AN ALGEBRA OF GENERALIZED FUNCTIONS ON AN OPEN INTERVAL: TWO-SIDED OPERATIONAL CALCULUS

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Let \((a, b)\) be any open sub-interval of the real line, such that \(-\infty \leq a < 0 < b \leq \infty\). Let \(L^{\text{loc}}(a, b)\) be the space of all the functions which are integrable on each interval \((a', b')\) with \(a < a' < b' < b\). There is a one-to-one linear transformation \(\mathcal{I}\) which maps \(L^{\text{loc}}(a, b)\) into a commutative algebra \(\mathcal{A}\) of (linear) operators. This transformation \(\mathcal{I}\) maps convolution into operator-multiplication; therefore, this transformation \(\mathcal{I}\) is a useful substitute for the two-sided Laplace transformation; it can be used to solve problems that are not solvable by the distributional transformations (Fourier or bi-lateral Laplace).

In essence, the theme of this paper is a commutative algebra \(\mathcal{A}\) of generalized functions on the interval \((a, b)\); besides containing the function space \(L^{\text{loc}}(a, b)\), the algebra \(\mathcal{A}\) contains every element of the distribution space \(\mathcal{D}'(a, b)\) which is regular on the interval \((a, 0)\). The algebra \(\mathcal{A}\) is the direct sum \(\mathcal{A} \oplus \mathcal{A}_+\), where \(\mathcal{A}\) (respectively, \(\mathcal{A}_+)\) \((a, 0)\) (respectively, to the interval \((0, b)\)). There is a subspace \(\mathcal{Y}\) of \(\mathcal{A}\) such that, if \(y \in \mathcal{Y}\), then \(y\) has an "initial value" \(\langle y, 0{-}\rangle\) and a "derivative" \(\partial_t y\) (which corresponds to the usual distributional derivative). If \(y\) is a function \(f(\ )\) which is locally absolutely continuous on \((a, b)\), then \(y\) belongs to \(\mathcal{Y}\), the initial value \(\langle y, 0{-}\rangle\) equals \(f(0)\), and \(\partial_t y\) corresponds to the usual derivative \(f'(\ )\). If \(y\) is a distribution (such as the Dirac distribution) whose support is a locally finite subset of the interval \((a, b)\), then both \(y\) and \(\partial_t y\) belong to the subspace \(\mathcal{Y}\). In case \(a = -\infty\) and \(b = \infty\), the subspace \(\mathcal{Y}\) contains the distribution space \(\mathcal{D}'_+\).

The resulting operational calculus takes into account the behavior of functions to the left of the origin (in case \(a = -\infty\) and \(b = \infty\), the whole real line is accounted for—whereas Mikusiński’s operational calculus only accounts for the positive axis). Since the functions are not subjected to growth restrictions, the transformation \(\mathcal{I}\) is a useful substitute for the two-sided Laplace transformation (no strips of convergence need to be considered: see Examples 2.21 and the four problems 6.3–6.7). Problems such as

\[
\frac{d^2}{dt^2} y + y = \sec \frac{\pi t}{2\alpha} \quad (-\alpha < t < \alpha)
\]
can be solved by calculations which duplicate the ones that would arise if the Laplace transformation could be applied to such problems.

The differential equation

\( \frac{d^2 y}{dt^2} + y = \sum_{k=\pm\infty} \delta(t - 2k\pi) \)

is solved in 6.7 in order to illustrate our operational calculus; the right-hand side of this equation represents a series of unit impulses starting at \( t = -\infty \). The differential equation (1) cannot be solved by the distributional Fourier transformation nor by the distributional two-sided Laplace transformation. When \( -\infty = a < t < b = \infty \) the equation

\[ y(t) = c_0 \cos t + c_1 \sin t + \left(1 + \left[ \frac{t}{2\pi} \right] \right) \sin t \]

defines the general solution of the equation (1).

The paper is subdivided as follows. § 1: the space of generalized functions, § 2: two-sided operational calculus, § 3: translation properties, § 4: the topological space \( \mathcal{V}_\omega \), § 5: derivative of an operator, § 6: four problems.

The concepts introduced in § 5 (initial value, derivative, antiderivative of an operator) are more general and more appropriate than the corresponding ones in my textbook [5].

0. Preliminaries. Henceforth, \( \omega \) is an open sub-interval \((\omega_-, \omega_+)\) of the real line \( \mathbb{R} \); we suppose that \( \omega_- < 0 < \omega_+ \). If \( h(\cdot) \) is a function on \( \omega \), we denote by \( h_+(\cdot) \) the function defined by

\[ h_+(t) = \begin{cases} 0 & \text{for } t < 0 \\ h(t) & \text{for } t \geq 0 \end{cases} \]

we set

\[ h_{11}(\cdot) = h(\cdot) - h_+(\cdot) \]

As usual, the support of a function \( f(\cdot) \) (denoted \( \text{Supp} f \)) is the complement of the largest open subset of \( \mathbb{R} \) on which \( f(\cdot) \) vanishes. Let \( e_t(\cdot) \) be the function defined by

\[ e_t(u) = \begin{cases} 1 & \text{for } 0 \leq u < t \\ -1 & \text{for } t < u < 0 \end{cases} \]

and by \( e_t(u) = 0 \) for all other values of \( u \). It will be convenient to denote by \( e_t \) the support of the function \( e_t(\cdot) \); thus, \( e_t \) is the interval with end-points 0 and \( t \):
Unless otherwise specified, suppose that \( f( ) \) and \( g( ) \) belong to \( L^\infty(\omega) \) (this is the space of all the complex-valued functions which are Lebesgue integrable on each interval \((a, b)\) with \( \omega_- < a < 0 < b < \omega_+ \)). We denote by \( f \wedge g( ) \) the function defined by

\[
(0.5) \quad f \wedge g(t) = \int_0^t f(t - u)g(u)\,du \quad \text{ (all } t \text{ in } \omega); \]
that is,

\[
(0.6) \quad f \wedge g(t) = \int_{\epsilon_t}^t f(t - u)e_t(u)g(u)\,du .
\]

**Remark 0.7.** Suppose that \( \omega_- \leq a \leq 0 \leq b < \omega_+ \):

\[
(0.8) \quad \text{if } a < t < b \text{ and } u \in e_t \text{ then } (t - u) \in e_t \subset (a, b) .
\]
This is easily verified.

**Remarks 0.9.** The following properties are direct consequences of (0.1)–(0.8):

\[
(0.10) \quad f \wedge g(t) = f_+ \wedge g(t) = f_+ \wedge g_+(t) \quad \text{ (for } t > 0),
\]
and

\[
(0.11) \quad f \wedge g(t) = f_{11} \wedge g(t) = f_{11} \wedge g_{11}(t) \quad \text{ (for } t < 0).
\]

**Final Remark 0.12.** If \( f_i( ) = f( ) \) and \( g_i( ) = g( ) \) almost-everywhere on \( \omega \), then \( f_i \wedge g_i( ) = f \wedge g( ) \) almost-everywhere on \( \omega \). This is another easy consequence of (0.5)–(0.8).

**Lemma 0.13.** If \( a \leq 0 \leq b \) and if \( f( ) = 0 \) almost-everywhere on the interval \((a, b)\), then \( f \wedge g( ) = 0 \) on \((a, b)\).

**Proof.** If \( t \in (a, b) \) it follows from (0.8) that

\[
u \in e_t \quad \text{ implies } (t - u) \in e_t \subset (a, b) ;
\]
therefore, \( (t - u) \in (a, b) \), whence our hypothesis \( (f( ) = 0 \) almost-everywhere on \((a, b)) \) gives \( f(t - u) = 0 \) for \( u \) almost-everywhere on the interval \( e_t \): the conclusion \( f \wedge g(t) = 0 \) now follows directly from (0.6).

**Lemma 0.14.** Suppose that \( a < 0 < b \). If \( f( ) = 0 \) on the interval \((\omega_-, b)\), then
If \( h(\cdot) \in L^{10\varepsilon}(\omega) \) and if \( h(\cdot) = 0 \) on the interval \((a, \omega_+)\), then

\[
(0.16) \quad h \ast g(t) = -\int_{t-a}^{0} h(t-\tau)g(\tau)d\tau \quad \text{(for} \omega_- < t < a) .
\]

**Proof.** First, the case \( b < t < \omega_+ \). From (0.5) we have

\[
(1) \quad f \ast g(t) = \int_{0}^{t-b} f(t-\tau)g(\tau)d\tau + \int_{t-b}^{t} f(t-u)g(u)du .
\]

From (0.8) we see that

\[
u \in [0, t) \quad \text{implies} \quad (t-u) \in e_t \subset \omega ,
\]

so that \((t-u) \in \omega\). If \( u > t-b \), then \( b > t-u \), whence \( (t-u) \in (\omega_-, b) \); consequently, our hypothesis \((f(\cdot) = 0 \) on \((\omega_-, b))\) gives \( f(t-u) = 0 \) whenever \( u > t-b \): Conclusion (0.15) is now immediate from (1).

Next, the case \( \omega_- < t < a \). From (0.5) we have

\[
(2) \quad h \ast g(t) = -\int_{t}^{t-a} h(t-u)g(u)du - \int_{t-a}^{0} h(t-\tau)g(\tau)d\tau .
\]

From (0.8) we again see that

\[
u \in (t, 0) \quad \text{implies} \quad (t-u) \in e_t \subset \omega ,
\]

so that \((t-u) \in \omega\). If \( u < t-a \) then \( t-u > a \), whence \( (t-u) \in (a, \omega_+) \); consequently, our hypothesis \((h(\cdot) = 0 \) on \((a, \omega_+))\) gives \( h(t-u) = 0 \) whenever \( u < t-a \): Conclusion (0.16) is now immediate from (2).

0.17. **Convolution.** If \( F(\cdot) \) and \( G(\cdot) \) belong to \( L^{1}(\mathbb{R}) \), then \( F \ast G(\cdot) \) is the function defined by

\[
F \ast G(x) = \int_{\mathbb{R}} F(x-u)G(u)du \quad \text{(all} \ x \ \text{in} \ \mathbb{R}) ;
\]

it is well-known that \( F \ast G(\cdot) \in L^{1}(\mathbb{R}) \) (see [1], p. 634). Further,

\[
(0.18) \quad \text{Supp} \ F \ast G \subset (\text{Supp} \ F) + (\text{Supp} \ G) ;
\]

see p. 385 in [2].

**Theorem 0.19.** If \( f(\cdot) \) and \( g(\cdot) \) belong to \( L^{10\varepsilon}(\omega) \), then \( f \ast g(\cdot) \) belongs to \( L^{10\varepsilon}(\omega) \), and

\[
(0.20) \quad f \ast g(\cdot) = g \ast f(\cdot) \quad \text{almost-everywhere on} \ \omega .
\]
Proof. Suppose that \( \omega_- < a < 0 < b < \omega_+ \). If \( h(\cdot) \in L^{1_{\text{loc}}}(\omega) \), we can define the function \( h_b(\cdot) \) by

\[
(1) \quad h_b(t) = \begin{cases} h(t) & \text{for } 0 < t < b \\ 0 & \text{otherwise.} \end{cases}
\]

Similarly, \( h_a(\cdot) \) is defined by

\[
(2) \quad h_a(t) = \begin{cases} h(t) & \text{for } a < t < 0 \\ 0 & \text{otherwise.} \end{cases}
\]

Note that both \( h_b(\cdot) \) and \( h_a(\cdot) \) belong to \( L^1(\mathbb{R}) \). Set

\[
(3) \quad F(\cdot) = -f_a \ast g_a(\cdot) + f_b \ast g_b(\cdot).
\]

The four functions on the right-hand side of (3) are all integrable on \( \mathbb{R} \); consequently, both \( f_a \ast g_a(\cdot) \) and \( f_b \ast g_b(\cdot) \) are integrable on \( \mathbb{R} \); from (3) it now follows that \( F(\cdot) \) is integrable on \( \mathbb{R} \). In consequence, if we can prove that

\[
(4) \quad F(t) = f \bigtriangleup g(t) \quad \text{for } a < t - 0 < b,
\]

then \( f \bigtriangleup g(\cdot) \) is integrable on the arbitrary sub-interval \((a, b)\) of the interval \( \omega \); our conclusion \( f \bigtriangleup g \in L^{1_{\text{loc}}}(\omega) \) is at hand; moreover, Conclusion (0.20) comes from (4)–(3) and the property \( F_1 \ast F_2(\cdot) = F_2 \ast F_1(\cdot) \) (see [1], p. 635). Accordingly, the proof will be accomplished by proving (4).

The proof of (4) is divided into two cases. \textit{First case:} \( a < t < 0 \). Since \( \text{Supp} f_b \) and \( \text{Supp} g_b \) are subsets of the interval \([0, \infty)\), we see from (0.18) that

\[
\text{Supp} f_b \ast g_b \subset [0, \infty);
\]

consequently, \( f_b \ast g_b(\cdot) \) vanishes for \( t < 0 \); therefore, (3) gives

\[
(5) \quad F(t) = -f_a \ast g_a(t) = -\int_a^t f_a(t - u)g(u)du
\]

(for \( a < t < 0 \)); the second equation comes from (2) and the fact that \( g_a(u) = 0 \) when \( u < a \) and when \( u > 0 \). From (5) it follows that

\[
F(t) = -\int_a^t f_a(t - u)g(u)du - \int_0^t f_a(t - \tau)g(\tau)d\tau;
\]

but \( a < u < t \) implies \( t - u > 0 \), so that \( f_a(t - u) = 0 \); therefore,

\[
(6) \quad F(t) = -\int_0^t f_a(t - \tau)g(\tau)d\tau;
\]

but \( 0 > \tau > t \) implies \( t < t - \tau < 0 \); in consequence, since \( a < t \), we
have $a < t - \tau < 0$, so that (2) gives $f_a(t - \tau) = f(t - \tau)$: Equation (6) becomes

$$F(t) = \int_{\epsilon t} f(t - u)e_t(u)g(u)du.$$ 

In view of (6.6), this concludes the proof of (4) in case $a < t < 0$.

**Second case.** $0 < t < b$. As in the first case, we observe that $f_a * g_a(t) = 0$; it is a question of proving that $F(t) = f_b * g_b(t)$: the reasoning is entirely analogous to the one used in the first case.

**THEOREM 0.21.** Suppose that the functions $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ all belong to $L^{10}(\omega)$. If the function $|f| \wedge (|g| \wedge |h|)(\cdot)$ is continuous on $\omega$ then

$$f \wedge (g \wedge h)(x) = (f \wedge g) \wedge h(x) \quad \text{for every } x \in \omega.$$ 

**Proof.** From (0.6) it follows that

$$F \wedge (G \wedge H)(x) = \int_{\epsilon x} \int_{\epsilon t} F(x - t)G(t - u)H(u)dudt.$$ 

Since $|f| \wedge (|g| \wedge |h|)(\cdot)$ is continuous on $\omega$ (by hypothesis), we therefore have $|f| \wedge (|g| \wedge |h|)(\cdot) < \infty$, so that (1) gives

$$\int_{\epsilon x} \int_{\epsilon t} |f(x - t)g(t - u)h(u)|dudt < \infty;$$

we may therefore apply Tonelli's Theorem [3, p. 131] to write

$$f \wedge (g \wedge h)(x) = \int_{\epsilon x} \int_{\epsilon u} f(x - t)g(t - u)h(u)dtdu,$$

where $x_u$ is the appropriate interval. Let us prove that

$$f \wedge (g \wedge h)(x) = \int_0^x h(u) \int_0^x f(x - t)g(t - u)dtdu.$$ 

In case $x > 0$ the double integral is taken over the interior of the triangle

$$\{(u, t): 0 < t < x \text{ and } 0 < u < t\};$$

consequently, the range of $t$ (in the integral (2)) is the interval $x_u = [u, x]$: this establishes (3). In case $x < 0$ the double integral is taken over the triangle

$$\{(u, t): x < t < 0 \text{ and } t < u < 0\};$$

The principle of this proof is due to R. B. Darst.
consequently, the range of \( t \) (in the integral (2)) is the interval \( x_u = [x, u] \); the integral (2) becomes
\[
\int^x f(x - t)g(t - u)h(u) \, dt \, du,
\]
which again establishes the equation (3). The change of variable \( \tau = t - u \) brings (3) into the form
\[
\int_0^u h(u) \int_0^{x-u} f(x - u - \tau)g(\tau) \, d\tau \, du;
\]
consequently, (0.5) gives
\[
\int_0^x h(u) [f \Lambda g(x - u)] \, du:
\]
Conclusion (0.22) is now immediate from (0.5).

**Definition 0.23.** For any integer \( n \geq 1 \) we denote by \( q_n(\cdot) \) the function defined by the equation \( q_n(0) = 0 \) and
\[
q_n(t) = \exp \left( \frac{-1}{nt} \right) \quad (for \ t \neq 0).
\]

**Theorem 0.24.** Suppose that \( f(\cdot) \) belongs to \( L^{1c}(\omega) \). If \( \omega_- \leq a \leq 0 \leq b \leq \omega_+ \) and if
\[
(4) \quad f \Lambda q_n(t) = 0 \quad for \ a < t < b \quad and \quad every \ integer \ n \geq 1,
\]
then \( f(\cdot) \) vanishes almost-everywhere on the interval \( (a, b) \).

**Proof.** From (4) and (0.20) it follows that
\[
0 = \lim_{n \to \infty} q_n \Lambda f(t) = \lim_{n \to \infty} \int_{e_t} q_n(t - u)e_t(u)f(u) \, du;
\]
since \( |q_n(\cdot)| \leq 1 \) we may apply the Lebesgue Dominated Convergence Theorem:
\[
(5) \quad 0 = \int_{e_t} \lim_{n \to \infty} \left[ \exp \frac{-1}{nt - u} \right] e_t(u)f(u) \, du = \int_{e_t} e_t(u)f(u) \, du.
\]
From (5) and (0.3)-(0.4) we see that
\[
0 = \int_a^t f \quad for \ 0 < t < b, \quad and \quad 0 = -\int_a^t f \quad for \ a < t < 0,
\]
which implies our conclusion: \( f(\cdot) \) vanishes almost-everywhere on the interval \( (a, b) \).
1. The space \( \mathcal{D}_0 \) of generalized functions. As before, \( \omega \) is an arbitrary sub-interval of \( \mathbb{R} = (-\infty, \infty) \) such that \( \omega \ni 0 \). If \( f(\cdot) \) and \( g(\cdot) \) are functions, the equation \( f(\cdot) = g(\cdot) \) will mean that the functions are equal almost-everywhere on the interval \( \omega \).

**Notation 1.0.** Let \( \mathcal{C}_0(\omega) \) be the space of all the functions which are continuous on \( \omega \) and which vanish at the origin.

**Notation 1.1.** We denote by \( 1(\cdot) \) the constant function defined by \( 1(t) = 1 \) for all \( t \) in \( \mathbb{R} \).

**Lemma 1.2.** If \( g(\cdot) \in L^{1c}(\omega) \) then \( 1 \mathcal{L} g(\cdot) \in \mathcal{C}_0(\omega) \).

**Proof.** From (0.5) we see that
\[
1 \mathcal{L} g(t) = \int_0^t 1(t-u)g(u)du = \int_0^t g(u)du.
\]
On the other hand, \( g(\cdot) \in L^1(a, b) \) whenever \( (a, b) \) is a compact sub-interval of the open set \( \omega \): the conclusion is now at hand.

**Lemma 1.4.** If \( \Psi(\cdot) \) is continuous on \( \omega \), then \( (1 \mathcal{L} \Psi)' = \Psi(\cdot) \).

**Proof.** The equations
\[
(1 \mathcal{L} \Psi)'(t) = \frac{d}{dt} (1 \mathcal{L} \Psi)(t) = \Psi(t)
\]
are immediate from (1.3) and the Fundamental Theorem of Calculus.

**Lemma 1.5.** Suppose that \( v(\cdot) \in \mathcal{C}_0(\omega) \). If \( v'(\cdot) \) has only countably many discontinuities and is integrable in each compact sub-interval of the open interval \( \omega \), then \( v(\cdot) = 1 \mathcal{L} v'(\cdot) \).

**Proof.** Take \( t \) in \( \omega \). If \( t > 0 \) the equations
\[
v(t) = v(t) - v(0) = \int_0^t v'(u)du = 1 \mathcal{L} v(t)
\]
are from \( v(0) = 0 \), [4, p. 143], and (1.3). If \( t < 0 \), the same reasoning yields
\[
v(t) = -(v(0) - v(t)) = -\int_t^0 v'(u)du = 1 \mathcal{L} v(t)
\]

**Theorem 1.6.** Let \( G(\cdot) \) be a function whose derivative is continuous on the interval \( \omega \). If \( f(\cdot) \in L^{1c}(\omega) \), then \( G \mathcal{L} f(\cdot) \in \mathcal{C}_0(\omega) \) and
\((1.7)\) \[ G \wedge f(\cdot) = G(0)(1 \wedge f)(\cdot) + 1 \wedge (G' \wedge f)(\cdot). \]

\textit{Proof.} Clearly, the function \(v(\cdot) = G(\cdot) - G(0)1(\cdot)\) belongs to \(C_0(\omega)\); consequently, \(1.5\) gives
\[ G(\cdot) - G(0)1(\cdot) = 1 \wedge G'(\cdot), \]
so that \(0.12\) implies
\[ (1) \quad G \wedge f(\cdot) - G(0)(1 \wedge f)(\cdot) = (1 \wedge G')(\wedge f)(\cdot). \]

From \(0.19\) it follows that \(\langle |G'| \wedge |f|\rangle(\cdot) \in L^{10\circ}(\omega)\); we can therefore conclude from \(1.2\) that the function \(1 \wedge (|G'| \wedge |f|)(\cdot)\) is continuous on \(\omega\), whence the equation
\[ (2) \quad (1 \wedge G')(\wedge f(\cdot) = 1 \wedge (G' \wedge f)(\cdot) \]
now remains from \(0.21\). Conclusion \((1.7)\) is immediate from \((1)-(2)\). It still remains to prove that \(G \wedge f(\cdot) \in C_0(\omega)\).

Set \(g(\cdot) = G' \wedge f(\cdot)\); Equation \((1.7)\) becomes
\[ (3) \quad G \wedge f(\cdot) = G(0)(1 \wedge f)(\cdot) + 1 \wedge g(\cdot). \]

From \(0.19\) we see that \(g(\cdot) \in L^{10\circ}(\omega)\); the conclusion \(G \wedge f(\cdot) \in C_0(\omega)\) is obtained from \((3)\) by setting \(g = f\) and then \(g = g,\) in \(1.2\).

1.8. The space of test-functions. Let \(W_\omega\) be the linear space of all the complex-valued functions which are infinitely differentiable on \(\omega\) and whose every derivative vanishes at the origin. Thus, \(w(\cdot) \in W_\omega\) if \(w(\cdot) \in C_0(\omega)\) and \(w^{(k)} \in C_0(\omega)\) for every integer \(k \geq 1\).

\textbf{Example 1.9.} Let \(q_n(\cdot)\) be the function defined in \(0.23\); it is easily verified that \(q_n^{(k)}(0) = 0\) for every integer \(k \geq 1\); therefore, \(q_n(\cdot) \in W_\omega\).

\textbf{Lemma 1.10.} \textit{If} \(f(\cdot) \in L^{10\circ}(\omega)\) \textit{and} \(q(\cdot) \in W_\omega\) \textit{then}
\[ (1.11) \quad q \wedge f(\cdot) \in C_0(\omega) \]
and
\[ (1.12) \quad (q \wedge f)'(\cdot) = q' \wedge f(\cdot). \]

\textit{Proof.} Since \(q'(\cdot) \in C_0(\omega)\), we can set \(G = q\) in \(1.6\) to obtain \((1.11)\) and the equations
\[ (4) \quad q \wedge f(\cdot) = q(0)(1 \wedge f)(\cdot) + 1 \wedge (q' \wedge f)(\cdot) = 1 \wedge (q' \wedge f)(\cdot) \]
now come from \((1.7)\) and \(q(0) = 0\) \(\text{since} q(\cdot) \in C_0(\omega)\)). Next, set
Equation (4) becomes

\[ (5) \quad \Psi(\omega) = q' \wedge f(\omega). \]

Equation (4) becomes

\[ (6) \quad q \wedge f(\omega) = 1 \wedge \Psi(\omega). \]

Setting \( G = q' \) in 1.6, we see from (5) that \( \Psi(\omega) \in \mathbb{C}_0(\omega) \); the equations

\[ (7) \quad (1 \wedge \Psi)'(\omega) = \Psi'(\omega) = q' \wedge f(\omega) \]

therefore follow from 1.4 and (5). Conclusion (1.12) is immediate from (6)-(7).

**Lemma 1.13.** If \( f(\omega) \in L^{\text{loc}}(\omega) \) and \( w(\omega) \in W_\omega \), then \( w \wedge f(\omega) \in W_\omega \), and

\[ (1.14) \quad (f \wedge w)'(\omega) = w' \wedge f(\omega) = f \wedge w'(\omega). \]

**Proof.** If the equation

\[ (8) \quad (w \wedge f)^{(k)}(\omega) = w^{(k)} \wedge f(\omega) \]

holds for \( k = n \), then it holds for \( k = n + 1 \); this is easily seen by observing that the equations

\[ [(w \wedge f)^{(n)}]'(\omega) = (w^{(n)} \wedge f)'(\omega) = w^{(n+1)} \wedge f(\omega) \]

come from (8) and (1.12). Since (8) holds for \( k = 0 \), it holds for any integer \( k \geq 0 \). From (8) and (1.11) (with \( q = w^{(k)} \)) it follows that

\[ (w \wedge f)^{(k)}(\omega) \in \mathbb{C}_0(\omega) \]

for any integer \( k \geq 0 \);

therefore, \( w \wedge f(\omega) \in W_\omega \). Conclusion (1.14) comes from (1.12) and (0.20).

**Definitions 1.15.** An operator is a linear mapping of \( W_\omega \) into \( W_\omega \). If \( A \) is an operator and \( w(\omega) \in W_\omega \), we denote by \( .A w(\omega) \) the function that the operator \( A \) assigns to \( w(\omega) \).

As usual, the product \( A_1 A_2 \) of two operators is defined by

\[ (1.16) \quad .A_1 A_2 w(\omega) = .A_1 (.A_2 w)(\omega) \quad \text{(every } w(\omega) \text{ in } W_\omega). \]

1.17. The space of generalized functions. Let \( \mathcal{N}_\omega \) be the set of all the operators \( A \) such that the equation

\[ (1.18) \quad .A (w_1 \wedge w_2)(\omega) = (.A w_1) \wedge w_2(\omega) \]

holds whenever \( w_1(\omega) \) and \( w_2(\omega) \) belong to \( W_\omega \).
DEFINITION 1.19. If \( f(\cdot) \in L^{10c}(\omega) \) we denote by \( f^* \) the operator which assigns to each \( w(\cdot) \) in \( W_\omega \) the function \( f \wedge w(\cdot) \):

\[
(1.20) \quad f^*w(\cdot) = f \wedge w(\cdot) \quad \text{(for each } w(\cdot) \text{ in } W_\omega).
\]

THEOREM 1.21. If \( f_1(\cdot) \) and \( f_2(\cdot) \) belong to \( L^{10c}(\omega) \), then

\[
(1.22) \quad f_1^*f_2^* = (f_1 \wedge f_2)^*.
\]

Proof. Take any \( w_2(\cdot) \) in \( W_\omega \). From 1.13 and (0.20) we see that \( |f_2| \wedge \omega w_2(\cdot) \in W_\omega \); consequently, we can set \( w = |f_2| \omega w_2 \) and \( f' = |f_1| \) in 1.13 to obtain

\[
|f_1| \wedge (|f_2| \wedge w_2)(\cdot) \in W_\omega:
\]

from 0.21 it therefore follows that

\[
(1.23) \quad f_1 \wedge (f_2 \wedge w_2)(\cdot) = (f_1 \wedge f_2) \wedge w_2(\cdot),
\]

which, in view of 1.19, means that

\[
.f^*(f_2^*w_2)(\cdot) = (f_1 \wedge f_2)^*w_2(\cdot).
\]

Since \( w_2(\cdot) \) is an arbitrary element of \( W_\omega \), Conclusion (1.22) is immediate from (1.16).

REMARK 1.24. If \( f(\cdot) \in L^{10c}(\omega) \) then \( f^* \in \mathcal{A}_\omega \). Indeed, \( f^* \) is an operator (by (1.20), (0.20), and 1.13): it only remains to prove that the equation (1.18) holds for \( A = f^* \). Setting \( f_1 = f \) and \( f_2 = w_1 \) in (1.23), we obtain

\[
.f \wedge (w_1 \wedge w_2)(\cdot) = (f \wedge w_1) \wedge w_2(\cdot);
\]

in view of (1.20), this becomes

\[
.f^*(w_1 \wedge w_2)(\cdot) = (f^*w_1) \wedge w_2(\cdot);
\]

therefore, (1.18) holds when \( A = f^* \).

DEFINITIONS 1.25. We denote by \( D \) the differentiation operator:

\[
(1.26) \quad Dw(\cdot) = w'(\cdot) \quad \text{(all } w(\cdot) \text{ in } W_\omega).
\]

Let \( I \) be the identity-operator:

\[
(1.27) \quad Iw(\cdot) = w(\cdot) \quad \text{(all } w(\cdot) \text{ in } W_\omega).
\]

If \( f(\cdot) \in L^{10c}(\omega) \), we denote by \( \{f(t)\} \) the operator defined by

\[
(1.28) \quad \{f(t)\}w(\cdot) = f \wedge w'(\cdot) \quad \text{(all } w(\cdot) \text{ in } W_\omega);
\]
the operator \{f(t)\} will be called the operator of the function \(f(\ )\).

**Remark 1.29.** \(\{1(t)\} = I\). Indeed, the equations

\[ \{1(t)\}w(\ ) = 1 \wedge w'(\ ) = w(\ ) \]

are from (1.28) and 1.5.

**Remark 1.30.** \(D \in \mathcal{A}_\omega\). Indeed, \(D\) is clearly an operator, and the equations

\[ .D(w_1 \wedge w_2)(\ ) = (w_1 \wedge w_2)'(\ ) = w_1' \wedge w_2(\ ) = (.Dw_1) \wedge w_2(\ ) \]

are from (1.26), (1.14), and (1.26).

**Definition 1.31.** Let \((a, b)\) be a sub-interval of \(\omega\) such that \(a \leq 0 \leq b\); if \(A \in \mathcal{A}_\omega\) and \(B \in \mathcal{A}_\omega\), we say that \(A\) agrees with \(B\) on \((a, b)\) if

\[ .Aw(t) = .Bw(t) \text{ for } a < t < b \text{ and for every } w(\ ) \text{ in } W_\omega. \]

**Theorem 1.32.** Suppose that \(f_k(\ ) \in L^{1_{\text{loc}}}(\omega)\) for \(k = 1, 2\). If \(\{f_1(t)\}\) agrees with \(\{f_2(t)\}\) on \((a, b)\), then \(f_1(\ ) = f_2(\ )\) almost-everywhere on the interval \((a, b)\). Conversely, if the functions are equal almost-everywhere on \((a, b)\), then their operators agree on \((a, b)\).

**Proof.** Set \(h(\ ) = f_1(\ ) - f_2(\ )\). By hypothesis, the relation

\[ \{h(t)\}w(t) = 0 \quad (\text{for } a < t < b) \]

holds for every \(w(\ )\) in \(W_\omega\); it will suffice to show that \(h(\ ) = 0\) almost-everywhere on \((a, b)\). Take any integer \(n \geq 1\), and let \(q_n(\ )\) be the function that was defined in 0.23; since \(q_n(\ ) \in W_\omega\) (see 1.9), it follows from 1.13 (with \(f = 1\)) that \(q_n \wedge 1(\ ) \in W_\omega\); in view of (0.20) we may therefore set \(w(\ ) = 1 \wedge q_n(\ )\) in (1) to obtain

\[ \{h(t)\}(1 \wedge q_n)(t) = 0 \quad (\text{for } a < t < b). \]

The equations

\[ \{h(t)\}(1 \wedge q_n)(\ ) = h \wedge (1 \wedge q_n)'(\ ) = h \wedge q_n(\ ) \]

are from (1.28) and 1.4. Combining (2) and (3), we see that \(h \wedge q_n(t) = 0\) for \(a < t < b\) and for every integer \(n \geq 1\); the conclusion \(h(\ ) = 0\) (almost-everywhere on \((a, b)\)) now comes from 0.24.

Conversely, suppose that \(f_1(\ ) = f_2(\ )\) almost-everywhere; this means that \(h(\ ) = 0\) almost-everywhere on \((a, b)\); we may therefore apply 0.13 to conclude that
consequently, (1.28) gives \( \{h(t)\}w(t) = 0 \), so that

\[ \{f_1(t)\}w(t) = \{f_2(t)\}w(t) \quad \text{for } a < t < b \text{ and } w(\ ) \in W_\omega; \]

this proves that \( \{f_1(t)\} \) agrees with \( \{f_2(t)\} \) on \( (a, b) \).

**Corollary 1.33.** Suppose that \( f_1(\ ) \) and \( f_2(\) ) belong to \( L^{1\infty}(\omega) \): \n
\[ f_1(\ ) = f_2(\ ) \text{ if (and only if) } \{f_1(t)\} = \{f_2(t)\}. \]

**Proof.** Set \( a = \omega_- \) and \( b = \omega_+ \) in 1.32: by definition, two operators are equal if they agree on \( (a, b) \); moreover, we agree that the equation \( f_1(\ ) = f_2(\ ) \) means that these functions are equal almost-everywhere on \( (a, b) \). The conclusion is now immediate from 1.32.

**Theorem 1.34.** The mapping \( f(\ ) \mapsto \{f(t)\} \) is an injective linear transformation of \( L^{1\infty}(\omega) \) into \( \mathcal{W}_\omega \) such that

\[ \{f(t)\} = f^*D. \]  

**Proof.** The equation (1.35) is immediate from (1.28), (1.16), and (1.26). On the other hand, it is easily verified that \( \mathcal{W}_\omega \) is an algebra (if \( A_k \in \mathcal{W}_\omega \) for \( k = 1, 2 \), then \( A_1A_2 \in \mathcal{W}_\omega \) since \( f^* \in \mathcal{W}_\omega \) (by 1.24), and since \( D \in \mathcal{W}_\omega \) (by 1.30), the conclusion \( \{f(t)\} \in \mathcal{W}_\omega \) comes from (1.35). From 1.33 we may now conclude that \( f(\ ) \mapsto \{f(t)\} \) is an injective transformation of \( L^{1\infty}(\omega) \) into \( \mathcal{W}_\omega \): the linearity is clear from (1.28).

**Lemma 1.36.** If \( B \in \mathcal{W}_\omega \) then the equation

\[ B(p_1 \land p_2)(\ ) = p_1 \land (Bp_2)(\ ) \]

holds for every \( p_1(\ ) \) and \( p_2(\ ) \) in \( W_\omega \).

**Proof.** The equations

\[ B(p_1 \land p_2)(\ ) = B(p_2 \land p_1)(\ ) = (Bp_2) \land p_1(\ ) \]

are from (0.20), (0.12), and (1.18); conclusion (1.37) is now immediate from (0.20).

**Theorem 1.38.** \( \mathcal{W}_\omega \) is a commutative algebra.

**Proof.** The multiplication of the algebra \( \mathcal{W}_\omega \) is the usual operator-multiplication (defined in (1.16)); it is easily verified that \( \mathcal{W}_\omega \) is
an algebra. Take $A_1$ and $A_2$ in $\mathcal{A}_\omega$; to prove the commutativity, it will suffice to demonstrate that $A_1A_2 = A_2A_1 = 0$. Let $q_1(\ )$ and $q_2(\ )$ be any two elements of $W_\omega$; we begin by observing that

(1) \[ A_1A_2(q_1 \land q_2')(\ ) = A_1[(A_2q_1) \land q_2](\ ) = (A_2q_1) \land (A_1q_2')(\ ); \]
these equations are from (1.16), (1.18), and (1.37) (with $p_1 = A_2q_1'$ and $p_2 = q_2'$). On the other hand, the equations

(2) \[ A_2A_1(q_1 \land q_2'(\ ) = A_2(q_1 \land A_1q_2'(\ )) = (A_2q_1) \land (A_1q_2')(\ ) \]
are from (1.16), (1.37), and (1.18). We now subtract (2) from (1) to obtain

(3) \[ A(q_1 \land q_2)(\ ) = 0, \text{ where } A = A_1A_2 - A_2A_1. \]

From (3) and (1.18) it results that

\[ 0 = (Aq_1) \land q_2'(\ ) = \{Aq_1(t)\}q_2(\ ) \quad (\text{all } q_2(\ ) \text{ in } W_\omega); \]
the last equation is from (1.28). Consequently, $0 = \{Aq_1(t)\}$; we may now infer from 1.33 that $0 = Aq_1(\ )$ for each $q_1(\ )$ in $W_\omega$: the desired conclusion $A = 0$ is at hand.

**Theorem 1.39.** If $A \in \mathcal{A}_\omega$ and $w(\ ) \in W_\omega$, then $\{Aw(t)\} = A\{w(t)\}$.

**Proof.** Let $w_2(\ )$ be an arbitrary element of $W_\omega$; the equations

(4) \[ \{Aw(t)\}w_2(\ ) = (Aw) \land w_2'(\ ) = A(w \land w_2')(\ ) \]
are from (1.28) and (1.18). On the other hand, the equations

(5) \[ A\{w(t)\}w_2(\ ) = A(\{w(t)\}w_2)(\ ) = A(w \land w_2)(\ ) \]
come from (1.16) and (1.28). Comparing (4) and (5):

(6) \[ \{Aw(t)\}w_2(\ ) = (A\{w(t)\})w_2(\ ). \]
Since (6) holds for every $w_2(\ )$ in $W_\omega$, the proof is complete.

**Theorem 1.40.** If $f_1(\ )$ and $f_2(\ )$ both belong to $L^{10c}(\omega)$, then

(7) \[ D[f_1 \land f_2(t)] = \{f_1(t)\}\{f_2(t)\}. \]

**Proof.** The equations

(8) \[ D[f_1 \land f_2(t)] = D(f_1 \land f_2)^*D = Df_1^*f_2^*D = (f_1^*D)(f_2^*D) \]
are obtained by using (1.35) (with $f = f_1 \land f_2$), by using (1.22), and by utilizing the commutativity and the associativity of the multiplication in $\mathcal{A}_\omega$. Conclusion (7) comes directly from (8) and two more
2. Two-sided operational calculus. If $c$ is a scalar (that is, a complex number), the equation \( \{c1(t)\} = cI \) comes from 1.29 and the linearity of the transformation \( f(\cdot) \mapsto \{f(\cdot)\} \); consequently, \( cI \in \mathcal{S}_\omega \) (recall that \( I \) is the identity: (1.27)). Since the correspondence \( c \mapsto cI \) is an algebraic isomorphism of the field of scalars into the algebra \( \mathcal{S}_\omega \), there is no reason to distinguish between the scalar \( c \) and the operator \( cI \):

\begin{equation}
(2.0) \quad c = cI = \{c1(t)\} \quad \text{for any scalar } c. \nonumber
\end{equation}

Since \( ct^1(t) = ct^1 \) for all \( t \) in \( \mathbb{R} \), it is natural to write \( \{ct^1\} \) instead of \( \{ct^1(t)\} \); in particular,

\begin{equation}
(2.1) \quad c = cI = \{c\} \quad \text{and } 1 = I = \{1\}. \nonumber
\end{equation}

Substituting \( f_1 = 1 \) into 1.40:

\begin{equation}
(2.2) \quad D\{1 \Lambda f_2(t)\} = \{f_2(t)\}. \nonumber
\end{equation}

We can also combine the linearity property with (2.1) to obtain

\begin{equation}
(2.3) \quad \{c_1f_1(t) + c_2f_2(t) + c_3\} = c_1\{f_1(t)\} + c_2\{f_2(t)\} + c_3; \nonumber
\end{equation}

of course, we suppose throughout that \( c_k \) \((k = 1, 2, 3)\) are scalars, and \( f_k(\cdot) \quad (k = 1, 2) \) belong to \( L^{\text{loc}}(\omega) \).

**Theorem 2.4.** Suppose that \( f(\cdot) \) is a function which is continuous on the interval \( \omega \). If \( f'(\cdot) \) has at most countably-many discontinuities and is integrable in each compact sub-interval of \( \omega \), then

\begin{equation}
(2.5) \quad \{f'(t)\} = D\{f(t)\} - f(0)D. \nonumber
\end{equation}

**Proof.** If \( v(\cdot) = f(\cdot) - f(0)1 \), then \( v'(\cdot) = f'(\cdot) \) and we may apply 1.5:

\begin{equation}
(1) \quad f(\cdot) - f(0)1 = v(\cdot) = 1 \Lambda f'(\cdot). \nonumber
\end{equation}

From (1) and (2.3) it follows that

\begin{equation}
(2) \quad \{f(t)\} - f(0) = \{1 \Lambda f'(t)\}. \nonumber
\end{equation}

Multiplying by \( D \) both sides of (2), we obtain

\[ D\{f(t)\} - f(0)D = D\{1 \Lambda f'(t)\} = \{f'(t)\}; \]

the last equation is from (2.2).

2.6. Invertibility. As usual, an operator \( A \) is called invertible
if $A \in \mathcal{A}_\omega$ and there exists an operator $X$ in $\mathcal{A}_\omega$ such that $AX = 1$. Suppose that $A$ is an invertible operator; since $\mathcal{A}_\omega$ is a commutative algebra, it is easily verified that there exists exactly one operator $A^{-1}$ such that $A^{-1} \in \mathcal{A}_\omega$ and $AA^{-1} = 1$. Setting $f(t) = t$ in 2.4, we obtain

\[(2.7) \quad \{1\} = D\{t\} ;\]

consequently, $D$ is an invertible operator, and $D^{-1} = \{t\}$.

**Theorem 2.8.** Suppose that $Y \in \mathcal{A}_\omega$ and $V \in \mathcal{A}_\omega$. If the equation $VY = R$ holds for some invertible $R$ in $\mathcal{A}_\omega$, then $V$ is invertible, and $Y = R/V$, where $R/V$ denotes $RV^{-1}$.

**Proof.** Easy; see 1.76 in [5].

**Remarks 2.9.** From (2.5) we see that

\[(2.10) \quad D\{\sin t\} = \{\cos t\},\]

whence $D^2\{\sin t\} = D\{\cos t\} = -\{\sin t\} + D$ (this last equation also comes from (2.5)); we may therefore use 2.8 to obtain

\[(2.11) \quad \{\sin t\} = \frac{D}{D^2 + 1}.\]

The equation

\[(2.12) \quad D^{-k} = \left\{\frac{t^k}{k!}\right\} \quad \text{(for any integer } k \geq 0)\]

is an easy consequence of (2.7) and (2.5).

2.13. Notation. We shall often write $f$ instead of $\{f(t)\}$. Consequently, (2.3) can be re-written in the form

\[(2.14) \quad \{c_1f_1(t) + c_2f_2(t) + c_3\} = c_1f_1 + c_2f_2 + c_3,\]

and 1.33 becomes

\[(2.15) \quad f_1 = f_2 \text{ if (and only if) } f_1(\ ) = f_2(\ ).\]

Combining 1.40 with (0.5):

\[(2.16) \quad f_1 \wedge f_2 = f_1D^{-1}f_2 = \left\{\int_0^t f_1(t - u)f_2(u)du\right\}.\]

Also, note that (2.2) gives

\[(2.17) \quad f_2 = D(1 \wedge f_2).\]
that is,
\begin{equation}
D^{-1}f_z = 1 \Lambda f_z;
\end{equation}
combining with (1.3):
\begin{equation}
\left\{ \int_0^t f_z \right\} = D^{-1}f_z.
\end{equation}

Finally, note that Theorem 1.39 becomes
\begin{equation}
Aw = Aw \quad \text{(for } A \in \mathcal{A}_\omega \text{ and } w( ) \in W_\omega).\nonumber
\end{equation}

**APPLICATION 2.21.** Given a function \( f( ) \) in \( L^{10c}(-\alpha, \alpha) \), let us solve the differential equation
\begin{equation}
y''(t) + y(t) = f(t) \quad (-\alpha < t < \alpha);
\end{equation}
for example, we could have \( f(t) = \sec(\pi t/2\alpha) \). To solve (1), set \( \omega = (-\alpha, \alpha) \), \( c_0 = y(0), c_1 = y'(0) \), and inject both sides of (1) into \( \mathcal{A}_\omega \); this gives \( D^2y + y = c_1D + c_0D^2 + f \); solving for \( y \):
\begin{align*}
y &= c_1 \frac{D}{D^2 + 1} + c_0D \frac{D}{D^2 + 1} + \frac{D}{D^2 + 1} D^{-1}f;
\end{align*}
we can now use (2.11), (2.10), and (2.16) to write
\begin{equation}
y = c_1 \sin \omega + c_0 \cos \omega + \left\{ \int_0^t (\sin (t - u))f(u)du \right\}.\nonumber
\end{equation}

3. **Translation properties.** In this section we shall describe some two-sided analogues of the translation properties described in [5].

If \( b \geq 0 \) we define the function \( T_b( ) \) by
\begin{equation}
T_b(t) = \begin{cases} 0 & \text{for } t < b \\ 1 & \text{for } t \geq b. \end{cases}
\end{equation}

If \( a < 0 \) we set
\begin{equation}
T_a(t) = \begin{cases} -1 & \text{for } t < a \\ 0 & \text{for } t \geq a. \end{cases}
\end{equation}

Observe that
\begin{equation}
T_x( ) = 0 \text{ on } (-|x|, |x|) \quad \text{(for any } x \text{ in } \mathbb{R}).
\end{equation}

Until further notice, let \( g( ) \) be a function in \( L^{10c}(\omega) \), and let \( g_x( ) \) be the function defined by
\begin{equation}
g_x(u) = T_x(u)g(u - x) \quad \text{(for } u \in \omega);\nonumber
\end{equation}
note that $g_x(\cdot) \in L^{10c}(\omega)$.

**Lemma 3.4.** If $b \geq 0$ then $1 \Lambda g_b(\cdot) = T_b \Lambda g(\cdot)$.

*Proof.* Observe that $g_b(\cdot) = 0 = T_b(\cdot)$ on the interval $(\omega_-, b)$; from 0.13 it therefore follows that

\[(1) \quad g_b \Lambda 1(t) = 0 = T_b \Lambda g(t) \quad \text{for } t \in (\omega_-, b) .\]

Next, suppose that $t > b$ and $t \in \omega$: the equation

\[1 \Lambda g_b(t) = \int_0^t 1(t - u)T_b(u)g(u - x)du\]

comes from (0.5) and (3.3); in view of (3.0), we see that

\[(2) \quad 1 \Lambda g_b(t) = \int_b^t g(u - x)du = \int_0^{t-b} g(\tau)d\tau = T_b \Lambda g(t) :\]

the second equation is obtained by the change of variable $\tau = u - b$; the last equation comes from (0.15) by setting $f = T_b$ in 0.14. The conclusion is immediate from (1)-(2).

**Theorem 3.5.** If $x \in \mathbb{R}$ then $1 \Lambda g_x(\cdot) = T_x \Lambda g(\cdot)$ and

\[(3.6) \quad g_x = gT_x .\]

*Proof.* In view of 3.4, it only remains to consider the case $x = a < 0$. Observe that $g_a(\cdot) = 0 = T_a(\cdot)$ on the interval $(a, \omega_+)$; from 0.13 it therefore follows that

\[(3) \quad g_a \Lambda 1(t) = 0 = T_a \Lambda g(t) \quad \text{for } t \in (a, \omega_+)) .\]

Next, suppose that $t < a$ and $t \in \omega$: as in the proof of 3.4, we see that

\[(4) \quad 1 \Lambda g_a(t) = -\int_t^a g(u - x)du = -\int_{t-a}^0 g(\tau)d\tau :\]

the second equation is obtained by the change of variable $\tau = u - a$. Note that $T_a(\cdot) = 0$ on the interval $(a, \omega_+)$: we can therefore set $h = T_a$ in 0.14 and use (0.16) to obtain

\[(5) \quad T_a \Lambda g(t) = -\int_{t-a}^0 T_a(t - \tau)g(\tau)d\tau = -\int_{t-a}^0 g(\tau)d\tau .\]

From (4)-(5) it results that $1 \Lambda g_a(t) = T_a \Lambda g(t)$ for $\omega_- < t < a$; the conclusion $1 \Lambda g_x(\cdot) = T_x \Lambda g(\cdot)$ is now immediate from (3). The equations
are from (2.17), from our conclusion (1 ∨ g( ) = T_x \land g), and from (2.17): this proves (3.6).

3.7. Particular cases. In view of (3.3), we can write (3.6) in the form

\begin{equation}
\{T_x(t)g(t - x)\} = T_xg \quad \text{(for } x \in \mathbb{R} \text{ and } g( ) \in L^{1_{loc}}(\omega)).
\end{equation}

This equation is a useful substitute for the Laplace-transform identity

\[ \mathcal{L}[T_x(t)g(t - x)] = e^{-sx}\mathcal{L}[g(t)]. \]

Let \( \mathcal{U}( ) \) be the function \( 1( ) - 1_{+}( ) \); that is,

\begin{equation}
\mathcal{U}( ) = 1( ) - T_0( ).
\end{equation}

From (0.1) and (3.0) it follows that \( g_{+}( ) = T_0( )g( ) \); but (3.8) then gives \( \{g_{+}(t)\} = T_0g \), so that

\begin{equation}
\{g_{+}(t)\} = g - T_0g = \mathcal{U}g \quad \text{(by (0.2) and (3.9))}.
\end{equation}

Setting \( g( ) = T_0( ) \) in (3.8) we see that \( T_0 = \{T_0(t)T_0(t)\} = T_0T_0 \), whence it results that

\begin{equation}
T_0\mathcal{U} = 0, \quad T_0^2 = T_0, \quad \text{and } \mathcal{U}^2 = \mathcal{U}.
\end{equation}

If \( A \in \mathcal{A}_\omega \) we set \( A_+ = T_0A \) and \( A_{\mathcal{U}} = \mathcal{U}A \); clearly, \( A = A_{\mathcal{U}} + A_+ \) and \( A_{\mathcal{U}}A_+ = 0 \). If \( B \in \mathcal{A}_\omega \) then

\begin{equation}
A_{\mathcal{U}}B = A_{\mathcal{U}}B_{\mathcal{U}} = \mathcal{U}(AB)
\end{equation}

and

\begin{equation}
A_+B = AB_+ = A_+B_+ = (AB)_+.
\end{equation}

Let \( (B,\mathcal{A}) \) denote the set \( \{BA: A \in \mathcal{A}\} \); it is easily seen that \( (\mathcal{U},\mathcal{A}) \) and \( (T_0,\mathcal{A}) \) are ideals in the algebra \( \mathcal{A}_\omega \), and \( \mathcal{A}_\omega \) is the direct sum of these ideals:

\begin{equation}
\mathcal{A} = (\mathcal{U},\mathcal{A}) \oplus (T_0,\mathcal{A}).
\end{equation}

Note that \( \text{sgn } t = -\mathcal{U}(t) + T_0(t) \), so that \( \text{sgn } = -\mathcal{U} + T_0 \). It is easily verified that \( \{|t|\} = D^{-1} \text{sgn} \), and

\begin{equation}
\{e^{a|t|}\} = \frac{D^2 + aD \text{sgn}}{D^2 - a^2}.
\end{equation}

If \( \alpha > 0 \) we set

\[ 1^\alpha( ) = -T_{-\alpha}( ) + T_\alpha( ); \]
from (3.8) it follows readily that
\[ 1^\alpha g = \{-1 - 1^\alpha(t)g(t + \alpha) + 1^\alpha(t)g(t - \alpha)\} \, . \]

If \( h(\ ) \) is a periodic function of period \( \alpha \), then
\[ h = \frac{[1 - 1^\alpha(t)h(t)]}{1 - 1^\alpha} \, . \]

Finally, if \( \alpha \geq 0 \) and \( \beta \geq 0 \) then \( 1^\alpha 1^\beta = 1^{\alpha + \beta} \) and
\[ (3.15) \quad 1^\alpha 1^\beta = 1^{\alpha + \beta} \, ; \]
we define \( 1^\alpha \) to be \( 1 \) in case \( \alpha = 0 \).

3.16. Other operational calculi. Mikusiński’s injection (of \( L^{10\alpha}(0, \infty) \) into the Mikusiński field) is an extension of the Laplace transformation; analogously, our injection \( f(\ ) \mapsto \{f(t)\} \) is comparable to the two-sided Laplace transformation. However, if \( \mathcal{L}\{f(t)\} \) denotes the Laplace transform of the function \( f(\ ) \), then
\[ \mathcal{L}\{e^{-t} - e^t\}(s) = \frac{2}{1 - s^2} = \mathcal{L}\{e^{-|t|}\}(s) ; \]
the first equation holds for \( s > 1 \), the second for \( 0 < s < 1 \). This contrasts with
\[ \{e^{-t} - e^t\} = \frac{2D}{1 - D^2} \neq \{e^{-|t|}\} \quad \text{(see (3.14))}. \]

A problem which is not Laplace-transformable is discussed in 6.7.

**Theorem 3.17.** If \( \alpha > 0 \) and \( h(\ ) \in L^{10\alpha}(\omega) \), then the equation
\[ (3.18) \quad \left\{ \sum_{k = -\infty}^{\infty} c_k T_{k\alpha}(t)g(t - k\alpha) \right\} = g\left\{ \sum_{k = -\infty}^{\infty} c_k T_{k\alpha}(t) \right\} \]
holds for any scalar-valued sequence \( c_k \) \((k = 0, \pm 1, \pm 2, \pm 3, \cdots)\).

**Proof.** Set
\[ (1) \quad g(T_\alpha)(\ ) = \sum_{k = -\infty}^{\infty} c_k g_{k\alpha}(\ ) \, . \]

Take any \( t \) in \( \omega \): there exists an integer \( m > 0 \) such that \( |t| < m\alpha \). Clearly,
\[ (2) \quad g(T_\alpha)(t) = \sum_{|k| < m} c_k g_{k\alpha}(t) + \sum_{|i| \geq m} c_i g_{i\alpha}(t) \, . \]
Since \( t \in (-m\alpha, m\alpha) \subset (-|i|\alpha, |i|\alpha) \) and since \( g_{i\alpha}(\ ) = 0 \) on the interval
(−|i|α, |i|α) (by (3.2) and (3.3)), we have \( g_{ic}(t) = 0 \): consequently, the series (1) converges, and (3.3) gives

\[
(3) \quad g(T_\alpha(t)) = \sum_{k=\infty}^{\infty} c_k T_{k\alpha}(t)g(t - k\alpha).
\]

The equations

\[
g(T_\alpha) = D\left\{ 1 \land g(T_\alpha) \right\} = D\left\{ \sum_{k=\infty}^{\infty} c_k (1 \land g_{k\alpha})(t) \right\}
\]

are from (2.17) and (1); from 3.5 it therefore follows that

\[
(4) \quad g(T_\alpha) = D\left\{ \sum_{k=\infty}^{\infty} c_k (T_{k\alpha} \land g)(t) \right\}.
\]

Equation (4) gives

\[
(5) \quad g(T_\alpha) = D\left\{ g \land \sum_{k=\infty}^{\infty} c_k T_{k\alpha}(t) \right\} = g\left\{ \sum_{k=\infty}^{\infty} c_k T_{k\alpha}(t) \right\}.
\]

the second equation is from 1.40. Conclusion (3.18) now comes from (3) and (5).

**Remark 3.19.** If \( c \) is a scalar and if \( \lambda \geq 0 \), the equation

\[
\frac{1^l h}{1 - cI^\alpha} = \left\{ \sum_{k=0}^{\infty} \frac{e^k(h(t + k\alpha + \lambda) + h(t - k\alpha - \lambda))}{1 - cI^\alpha} \right\}
\]

is not hard to verify; it is the two-sided analogue of Theorem 5.29 in [5].

**Theorem 3.20.** If \( x \in \mathbb{R} \) and \( w(\cdot) \in W_\omega \) then

\[
(3.21) \quad T_x w(t) = T_x(t)w(t - x) \quad \text{for } t \in \omega.
\]

**Proof.** The equations

\[
\{T_x(t)w(t - x)\} = T_x w = \cdot T_x w
\]

come from (3.8) and (2.20); Conclusion (3.21) now follows from (2.15).

**Lemma 3.22.** If \( R \in X_\omega \) and \( w(\cdot) \in W_\omega \) then

\[
(3.23) \quad R_{\|} w(\cdot) = [Rw]_{\|}(\cdot).
\]

**Proof.** Setting \( g = \cdot Rw \) in (3.9.1), we obtain

\[
(1) \quad \{[Rw]_{\|}(t)\} = \|\{Rw(t)\} = \|R\{w(t)\}:
\]

the last equation is from 1.39. Since \( B_{\|} = \|B \) (by definition), Equa-
tion (1) becomes
\[ ([.Rw]_{U}(t)) = R_{U}[w(t)] = [.R_{U}w(t)] : \]
the second equation is from 1.39. Conclusion (3.23) is immediate from (2) and 1.33.

**Theorem 3.24.** If \( A \in \mathcal{X}_\omega \) and \( B \in \mathcal{X}_\omega \), then
\[ A_{U} = B_{U} \text{ if (and only if) } A \text{ agrees with } B \text{ on } (\omega_-, 0). \]

*Proof.* Recall that \( (\omega_-, 0) = \omega \cap (-\infty, 0) \). Let \( w( ) \) be any element of \( W_\omega \); the equations
\[ (3) \quad [.A_\omega w(U( ) = .A_{U}w( ) = .B_{U}w( ) = [.B_\omega w(U( ) \]
are from (3.23), our hypothesis \( A_{U} = B_{U} \), and (3.23). Since \( h_{U}(t) = h(t) \) for \( t < 0 \) (see (0.1)-(0.2)), Equation (3) implies
\[ (4) \quad .A_\omega w(t) = .B_\omega w(t) \quad \text{ (for } \omega_- < t < 0). \]

From (4) and 1.31 we see that \( A \) agrees with \( B \) on \( (\omega_-, 0) \). Conversely, if \( A \) agrees with \( B \) on \( (\omega_-, 0) \), then (4) holds, whence the equation \( [.A_\omega w(U( ) = [.B_\omega w(U( ) \): combining this with (3.23), we obtain
\[ .A_{U}w( ) = .B_{U}w( ) \quad \text{ (for every } w( ) \text{ in } W_\omega), \]
which gives \( A_{U} = B_{U} \).

**Theorem 3.25.** The space \((T_0, \mathcal{X})\) consists of all the elements of \( \mathcal{X}_\omega \) which agree with \( 0 \) on \( (\omega_-, 0) \). Moreover,
\[ (3.26) \quad B \in (T_0, \mathcal{X}) \iff B_{U} = 0 \iff B = B_+ . \]

*Proof.* We begin with (3.26). If \( B \in (T_0, \mathcal{X}) \) then \( B = \tau_0A \) for some \( A \) in \( \mathcal{X}_\omega \); therefore, \( UB = 0 \) (by (3.10)); this gives \( B_{U} = 0 \); since \( B = B_{U} + B_+ \), the equation \( B_{U} = 0 \) implies \( B = B_+ \); if \( B = B_+ \), then \( B = \tau_0B \), whence \( B \in (T_0, \mathcal{X}) \). This proves (3.26).

If \( B \in (T_0, \mathcal{X}) \) then \( B_{U} = 0 \) (by (3.26)), which implies that \( B \) agrees with \( 0 \) on the interval \( (\omega_-, 0) \) (by 3.24). Conversely, if \( B \) agrees with \( 0 \) on the interval \( (\omega_-, 0) \), then \( B_{U} = 0 \) (by (3.24)): the conclusion \( B \in (T_0, \mathcal{X}) \) now comes from (3.26).

**Theorem 3.27.** If \( B \in \mathcal{X}_\omega \) is such that the equation \( f = B_{U} \) holds for some \( f( ) \) in \( L^{10\omega}(\omega) \), then \( f \) agrees with \( B \) on the interval \( (\omega_-, 0) \).

*Proof.* The equations
are from the definition \((f_{11} = \bigcup f)\), from our hypothesis, from the definition \((B_{11} = \bigcup B)\), from (3.10), and again from the definition \((B_{11} = \bigcup B)\). From (3.28) and 3.24 we see that \(f\) agrees with \(B\) on the interval \((\omega_-, 0)\).

4. The topological space \(\mathscr{X}_\omega\). Let the function space \(W_\omega\) be endowed with the topology of pointwise convergence on the interval \(\omega\): this enables us to topologize \(\mathscr{X}_\omega\) by endowing it with the product topology (recall that \(\mathscr{X}_\omega\) consists of mappings of \(W_\omega\) into the topological space \(W_\omega\)). Consequently, the equation

\[
B = \lim_{\lambda \to \gamma} A_\lambda \quad \text{(for } B \text{ and } A_\lambda \text{ in } \mathscr{X}_\omega) \]

means that

\[
(1) \quad .Bw(t) = \lim_{\lambda \to \gamma} .A_\lambda w(t) \quad \text{(for } t \in \omega \text{ and } w( ) \in \omega_\omega). \]

It is immediately clear that \(\mathscr{X}_\omega\) is a locally convex Hausdorff vector space: in fact, H. Shultz has proved that it is sequentially complete and that the multiplication of the algebra \(\mathscr{X}_\omega\) is sequentially continuous.

We denote by \(\lim A_\lambda \) the mapping that assigns to each \(w( )\) in \(W_\omega\) the function \(.Bw( )\) defined by (1):

\[
(4.1) \quad \left(\lim_{\lambda \to \gamma} A_\lambda\right)w( ) = \lim_{x \to \gamma} .A_\lambda w( ) \quad \text{(every } w( ) \text{ in } W_\omega). \]

If \(x \mapsto F(x)\) is a mapping into \(\mathscr{X}_\omega\), we set

\[
(4.2) \quad \frac{d}{dx} F(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [F(x + \varepsilon) - F(x)]; \]

in view of (4.1), this means that \(dF(x)/dx\) is the operator defined for any \(w( )\) in \(W_\omega\) by

\[
(4.3) \quad .\left(\frac{d}{dx} F(x)\right)w( ) = \frac{\partial}{\partial x} (\cdot F(x)w( )). \]

Theorem 4.4. If \(x \in \mathbf{R}\), then \(\left(\frac{d}{dx}\right)_{\tau_x} = -\tau_x D\).

Proof. Take any \(w( )\) in \(W_\omega\), take any \(t \neq x\) in \(\omega\); from (4.3) we see that

\[
(2) \quad .\left(\frac{d}{dx} \tau_x\right)w(t) = \frac{\partial}{\partial x} (\cdot \tau_x w(t)) = \frac{\partial}{\partial x} \tau_x(t)w(t - x): \]
the second equation is from (3.21). Set $E_1 = \{x: x > t\}$ and $E_2 = \{x: x < t\}$; note that the function $x \mapsto T_x(t)$ is constant on $E_k$ when $k = 1, 2$; consequently, since $x \neq t$ then $x \in E_k$ for some $k$, whence $\partial T_x(t)/\partial x = 0$; we can use this to infer from (2) that

$$\left(\frac{d}{dx} T_x\right) w(t) = T_x(t) \frac{\partial}{\partial x} w(t - x) = -T_x(t)w'(t - x) \quad (\text{all } t \neq x).$$

Consequently, we may use (3.21) to write

$$\left(\frac{d}{dx} T_x\right) w(\cdot) = -T_x w'(\cdot) \quad (\text{all } w(\cdot) \text{ in } W_\omega).$$

Calling $B = dT_x/dx$, this gives $Bw(\cdot) = -T_x Dw(\cdot)$, whence the conclusion $B = -T_x D$.

**Corollary 4.5.** if $x \in \mathbb{R}$ then $D T_x = \lim_{\varepsilon \to 0^+} (1/\varepsilon)(T_{x+\varepsilon} - T_x)$.

**Proof.** From 4.4 and (4.2) it follows that

$$-T_x D = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (T_{x+\varepsilon} - T_x),$$

which implies directly our conclusion.

**Remark 4.6.** Corollary 4.5 indicates that $D T_x$ corresponds to the Dirac delta distribution $\delta_x$ concentrated at the point $x$.

**Theorem 4.7.** If $F_k(\cdot)$ ($k = 0, \pm 1, \pm 2, \pm 3, \cdots$) is a sequence in $L^1_{\text{loc}}(\omega)$, then

$$\sum_{k=-\infty}^{\infty} T_{ka} F_k = \left\{ \sum_{k=-\infty}^{\infty} T_{ka}(t) F_k(t - ka) \right\}.$$  \hspace{1cm} (4.8)

**Proof.** Let $T_{ka} F_k(\cdot)$ be the function defined by

$$T_{ka} F_k(t) = T_{ka}(t) F_k(t - ka). \hspace{1cm} (1)$$

Set

$$f_\alpha(\cdot) = \sum_{k=-\infty}^{\infty} T_{ka} F_k(\cdot). \hspace{1cm} (2)$$

For any integer $n \geq 1$, observe that

$$f_\omega(\cdot) = f_n(\cdot) + \sum_{|i| > n} T_{ia} F_i(\cdot); \hspace{1cm} (3)$$

since $(-n\alpha, n\alpha) \subseteq (-|i|\alpha, |i|\alpha)$ and since $T_{ia} F_i(\cdot) = 0$ on the interval $(-|i|\alpha, |i|\alpha)$ (because of (3.2) and (1)), we may conclude that $T_{ia} F_i(\cdot) = 0$ on ($-$
0 on the interval \((-na, na)\): consequently, (3) becomes

\[(4) \quad f_\omega(t) = f_n(t) \quad \text{on} \quad (-na, na) \quad \text{for any integer} \quad n \geq 1.\]

If \(t \in \omega\) there exists an integer \(m \geq 1\) such that \(t \in (-m\alpha, m\alpha)\): from (4), (2), and (1) we see that

\[(5) \quad \sum_{k=-\infty}^{\infty} T_{k\alpha}(t) F_k(t - k\alpha) = f_\omega(t) = \sum_{k=-m}^{\infty} T_{k\alpha} F_k(t).\]

On the other hand,

\[(6) \quad f_n = \left\{ \sum_{k=-n}^{n} T_{k\alpha} F_k(t) \right\} = \sum_{k=-k}^{n} T_{k\alpha} F_k;\]

the second equation is from (3.8) and (1).

In view of (5)-(6), the proof of (4.8) will be accomplished by showing that

\[(7) \quad \lim_{n \to \infty} f_n = f_\omega.\]

To that effect, take any \(w(\ )\) in \(W_\omega\), and any \(t\) in the interval \(\omega\); we must prove that

\[(8) \quad \lim_{n \to \infty} f_n w(t) = f_\omega w(t).\]

Observe that there exists an integer \(m \geq 1\) such that \(|t| < m\alpha\); suppose that \(n \geq m\); from (4) and 1.32 it follows that the operators \(f_n\) and \(f_\omega\) agree on \((-na, na)\): therefore, 1.31 gives

\[(9) \quad f_n w(t) = f_\omega w(t) \quad \text{for all} \quad n \geq m;\]

this is because \(w(\ ) \in W_\omega\) and \(-m\alpha < t < m\alpha\). Conclusion (8) is immediate from (9).

**Remark 4.9.** Let \(c_k\ (k = 0, \pm 1, \pm 2, \pm 3, \cdots)\) be a scalar-valued sequence. Setting \(F_k(\ ) = c_k\) in (4.8), we obtain

\[(4.10) \quad \sum_{k=-\infty}^{\infty} c_k T_{k\alpha} = \left\{ \sum_{k=-\infty}^{\infty} c_k T_{k\alpha}(t) \right\};\]

combining with (3.18):

\[(4.11) \quad \left\{ \sum_{k=-\infty}^{\infty} c_k T_{k\alpha}(t) g(t - k\alpha) \right\} = \sum_{k=-\infty}^{\infty} c_k T_{k\alpha} g.\]

Obviously, if \(g(\ )\) is a periodic function of period \(\alpha > 0\), then (4.11) becomes

\[(4.12) \quad g \sum_{k=-\infty}^{\infty} c_k T_{k\alpha} = \left\{ g(t) \sum_{k=-\infty}^{\infty} c_k T_{k\alpha}(t) \right\}.\]
5. Derivative of an operator. Given $A \in \mathcal{A}$ and $B \in \mathcal{A}$, let us indicate by $A \subset B$ the existence of a number $a < 0$ such that $A$ agrees with $B$ on the interval $(a, 0)$. The notion of “agreeing with” has been defined in 1.31. Recall that $F = \{F(t)\}$ (see 2.13); as usual, $F(0-)$ denotes the limit of $F(t)$ as $t$ approaches zero through negative values.

**Theorem 5.0.** Suppose that $B \in \mathcal{A}$. There is at most one scalar $c_1$ such that the equation $c_1 = f_i(0-)$ holds for some function $f_i(\cdot)$ in $L^{1\infty}(\omega)$ with $f_i \subset B$.

**Proof.** Suppose that the equation $c_2 = f_i(0-)$ holds for some function $f_2(\cdot)$ in $L^{1\infty}(\omega)$ with $f_2 \subset B$; we must prove that $c_1 = c_2$. By definition, there exists an interval $(a_k, 0)$ such that $f_k$ agrees with $B$ on the interval $(a_k, 0)$ (for $k = 1, 2$); from 1.31 we now see that $f_i$ agrees with $f_2$ on $(a, 0)$, where $a$ is the largest of the two negative numbers $a_1$ and $a_2$; from 1.32 it follows that $f_i(\cdot) = f_2(\cdot)$ on $(a, 0)$, whence $f_i(0-) = f_2(0-)$: this proves that $c_1 = c_2$.

5.1. Derivable operators. An operator $B$ is said to be derivable if $B \in \mathcal{A}$ and if there exists a function $f_i(\cdot)$ in $L^{1\infty}(\omega)$ such that $|f_i(0-)| < \infty$ and $f_i \subset B$.

5.2. Initial value of an operator. If $B$ is derivable, we denote by $\langle B, 0- \rangle$ the unique scalar $c_1$ such that the equation $c_1 = f_i(0-)$ holds for some function $f_i(\cdot)$ in $L^{1\infty}(\omega)$ such that $f \subset B$; we also set

$$\partial_i B = DB - \langle B, 0- \rangle D.$$  

(5.3)

The uniqueness of $c_1$ comes from 5.0, while the existence of $c_1$ can be verified by setting $c_1 = f_i(0-)$ in 5.1.

**Remarks 5.4.** If $f(\cdot)$ is a function in $L^{1\infty}(\omega)$ such that $|f(0-)| < \infty$, then the operator $f$ is derivable, and $\langle f, 0- \rangle = f(0-)$ (this is immediate from 5.1); from (5.3) we see that

$$\partial_i f = Df - f(0-) D.$$  

5.5. Suppose that $f(\cdot)$ is continuous on $\omega$; if $f'(\cdot)$ has at most countably-many discontinuities and is integrable on each compact sub-interval of the open interval $\omega$, then

$$\partial_i f = \{f'(t)\} \quad \text{and} \quad \langle f, 0- \rangle = f(0):$$  

this follows immediately from 2.4, 2.13, and 5.4.
5.6. Suppose that $B \in \mathcal{S}_\omega$. If $f(\cdot) \in L^1_{\text{loc}}(\omega)$ is such that $|f(0^-)| < \infty$ and $f \subseteq B$, then $B$ is derivable and $\langle B, 0^- \rangle = f(0^-)$: this follows directly from 5.0-5.2.

5.7. If $B \in \mathcal{S}_\omega$ is such that the equation $B \mu = f$ holds for some function $f(\cdot) \in L^1_{\text{loc}}(\omega)$ such that $|f(0^-)| < \infty$, then $B$ is derivable and $\langle B, 0^- \rangle = f(0^-)$. This is immediate from 3.27 and 5.6.

**Theorem 5.8.** Suppose that $\alpha > 0$. If $A_k$ $(k = 0, \pm 1, \pm 2, \pm 3, \ldots)$ is a sequence in $\mathcal{S}_\omega$ such that the equation

\[
B = \sum_{k=-\infty}^{\infty} \mathcal{T}_{k\alpha} A_k
\]

defines an element $B$ of $\mathcal{S}_\omega$, then $B$ is derivable, $\langle B, 0^- \rangle = 0$, and $\partial_\nu B = DB$.

**Proof.** Take any $w(\cdot)$ in $W_\omega$. From (1) and (3.21) it follows that

\[
.B w(t) = \mathcal{T}_\alpha(t).A_0 w(t) + \sum_{k \neq 0} \mathcal{T}_{k\alpha}(t).A_k w(t - k\alpha)\quad (\text{for } t \in \omega).
\]

If $k \neq 0$ we see from (3.2) that $\mathcal{T}_{k\alpha}(\cdot) = 0$ on $(-\alpha, \alpha)$: consequently, the equation (2) implies that

\[
.B w(t) = \mathcal{T}_\alpha(t).A_0 w(t)\quad (\text{for } |t| < \alpha).
\]

Since $\mathcal{T}_\alpha(\cdot) = 0$ on $(-\alpha, 0)$, it now follows from (3) that $Bw(t) = 0$ for $-\alpha < t < 0$ and for any $w(\cdot)$ in $W_\omega$: therefore, the operator $0$ agrees with $B$ on $(-\alpha, 0)$, whence $0 \subseteq B$; the conclusion $\langle B, 0^- \rangle = 0$ now follows from 5.6; in view of (5.3), the proof is concluded.

**Theorem 5.9.** Suppose that $x \in \mathbb{R}$. Each element of $(\mathcal{T}_x \mathcal{S})$ is infinitely derivable; in fact,

\[
\langle B, 0^- \rangle = 0 \quad \text{and} \quad \partial^k_x B = D^k B \quad (\text{for each integer } k \geq 1)
\]

whenever $B \in (\mathcal{T}_x \mathcal{S})$.

**Proof.** Note that $(\mathcal{T}_x \mathcal{S})$ is the set $\{\mathcal{T}_x A : A \in \mathcal{S}_\omega\}$. If $B$ is an element of $(\mathcal{T}_x \mathcal{S})$, then $B = \mathcal{T}_x A$ for some $A$ in $\mathcal{S}_\omega$: clearly, $B$ can be written in the form (1) (set $\alpha = |x|$ and $A_k = A$ for $k = \text{sgn} x$ and $A_k = 0$ for other values of $k$): the conclusion $\langle B, 0^- \rangle = 0$ now comes from 5.8. Since $\partial^k_x B = B$ (by definition) for $k = 0$, we proceed by induction on $k \geq 1$. To that effect, we assume that $\partial^k_x B = D^k B$: clearly,
(4) \[ \partial_t^{n+1}B = \partial_t(D^nB) = D^{n+1}B + \langle D^nB, 0-D \rangle. \]

On the other hand, \( D^nB = D^nT_xA = T_xD^nA \); consequently, \( D^nB \) belongs to \( (T_xA) \), whence \( \langle D^nB, 0-D \rangle = 0 \) (by what we established at the beginning of this proof); therefore (4) gives \( \partial_t^{n+1}B = D^{n+1}B \). The induction proof is completed.

**Note 5.11.** Both \( T_x \) and the Dirac delta distribution \( D\delta \) belong to the space \( (T_xA) \). If \( B = B_\pm \) or if \( B_{\perp} = 0 \) then \( B \) belongs to \( (T_xA) \): see 3.25.

**Theorem 5.12.** Set \( a = \omega_- \) and suppose that \( B \in \mathcal{X}_a \). If the equation \( B_{\perp} = f \) holds for some function \( f(\ ) \) in \( L'(a, 0) \), there exists a unique scalar \( c_1 \) such that the equation

(5) \[ c_1 = \int_a^0 f_1(u)du \]

holds for some \( f_1(\ ) \) in \( L'(a, 0) \) with \( f_1 = B_{\perp} \).

**Proof.** Clearly, such a scalar exists. If

(6) \[ c_2 = \int_a^0 f_2(u)du \]

for \( f_2(\ ) \) in \( L'(a, 0) \) and \( f_2 = B_{\perp} \), then both \( f_1 \) and \( f_2 \) agree with \( B \) on \( (a, 0) \) (by 3.27): therefore, \( f_1(\ ) \) equals \( f_2(\ ) \) almost-everywhere on \( (a, 0) \) (by 1.32); the conclusion \( c_1 = c_2 \) now comes from (5)–(6).

**5.13. The anti-derivative.** Let \( B \) be as in 5.12. We set

(7) \[ \int_a^t B = D^{-1}B + c_1. \]

In a subsequent paper we shall prove that

\[ \langle \int_a^t B, 0-D \rangle = c_1 \quad \text{and} \quad \partial_t \int_a^t B = B. \]

In case \( B = f \) with \( f(\ ) \in L'(a, 0) \) and \( f(\ ) \in L^{loc}(\omega) \), it follows immediately from (2.19) and (3) (7) that

\[ \int_a^t f = \left\{ \int_a^t f(u)du \right\}. \]

**6. Four problems.** Recall that \( DT_x \) corresponds to the Dirac delta distribution concentrated at the point \( x \) (see 4.6), it is infinitely derivable (see 5.11). If an operator \( A \) is twice derivable, it follows directly from (5.3) that
\[ \partial_t^2 A = D^2 A - \langle A, 0 \rangle D^2 - \langle \partial_t A, 0 \rangle D . \]

We shall need two more facts. Each operator \( A \) in \( \mathcal{A} \) can be written as a sum

\[ A = A_\| + A_+, \quad \text{where} \quad A_+ = A \tau_0 \quad \text{(see 3.7)} ; \]

moreover, if \( g(\cdot) \in L^{\infty}(\omega) \) then

\[ g\tau_0 = \{\tau_0(t)g(t)\} \quad \text{(see (3.8))} . \]

6.3. First problem. Given two scalars \( m \) and \( a \), to find an operator \( y \) such that

\[ m \partial_t y = D\tau_0 \quad \text{and} \quad \langle y, 0 \rangle = a . \]

Definition (5.3) gives \( mDy - maD = D\tau_0 \), whence \( y(\cdot) = a + m^{-1}\tau_0(\cdot) \). This same problem has been discussed in [5, p. 38].

6.5. Second problem. The equations

\[ i = \partial_t q \quad \text{and} \quad q = CE \]

relate the current \( i \) to the change \( q \) in a simple electric circuit having capacitance \( C \), impressed electromotive force \( E \), no inductance, and no resistance (see 7.19 in [5]). From (1) and (5.3) it follows that

\[ i = CDE - \langle q, 0 \rangle D . \]

Multiplying by \( \tau_0 \) both sides of (2), we can use (6.1) to write

\[ i_+ = CDE_+ - \langle q, 0 \rangle D\tau_0 . \]

If there is a short-circuit at the time \( t = 0 \), then \( E_+ = 0 \), so that (3) gives the answer \( i_+ = -\langle q, 0 \rangle D\tau_0 \); this is an impulse whose magnitude is the negative of the initial charge \( \langle q, 0 \rangle \).

6.6. Third problem. Given a scalar \( c \), to find an operator \( y \) such that

\[ \partial_t^2 y + y = \partial_t(D\tau_0) \quad \text{and} \quad \langle \partial_t y, 0 \rangle = \langle y, 0 \rangle = c . \]

Since \( \partial_t(D\tau_0) = D^2\tau_0 \) (by 5.9), we can use (6.0) to write

\[ (D^2 + 1)y = D^2\tau_0 + \langle y, 0 \rangle D^2 + \langle \partial_t y, 0 \rangle D ; \]

we now use the initial conditions and solve for \( y \):

\[ y = \frac{D^2}{D^2 + 1}\tau_0 + c\left(\frac{D^2}{D^2 + 1} + \frac{D}{D^2 + 1}\right) . \]
From (4) and (2.10)-(2.11) it results that
\[ y = \{\cos t\} T_0 + c(\sin + \cos), \]
whence our conclusion \( y(\cdot) = T_0(\cdot) \cos + c(\sin + \cos) \) now comes directly from (6.2) and 1.33.

**Last problem 6.7.** To find an element \( y \) of \( \mathcal{A}_\omega \) such that
\[
\frac{\partial^2}{\partial t^2} y + y = \sum_{k=-\infty}^{\infty} D T_{2k\pi}.
\]

Setting \( c_0 = \langle y, 0- \rangle \) and \( c_1 = \langle \partial_t y, 0- \rangle \), we see from (6.0) that
\[
(D^2 + 1)y = c_0 D^2 + c_1 D + D \sum_{k=-\infty}^{\infty} T_{2k\pi}.
\]

Solving (6) for \( y \), we obtain \( y = c_0 \cos + c_1 \sin + y_p \), where
\[
y_p = \frac{D}{D^2 + 1} \sum_{k=-\infty}^{\infty} T_{2k\pi} = \{\sin t\} \sum_{k=-\infty}^{\infty} T_{2k\pi}:
\]

the second equation is from (2.11). From (7) and (4.12) it now follows that
\[
y_p = \left\{ \sin t \sum_{k=-\infty}^{\infty} T_{2k\pi}(t) \right\}.
\]

From (8) and (2.15) we can now write
\[
y_p(t) = \sin t \sum_{k=-\infty}^{\infty} T_{2k\pi}(t) = \left(1 + \left[\frac{t}{2\pi}\right]\right) \sin t;
\]
as usual, \( \lfloor t/2\pi \rfloor \) is the greatest integer \( < t/2\pi \) (the last equation follows directly from the definition of \( T_x(\cdot) \)). In case \( \omega = \mathbb{R} \), the answer (9) to the problem (5) cannot be obtained by the Fourier transformation nor by the distributional two-sided Laplace transformation.

**Added in proof.** There still remains to connect the theory presented in this paper with the theory of distributions; this has been done in the Research Announcement "An algebra of generalized functions on an open interval; two-sided operational calculus" (by Gregers Krabbe), Bull. Amer. Math. Soc. 77 (1971), 78–84.

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