definition (5.3) gives $mDy - maD = D\mathsf{T}_0$, whence $y(\ ) = a + m^{-1}\mathsf{T}_0(\ )$. This same problem has been discussed in [5, p. 38].

6.5. *Second problem.* The equations

(1) $$i = \partial_t q \quad \text{and} \quad q = CE$$

relate the current i to the change q in a simple electric circuit having capacitance C, impressed electromotive force E, no inductance, and no resistance (see 7.19 in [5]). From (1) and (5.3) it follows that

(2) $$i = CDE - \langle q, 0-\rangle D\ .$$

Multiplying by T_0 both sides of (2), we can use (6.1) to write

(3) $$i_+ = CDE_+ - \langle q, 0-\rangle D\mathsf{T}_0\ .$$

If there is a short-circuit at the time $t = 0$, then $E_+ = 0$, so that (3) gives the answer $i_+ = -\langle q, 0-\rangle D\mathsf{T}_0$: this is an impluse whose magnitude is the negative of the initial charge $\langle q, 0-\rangle$.

6.6. *Third problem.* Given a scalar c, to find an operator y such that

$$\partial_t^2 y + y = \partial_t(D\mathsf{T}_0) \quad \text{and} \quad \langle \partial_t y, 0-\rangle = \langle y, 0-\rangle = c\ .$$

Since $\partial_t(D\mathsf{T}_0) = D^2\mathsf{T}_0$ (by 5.9), we can use (6.0) to write

$$(D^2 + 1)y = D^2\mathsf{T}_0 + \langle y, 0-\rangle D^2 + \langle \partial_t y, 0-\rangle D\,;$$

we now use the initial conditions and solve for y:

(4) $$y = \frac{D^2}{D^2+1}\mathsf{T}_0 + c\left(\frac{D^2}{D^2+1} + \frac{D}{D^2+1}\right).$$

From (4) and (2.10)–(2.11) it results that

$$y = \{\cos t\}\mathsf{T}_0 + c(\sin + \cos) ,$$

whence our conclusion $y(\ ) = \mathsf{T}_0(\ )\cos + c(\sin + \cos)$ now comes directly from (6.2) and 1.33.

Last problem 6.7. To find an element y of $\mathscr{A}_\omega$ such that

$$\partial_t^2 y + y = \sum_{k=-\infty}^{\infty} D\mathsf{T}_{2k\pi} . \tag{5}$$

Setting $c_0 = \langle y, 0-\rangle$ and $c_1 = \langle \partial_t y, 0-\rangle$, we see from (6.0) that

$$(D^2 + 1)y = c_0 D^2 + c_1 D + D \sum_{k=-\infty}^{\infty} \mathsf{T}_{2k\pi} . \tag{6}$$

Solving (6) for y, we obtain $y = c_0 \cos + c_1 \sin + y_p$, where

$$y_p = \frac{D}{D^2+1} \sum_{k=-\infty}^{\infty} \mathsf{T}_{2k\pi} = \{\sin t\} \sum_{k=-\infty}^{\infty} \mathsf{T}_{2k\pi} : \tag{7}$$

the second equation is from (2.11). From (7) and (4.12) it now follows that

$$y_p = \left\{\sin t \sum_{k=-\infty}^{\infty} \mathsf{T}_{2k\pi}(t)\right\} . \tag{8}$$

From (8) and (2.15) we can now write

$$y_p(t) = \sin t \sum_{k=-\infty}^{\infty} \mathsf{T}_{2k\pi}(t) = \left(1 + \left[\frac{t}{2\pi}\right]\right) \sin t ; \tag{9}$$

as usual, $[t/2\pi]$ is the greatest integer $< t/2\pi$ (the last equation follows directly from the definition of $\mathsf{T}_x(\ )$). In case $\omega = \mathbf{R}$, the answer (9) to the problem (5) cannot be obtained by the Fourier transformation nor by the distributional two-sided Laplace transformation.

Added in proof. There still remains to connect the theory presented in this paper with the theory of distributions; this has been done in the Research Announcement "An algebra of generalized functions on an open interval; two-sided operational calculus" (by Gregers Krabbe), Bull. Amer. Math. Soc. 77 (1971), 78–84.

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