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ON THE ABSOLUTE HAUSDORFF SUMMABILITY OF A FOURIER SERIES

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In this paper a theorem on the absolute Hausdorff summability of a series associated with a Fourier series has been established. This theorem unifies and extends various known results.

1. Let μ_n be a sequence of real or complex numbers and write

$$\Delta^0 \mu_n = \mu_n, \Delta^p \mu_n = \Delta^{p-1} \mu_n - \Delta^{p-1} \mu_{n+1}, \quad p \geq 1.$$

If S_n denotes the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$, the transformation

$$t_m = \sum_{n=0}^m \binom{m}{n} (\Delta^{m-n} \mu_n) S_n$$

defines the sequence $\{t_m\}$ of (H, μ) means or the Hausdorff means [3, 12] of the sequence $\{S_n\}$. The series $\sum a_n$ is said to be summable (H, μ) to the sum s if $\lim_{m \rightarrow \infty} t_m = s$ and is said to be absolutely summable (H, μ) or summable $|H, \mu|$ if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < C^1.$$

In order that (H, μ) should be a convergence preserving transformation it is necessary and sufficient that μ_n should be a moment constant, that is, there exists a function $\chi(x)$ of bounded variation in $0 \leq x \leq 1$, such that

$$\mu_n = \int_0^1 x^n d\chi(x), \quad n = 0, 1, 2, \dots$$

We may suppose without loss of generality that $\chi(0) = 0$. If also, $\chi(1) = 1$ and $\chi(+0) = \chi(0) = 0$, so that $\chi(x)$ is continuous at the origin, then μ_n is a regular moment constant and (H, μ) is a regular Hausdorff transformation [3]. It is known that the sequence to sequence Hausdorff transformation is absolute convergence preserving or absolutely regular if and only if it is a convergence preserving or regular transformation of the same type [4, 8, 9].

In the case in which

$$\chi(x) = 1 - (1 - x)^\delta, \quad \delta > 0,$$

¹ Throughout the paper C denotes a positive constant not necessarily the same at each occurrence.

the method (H, μ) reduces to the well known Cesàro method (C, δ) [3, 12].

2. Let $f(t)$ be a periodic function with period 2π and integrable in the Lebesgue sense in $(-\pi, \pi)$. Let the Fourier series of $f(t)$ be

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t),$$

it being assumed that the constant term is zero.

The ε th forward and backward fractional integrals of a function $g(x)$, which is Lebesgue integrable in $(0, 1)$, are respectively defined as

$$g_{\varepsilon}^{+}(x) = \frac{1}{\Gamma(\varepsilon)} \int_0^x (x-u)^{\varepsilon-1} g(u) du,$$

and

$$g_{\varepsilon}^{-}(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^1 (u-x)^{\varepsilon-1} g(u) du.$$

These integrals exist almost everywhere for $\varepsilon > 0$.

We write

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\};$$

$$\Phi_0(t) = \phi(t);$$

$$\Phi_{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u) du, \quad \alpha > 0;$$

$$\phi_{\alpha}(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_{\alpha}(t), \quad \alpha \geq 0;$$

$$M(n, x, t) = \sum_{\nu=1}^n \nu \binom{n}{\nu} \varepsilon(\nu) x^{\nu} (1-x)^{n-\nu} \cos \nu t;$$

$$N(n, x, t) = \sum_{\nu=1}^n \binom{n}{\nu} \varepsilon(\nu) x^{\nu} (1-x)^{n-\nu} \sin \nu t;$$

$$L(\chi; n, t) = \frac{2}{\pi} \sum_{\nu=1}^n \binom{n}{\nu} \varepsilon(\nu) \sin \nu t \int_0^1 x^{\nu} (1-x)^{n-\nu} d\chi(x);$$

$$I(\chi; n, u) = \frac{1}{\Gamma(1-\alpha)} \int_u^{\pi} (t-u)^{-\alpha} \frac{d}{dt} L(\chi; n, t) dt;$$

$$J(\chi; n, u) = \frac{1}{\Gamma(1+\alpha)} \int_0^u v^{\alpha} \frac{d}{dv} I(\chi; n, v) dv.$$

3. In this paper we establish the following:

THEOREM. *Let $\varepsilon(t)$ be a positive and monotonic nondecreasing*

function of t such that

$$(3.1) \quad \sum_{n=[1/t]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha}t^{\gamma-\alpha}} = O\left(\varepsilon\left(\frac{k}{t}\right)\right), \quad (0 < t < \pi, k > \pi)$$

and

$$(3.2) \quad \int_0^{\pi} \varepsilon\left(\frac{k}{t}\right) |d\phi_{\alpha}(t)| < C.$$

If

$$\begin{aligned} \text{either (a)} \quad & \chi(u) = g_{1+\gamma}^+(u) + C, \quad (0 \leq \alpha < \gamma < 1) \\ \text{or (b)} \quad & \chi(u) = g_{1+\gamma}^-(u) + C, \end{aligned}$$

for some function $g(u)$ which is Lebesgue integrable in $(0, 1)$, then the series $\sum \varepsilon(n)A_n(t)$ is summable $|H, \mu|$ at the point $t = x$, it being assumed that the transformation (H, μ) is convergence preserving.

Taking $\varepsilon(t) = 1$ and $\alpha = 0$ the above theorem reduces to a recent result on the absolute Hausdorff summability of a Fourier series ([11], Theorem 1) which in turn includes² a result of Bosanquet ([1], Theorem 1) on the absolute Cesàro summability of a Fourier series and the case $0 < \alpha < 1$ covers another result on the absolute Cesàro summability of a Fourier series ([2], Theorem 1). Also for $\alpha = 0$ choosing $\varepsilon(t) = t^{\beta}$ and $\gamma = \beta + \delta$ ($\beta > 0, \delta > 0$) we get another result ([10], Theorem 1) on the absolute Hausdorff summability which is known to include a theorem on the absolute Cesàro summability of the series $\sum n^{\beta}A_n(t)$ due to Mohanty ([6], Theorem 1). Further choosing $\varepsilon(t) = \log(1 + t)$ we get (cf. [7])

THEOREM A. *If*

$$\int_0^{\pi} \log \frac{k}{t} |d\phi_{\alpha}(t)| < C \quad (k > \pi)$$

and

$$\begin{aligned} \text{either (a)} \quad & \chi(u) = g_{1+\gamma}^+(u) + C, \quad (0 \leq \alpha < \gamma < 1) \\ \text{or (b)} \quad & \chi(u) = g_{1+\gamma}^-(u) + C, \end{aligned}$$

for some function $g(u)$ which is Lebesgue integrable in $(0, 1)$, then the series $\sum \log(n + 1)A_n(t)$ is summable $|H, \mu|$, at the point $t = x$, it being assumed that the transformation (H, μ) is convergence preserving.

4. We require the following lemmas for the proof of our theorem.

² While this paper was in press, a paper due to B. Kuttner and N. Tripathi (Quart. J. Math., **22** (1971), 229-308) appeared in which it is shown that Tripathi's theorem can be deduced from the result of Bosanquet.

LEMMA 1 [4]. Let $\{t_n\}$ and $\{S_n\}$ be the partial sums of the series $\sum b_n$ and $\sum a_n$ respectively. Then the sequence to sequence transformation

$$t_m = \sum_{n=0}^m \binom{m}{n} (\Delta^{m-n} \mu_n) S_n$$

can be put in the series to series form as

$$b_m = \frac{1}{m} \sum_{n=1}^m \binom{m}{n} (\Delta^{m-n} \mu_n) n a_n ,$$

$$b_0 = a_0 .$$

LEMMA 2 [5]. If $g(x)$ and $h(x)$ are Lebesgue integrable in $(0, 1)$, then for $\varepsilon > 0$

$$\int_0^1 g_\varepsilon^+(x) h(x) dx = \int_0^1 g(x) h_\varepsilon^-(x) dx .$$

LEMMA 3.

$$I_1 \equiv \int_0^x N(n, 1 - v, t) dv = O\left(\frac{\varepsilon(n)}{nt}\right)$$

and

$$I_2 \equiv \int_0^x M(n, 1 - v, t) dv = O\left(\frac{\varepsilon(n)}{t}\right) .$$

uniformly for x in $(0, 1)$.

Proof. By Abel's transformation, we have

$$\begin{aligned} I_1 &= \int_0^x \left(\sum_{\nu=1}^n \binom{n}{\nu} \varepsilon(\nu) (1 - \nu)^\nu v^{n-\nu} \sin \nu t \right) dv \\ &= \int_0^x \left[\sum_{\nu=1}^{n-1} \Delta_\nu \left\{ \binom{n}{\nu} \varepsilon(\nu) (1 - v)^\nu v^{n-\nu} \right\} \sum_{r=1}^\nu \sin rt \right. \\ &\quad \left. + \binom{n}{n} \varepsilon(n) (1 - v)^n \sum_{\nu=1}^n \sin \nu t \right] dv \\ &= O\left(\frac{1}{t}\right) \sum_{\nu=1}^{n-1} |\Delta_\nu(\varepsilon(\nu) p_\nu(x))| + O\left(\frac{\varepsilon(n)}{nt}\right) , \end{aligned}$$

where

$$p_\nu(x) = \binom{n}{\nu} \int_0^x v^{n-\nu} (1 - v)^\nu dv , \quad (1 \leq \nu \leq n - 1) .$$

We observe that $\varepsilon(\nu) p_\nu(x)$ ($1 \leq \nu \leq n - 1$) is a nondecreasing function of ν for fixed x , since by hypothesis $\varepsilon(\nu)$ is non-decreasing and

$$\begin{aligned}
 p_\nu(x) &= \binom{n}{\nu} \int_0^x v^{n-\nu} (1-v)^\nu dv \\
 &= \binom{n}{\nu} \left[-\frac{v^{n-\nu} (1-v)^{\nu+1}}{\nu+1} \right]_0^x + \binom{n}{\nu} \frac{(n-\nu)}{(\nu-1)} \int_0^x v^{n-\nu-1} (1-v)^{\nu+1} dv \\
 &= \binom{n}{\nu+1} \int_0^x v^{n-\nu-1} (1-v)^{\nu+1} dv - \frac{1}{(\nu+1)} \binom{n}{\nu} x^{n-\nu} (1-x)^{\nu+1} \\
 &= p_{\nu+1}(x) - \frac{1}{(\nu+1)} \binom{n}{\nu} x^{n-\nu} (1-x)^{\nu+1} \\
 &< p_{\nu+1}(x).
 \end{aligned}$$

Hence

$$\begin{aligned}
 I_1 &= O\left(\frac{1}{t}\right) \sum_{\nu=1}^{n-1} [\varepsilon(\nu+1)p_{\nu+1}(x) - \varepsilon(\nu)p_\nu(x)] + O\left(\frac{\varepsilon(n)}{nt}\right) \\
 &= O\left(\frac{\varepsilon(n)}{t}\right) \int_0^x (1-v)^n dv + O\left(\frac{\varepsilon(n)}{nt}\right) \\
 &= O\left(\frac{\varepsilon(n)}{nt}\right).
 \end{aligned}$$

I_2 can be similarly estimated. Hence the lemma.

LEMMA 4. For $0 < \gamma < 1$,

$$J_1 \equiv \int_0^{1-x} (1-x-u)^{\gamma-1} N(n, 1-u, t) du = O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right)$$

and

$$J_2 \equiv \int_0^{1-x} (1-x-u)^{\gamma-1} M(n, 1-u, t) du = O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right)$$

uniformly for x in $(0, 1)$.

Proof. Since

$$\begin{aligned}
 (4.1) \quad |N(n, 1-u, t)| &\leq \varepsilon(n) \sum_{\nu=1}^n \binom{n}{\nu} (1-u)^\nu u^{n-\nu} \\
 &\leq \varepsilon(n)
 \end{aligned}$$

we have

$$\begin{aligned}
 (4.2) \quad J_1 &= \int_0^{1-x} (1-x-u)^{\gamma-1} N(n, 1-u, t) du \\
 &= O(\varepsilon(n)) \int_0^{1-x} (1-x-u)^{\gamma-1} du \\
 &= O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right),
 \end{aligned}$$

if $x > 1 - 1/nt$.

On the other hand if $x < 1 - 1/nt$, write

$$\begin{aligned}
 J_1 &= \int_0^{1-x} (1-x-u)^{\gamma-1} N(n, 1-u, t) du \\
 &= \int_0^{1-x-1/nt} (1-x-u)^{\gamma-1} N(n, 1-u, t) du \\
 &\quad + \int_{1-x-1/nt}^{1-x} (1-x-u)^{\gamma-1} N(n, 1-u, t) du \\
 &= J_{1,1} + J_{1,2},
 \end{aligned}
 \tag{4.3}$$

say. Since $\gamma < 1$ and $(1-x-u)^{\gamma-1}$ is an increasing function of u ,

$$\begin{aligned}
 J_{1,1} &= \frac{1}{(nt)^{\gamma-1}} \int_0^{1-x-1/nt} N(n, 1-u, t) du, \quad \left(0 \leq \eta \leq 1-x-\frac{1}{nt}\right) \\
 &= O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right)
 \end{aligned}
 \tag{4.4}$$

by the application of the estimate I_1 of Lemma 3.

And using the estimate (4.1) we have

$$J_{1,2} = O(\varepsilon(n)) \int_{1-x-1/nt}^{1-x} (1-x-u)^{\gamma-1} du = O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right).
 \tag{4.5}$$

A combination of the estimates in (4.4) and (4.5), in view of (4.3), yields

$$J_1 = O\left(\frac{\varepsilon(n)}{n^\gamma t^\gamma}\right)
 \tag{4.6}$$

when $x < 1 - 1/nt$. Hence, in view of the estimates in (4.2) and (4.6), the first part of the lemma follows. The second part follows on similar lines.

5. Proof of the theorem. In view of the definition and Lemma 1, the absolute Hausdorff summability of the series $\sum \varepsilon(n)A_n(x)$ is equivalent to the absolute convergence of the series $\sum_{n=1}^\infty b_n$, where

$$b_n = \frac{1}{n} \sum_{\nu=1}^n \nu \binom{n}{\nu} (A^{n-\nu} \mu_\nu) \varepsilon(\nu) A_\nu(x).$$

We first consider the case $\alpha = 0$.

Since

$$\begin{aligned}
 A_\nu(x) &= \frac{2}{\pi} \int_0^\pi \phi(t) \cos \nu t dt \\
 &= -\frac{2}{\pi} \int_0^\pi \frac{\sin \nu t}{\nu} d\phi(t),
 \end{aligned}$$

and the transformation (H, μ) is convergence preserving,

$$\begin{aligned} b_n &= -\frac{2}{n\pi} \int_0^\pi d\phi(t) \int_0^1 \left(\sum_{\nu=1}^n \binom{n}{\nu} \right) \varepsilon(\nu) x^\nu (1-x)^{n-\nu} \sin \nu t \, d\chi(x) \\ &= -\frac{2}{n\pi} \int_0^\pi d\phi(t) \int_0^1 N(n, x, t) d\chi(x), \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{n=1}^\infty |b_n| &\leq \frac{2}{\pi} \int_0^\pi |d\phi(t)| \left[\sum_{n=1}^{[1/t]} \frac{1}{n} \left| \int_0^1 N(n, x, t) d\chi(x) \right| \right. \\ &\quad \left. + \sum_{n=[1/t]+1}^\infty \frac{1}{n} \left| \int_0^1 N(n, x, t) d\chi(x) \right| \right]. \end{aligned}$$

Since

$$\int_0^\pi \varepsilon\left(\frac{k}{t}\right) |d\phi(t)| < C,$$

it is clear that we have to show that uniformly in $0 < t \leq \pi$,

$$(5.1) \quad \sum_1 \equiv \sum_{n=1}^{[1/t]} \frac{1}{n} \left| \int_0^1 N(n, x, t) d\chi(x) \right| = O\left(\varepsilon\left(\frac{k}{t}\right)\right),$$

and

$$(5.2) \quad \sum_2 \equiv \sum_{n=[1/t]+1}^\infty \frac{1}{n} \left| \int_0^1 N(n, x, t) d\chi(x) \right| = O\left(\varepsilon\left(\frac{k}{t}\right)\right).$$

Clearly

$$\begin{aligned} |N(n, x, t)| &\leq \sum_{\nu=1}^n \binom{n}{\nu} \varepsilon(\nu) x^\nu (1-x)^{n-\nu} t \\ &\leq nt\varepsilon(n) \sum_{\nu=1}^n \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \\ &\leq nt\varepsilon(n), \end{aligned}$$

and therefore

$$\begin{aligned} \sum_1 &= \sum_{n=1}^{[1/t]} \frac{1}{n} \left| \int_0^1 N(n, x, t) d\chi(x) \right| \\ &\leq \sum_{n=1}^{[1/t]} \frac{nt\varepsilon(n)}{n} \int_0^1 |d\chi(x)| \\ &= O\left(\varepsilon\left(\frac{k}{t}\right)\right), \end{aligned}$$

the function $\chi(x)$ being of bounded variation in $(0, 1)$. This completes the proof of the estimate in (5.1). We now proceed to establish (5.2).

Putting

$$\chi(x) = g_{1+\gamma}^+(x) + C$$

we have

$$\begin{aligned} \sum_2 &\equiv \sum_{n=[1/t]+1}^{\infty} \frac{1}{n} \left| \int_0^1 N(n, x, t) d\chi(x) \right| \\ &= \sum_{n=[1/t]+1}^{\infty} \frac{1}{n} \left| \int_0^1 g_{\gamma}^+(x) N(n, x, t) dx \right| \\ (5.3) \quad &= \sum_{n=[1/t]+1}^{\infty} \frac{1}{n} \left| \int_0^1 g(x) N_{\gamma}^-(n, x, t) dx \right| \\ &\quad \text{(by the application of Lemma 2)} \\ &\leq \int_0^1 |g(x)| \left(\sum_{n=[1/t]+1}^{\infty} \frac{1}{n} |N_{\gamma}^-(n, x, t)| \right) dx, \end{aligned}$$

where $N_{\gamma}^-(n, x, t)$ means the γ th backward fractional integral of $N(n, x, t)$ regarded as a function of x . By the application of Lemma 4 we get

$$\begin{aligned} (5.4) \quad N_{\gamma}^-(n, x, t) &= \frac{1}{\Gamma(\gamma)} \int_x^1 (u-x)^{\gamma-1} N(n, u, t) du \\ &= O\left(\frac{\varepsilon(n)}{n^{\gamma} t^{\gamma}}\right), \end{aligned}$$

and therefore

$$\begin{aligned} \sum_2 &= O(1) \sum_{n=[1/t]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma} t^{\gamma}} \int_0^1 |g(x)| dx \\ &= O(1) \sum_{n=[1/t]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma} t^{\gamma}} \\ &= O\left(\varepsilon\left(\frac{k}{t}\right)\right), \end{aligned}$$

and this completes the proof of (5.2). The proof of the estimate in (5.2) in the case when

$$\chi(x) = g_{1+\gamma}^-(x) + C$$

follows on similar lines.

We now consider the case $0 < \alpha < 1$. We have

$$\begin{aligned} (5.5) \quad b_n &= \frac{1}{n} \int_0^{\pi} \phi(t) \frac{d}{dt} \left(\frac{2}{\pi} \sum_{\nu=1}^n \binom{n}{\nu} \varepsilon(\nu) \sin \nu t \int_0^1 x^{\nu} (1-x)^{n-\nu} d\chi(x) \right) dt \\ &= \frac{1}{n} \int_0^{\pi} \phi(t) \frac{d}{dt} L(\chi; n, t) dt \\ &= \frac{1}{n} \int_0^{\pi} \frac{d}{dt} L(\chi; n, t) \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} d\Phi_{\alpha}(u) \right) dt \\ &= \frac{1}{n\Gamma(1-\alpha)} \int_0^{\pi} d\Phi_{\alpha}(u) \int_u^{\pi} (t-u)^{-\alpha} \frac{d}{dt} L(\chi; n, t) dt \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{n} \int_0^\pi I(\chi; n, u) d\phi_\alpha(u) \\
 &= \frac{1}{n} \Phi_\alpha(\pi) I(\chi; n, \pi) - \frac{1}{n} \Phi_\alpha(\pi) J(\chi; n, \pi) + \frac{1}{n} \int_0^\pi J(\chi; n, u) d\phi_\alpha(u),
 \end{aligned}$$

and therefore in order to prove the theorem we have to show that

$$(5.6) \quad \sum_{n=1}^\infty n^{-1} |I(\chi; n, \pi)| < C,$$

$$(5.7) \quad \sum_{n=1}^\infty n^{-1} |J(\chi; n, \pi)| < C$$

and

$$(5.8) \quad \sum_{n=1}^\infty n^{-1} |J(\chi; n, u)| = O\left(\varepsilon\left(\frac{k}{u}\right)\right),$$

uniformly in $0 < u \leq \pi$.

We show that all the above estimates are true with

$$\chi(x) = g_{i+\gamma}^+(x) + C.$$

The method of proof for $\chi(x) = g_{i+\gamma}^-(x) + C$ will be similar.

For sake of brevity we write g^+ for $g_{i+\gamma}^+$. We have

$$\begin{aligned}
 I(g^+; n, u) &= \frac{1}{\Gamma(1-\alpha)} \int_u^{u+n-1} (t-u)^{-\alpha} \frac{d}{dt} L(g^+; n, t) dt \\
 &\quad + \frac{1}{\Gamma(1-\alpha)} \int_{u+n-1}^\pi (t-u)^{-\alpha} \frac{d}{dt} L(g^+, n, t) dt \\
 &= I_1(g^+; n, u) + I_2(g^+; n, u),
 \end{aligned}$$

say. Now

$$\begin{aligned}
 I_1(g^+; n, u) &= \frac{2}{\pi \Gamma(1-\alpha)} \int_u^{u+n-1} (t-u)^{-\alpha} dt \int_0^1 g_\gamma^+(x) \\
 &\quad \times \left(\sum_{\nu=1}^n \nu \binom{n}{\nu} \varepsilon(\nu) x^\nu (1-x)^{n-\nu} \cos \nu t \right) dx \\
 &= \frac{2}{\pi \Gamma(1-\alpha)} \int_u^{u+n-1} (t-u)^{-\alpha} dt \int_0^1 g_\gamma^+(x) M(n, x, t) dx \\
 &= \frac{2}{\pi \Gamma(1-\alpha)} \int_u^{u+n-1} (t-u)^{-\alpha} dt \int_0^1 g(x) M_\gamma^-(n, x, t) dx^3 \\
 &= \frac{2}{\pi \Gamma(1-\alpha) \Gamma(\gamma)} \int_u^{u+n-1} (t-u)^{-\alpha} dt \int_0^1 g(x) dx
 \end{aligned}$$

³ $M_\gamma^-(n, x, t)$ is γ th backward fractional integral of $M(n, x, t)$ regarded as a function of x .

$$\begin{aligned} & \times \int_0^{1-x} (1-x-v)^{r-1} M(n, 1-v, t) dv \\ &= O\left(\frac{\varepsilon(n)}{n^{r-1}}\right) \int_u^{u+n^{-1}} t^{-r} (t-u)^{-\alpha} dt \int_0^1 |g(x)| dx \\ &= O\left(\frac{\varepsilon(n)}{n^{r-\alpha} u^r}\right), \end{aligned}$$

by the application of the estimate of J_2 in Lemma 4 and using the fact that $g(x)$ is Lebesgue integrable in $(0, 1)$.

Also

$$\begin{aligned} I_2(g^+; n, u) &= \frac{1}{\Gamma(1-\alpha)} \int_{u+n^{-1}}^\pi (t-u)^{-\alpha} \frac{d}{dt} L(g^+; n, t) dt \\ &= \frac{n^\alpha}{\Gamma(1-\alpha)} \int_{u+n^{-1}}^\zeta \frac{d}{dt} L(g^+; n, t) dt, \quad (u+n^{-1} < \zeta < \pi) \end{aligned}$$

where

$$\begin{aligned} L(g^+; n, t) &= \frac{2}{\pi} \int_0^{1/\nu} \left(\sum_{\nu=1}^n \binom{n}{\nu} \right) \varepsilon(\nu) x^\nu (1-x)^{n-\nu} \sin \nu t \, g_\nu^+(x) dx \\ &= \frac{2}{\pi} \int_0^1 g_\nu^+(x) N(n, x, t) dx \\ &= \frac{2}{\pi} \int_0^1 g(x) N_\nu^-(n, x, t) dx \\ &= \frac{2}{\pi \Gamma(\gamma)} \int_0^1 g(x) dx \int_0^{1-x} (1-x-v)^{r-1} N(n, 1-v, t) dv \\ &= O\left(\frac{\varepsilon(n)}{n^r t^r}\right) \int_0^1 |g(x)| dx \\ &= O\left(\frac{\varepsilon(n)}{n^r t^r}\right), \end{aligned}$$

by the application of the estimate of J_1 in Lemma 4 and the Lebesgue integrability of $g(x)$ in $(0, 1)$. Hence

$$I_2(g^+; n, u) = O\left(\frac{\varepsilon(n)}{n^{r-\alpha} u^r}\right).$$

Thus we have shown that

$$(5.9) \quad I(g^+; n, u) = O\left(\frac{\varepsilon(n)}{n^{r-\alpha} u^r}\right).$$

Using this estimate we have

$$\sum_{n=1}^\infty n^{-1} |I(g^+; n, \pi)| \leq C \sum_{n=1}^\infty \frac{\varepsilon(n)}{n^{1+r-\alpha}} < C,$$

and this completes the proof of (5.6).

If in particular we suppose that $\phi(t) = 1$ for all t , in which case $\phi_\alpha(t) = 1$ for all t and $b_n = 0$ for every n , we obtain from (5.5) and the estimate in (5.9)

$$O = O\left(\frac{\varepsilon(n)}{n^{1+\gamma-\alpha}}\right) - \frac{J(g^+; n, \pi)}{n}$$

and therefore

$$(5.10) \quad J(g^+; n, \pi) = O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha}}\right).$$

Hence

$$\sum_{n=1}^{\infty} n^{-1} |J(g^+; n, \pi)| \leq C \sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha}} < C,$$

and this establishes the estimate in (5.7). Now it remains to establish (5.8). We note that

$$\begin{aligned} I(\chi; n, u) &= \frac{1}{\Gamma(1-\alpha)} \left(\int_u^{u+n-1} + \int_{u+n-1}^\pi \right) (t-u)^{-\alpha} \frac{d}{dt} L(\chi; n, t) dt \\ &= O(n\varepsilon(n)) \int_u^{u+n-1} (t-u)^{-\alpha} dt \int_0^1 \left(\sum_{\nu=1}^n \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \right) |d\chi(x)| \\ &\quad + \frac{2}{\pi} \cdot \frac{n^\alpha}{\Gamma(1-\alpha)} \sum_{\nu=1}^n \nu \binom{n}{\nu} \varepsilon(\nu) \int_0^1 x^\nu (1-x)^{n-\nu} d\chi(x) \\ &\quad \times \int_{u+n-1}^\zeta \cos \nu t dt \quad (u+n-1 < \zeta < \pi) \\ &= O(n^\alpha \varepsilon(n)) + O(n^\alpha \varepsilon(n)) \int_0^1 |d\chi(x)| \sum_{\nu=1}^n \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \\ &= O(n^\alpha \varepsilon(n)), \end{aligned}$$

and therefore

$$\begin{aligned} (5.11) \quad J(\chi; n, u) &= \frac{1}{\Gamma(1+\alpha)} [v^\alpha I(\chi; n, v)]_0^u - \frac{\alpha}{\Gamma(1+\alpha)} \int_0^u v^{\alpha-1} I(\chi; n, v) dv \\ &= O(n^\alpha u^\alpha \varepsilon(n)) + O(n^\alpha \varepsilon(n)) \int_0^u v^{\alpha-1} dv \\ &= O(n^\alpha u^\alpha \varepsilon(n)). \end{aligned}$$

Also

$$\Gamma(\alpha + 1) [J(\chi; n, \pi) - J(\chi; n, u)] = [v^\alpha I(\chi; n, v)]_u^\pi - \alpha \int_u^\pi v^{\alpha-1} I(\chi; n, v) dv$$

and therefore

$$\begin{aligned}
 J(g^+; n, u) &= J(g^+; n, \pi) - \frac{1}{\Gamma(1 + \alpha)} [v^\alpha I(g^+; n, v)]_u^\pi \\
 &\quad + \frac{\alpha}{\Gamma(\alpha + 1)} \int_u^\pi v^{\alpha-1} I(g^+; n, v) dv \\
 (5.12) \qquad &= O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha}}\right) + O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha} u^{\gamma-\alpha}}\right) + O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha}}\right) \int_u^\pi v^{-\gamma+\alpha-1} dv \\
 &= O\left(\frac{\varepsilon(n)}{n^{\gamma-\alpha} u^{\gamma-\alpha}}\right)
 \end{aligned}$$

using the estimates in (5.9) and (5.10). Now by the application of the estimate in (5.11) and (5.12) we have

$$\begin{aligned}
 \sum_{n=1}^{\infty} n^{-1} |J(g^+; n, u)| &= \sum_{n=1}^{[1/u]} n^{-1} |J(g^+; n, u)| + \sum_{n=[1/u]+1}^{\infty} n^{-1} |J(g^+; n, u)| \\
 &= O(u^\alpha) \sum_{n=1}^{[1/u]} n^{\alpha-1} \varepsilon(n) + O(1) \sum_{n=[1/u]+1}^{\infty} \frac{\varepsilon(n)}{n^{1+\gamma-\alpha} u^{\gamma-\alpha}} \\
 &= O\left(\varepsilon\left(\frac{k}{u}\right)\right),
 \end{aligned}$$

uniformly in $0 < u \leq \pi$. This completes the proof of the estimate in (5.8). Hence the theorem.

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