INTEGRATED ORTHONORMAL SERIES

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Throughout this paper the author defines

\[ F_\alpha(t) = \sum_{m=1}^{\infty} |\phi_m(t)|^\alpha = \sum_{m=1}^{\infty} \left| \int_a^t \phi_m(x) dx \right|^\alpha \]

where \( 0 < \alpha \leq 2, a \leq t \leq b, \) and \( \{\phi_m\} \) is a sequence in \( L^1[a, b], \) usually orthonormal. In this paper, \( F_\alpha(t) \) is studied for the Haar, Walsh, trigonometric, and general orthonormal sequences. For instance, it is proved that for the Haar system \( F_\alpha(t) \) satisfies a Lipschitz condition of order \( \alpha/2 \) in \([0, 1]\) and that this result is best possible for any complete orthonormal sequence. An application is also given regarding the absolute convergence of Walsh series.

Previously, Bosanquet and Kestelman essentially proved [3, p. 91]

**Theorem A.** Let \( \{\varphi_m\} \) be orthonormal. Then the Fourier coefficients of every absolutely continuous function are absolutely convergent if and only if \( F_\alpha(t) \in L^\alpha[a, b]. \)

Also, applying Parseval's equality to the characteristic function of \([a, t],\) we obtain

**Theorem B.** Let \( \{\varphi_m\} \) be orthonormal. Then \( \{\varphi_m\} \) is complete in \( L^2[a, b] \) if and only if \( F_\alpha(t) = t - a, \) \( a \leq t \leq b. \)

For certain systems, such as the Haar system, the following extension of Theorem A is possible.

**Theorem 1.** Assume \( \{\varphi_m\} \) is orthonormal, \( \Phi_m(t) \) has constant sign on \([a, b]\) for each \( m = 1, 2, \ldots, \) and \( \sum |\Phi_m(b)| < \infty. \) Then the Fourier coefficients of every absolutely continuous function \( f(t), \) such that \( f''(t) \in L^p, \) are absolutely convergent if and only if \( F_{\alpha}(t) \in L^\alpha, 1 \leq p \leq \infty, p^{-1} + q^{-1} = 1. \)

**Proof.** Necessity. Integrating by parts we obtain

\[ \int_a^b f''(t) \sum_{m=1}^{\infty} |\phi_m(t)| dt \]

exists for every \( f'' \in L^p. \) Hence, \( F_{\alpha}(t) \in L^\alpha \) [7, p. 166].

**Sufficiency.** By Hölder's inequality
If an orthonormal sequence \( \{\varphi_m\} \) is not complete we still obtain \( F_2(t) \) continuous since the "completed" series converges to a continuous function and hence (i.e. by Dini's theorem) the convergence must be uniform. In fact, we have

**Theorem 2.** If \( \{\varphi_m\} \) is orthonormal, then \( F_2(t) \in \text{Lip} (1/2) \).

**Proof.** Let \( x, y \in [a, b] \). Using Bessel's inequality, we obtain

\[
|F_2(x) - F_2(y)| = \left| \sum_{m=1}^{\infty} [\varphi_m(x)]^2 - [\varphi_m(y)]^2 \right| \\
\leq \sum_{m=1}^{\infty} |\varphi_m(x) - \varphi_m(y)| \left\{ |\varphi_m(x)| + |\varphi_m(y)| \right\} \\
\leq \left\{ \sum_{m=1}^{\infty} [\varphi_m(x) - \varphi_m(y)]^2 \right\}^{1/2} \left\{ \sum_{m=1}^{\infty} [\varphi_m(x)]^2 \right\}^{1/2} \\
+ \left\{ \sum_{m=1}^{\infty} [\varphi_m(x) - \varphi_m(y)]^2 \right\}^{1/2} \left\{ \sum_{m=1}^{\infty} [\varphi_m(y)]^2 \right\}^{1/2} \\
\leq 2|b - a|^{1/2} |x - y|^{1/2} .
\]

**Remark 1.** This result is best possible in the following sense: For every \( \varepsilon > 0 \) if we set \( \varphi_i(x) = (1 - x)^{i(1-1/2)} \), \( 0 \leq x < 1 \), then \( \varphi_i \in L^2[0, 1] \) but \( [\varphi_i(t)]^2 \in \text{Lip} (1/2 + \varepsilon) \).

**Remark 2.** It would be interesting to know if \( F_2(t) \) is absolutely continuous and if \( F_2'(t) \in L^2 \) for any orthonormal sequence \( \{\varphi_m\} \).

**Theorem 3.** For any complete orthonormal system \( \{\varphi_m\} \), \( F_2(t) \in \text{Lip} (\alpha/2 + \varepsilon) \) for any \( \varepsilon > 0 \).

**Proof.** Let \( t \in [a, b] \). By Parseval's equality

\[
[F_2(t)]^{1/2} \geq [F_2(t)]^{1/2} = (t - a)^{1/2} , \quad 0 < \alpha \leq 2 ,
\]

since for any nonnegative sequence \( \{a_m\} \), \( [\sum a_m^\alpha]^{1/\alpha} \) is a non-increasing function of \( \alpha \) for \( \alpha > 0 \).

We will now determine which Lipschitz class \( F_2(t) \) belongs to for the Haar, Walsh, and trigonometric systems.

**Definition.** If \( 0 < \alpha \leq 1 \), set

\[
N_\alpha(f) = \sup |f(x) - f(y)| |x - y|^{-\alpha} \quad \text{for} \ x \neq y \quad \text{and} \quad x, y \in [a, b] .
\]

**Lemma 1.** Let \( \alpha > 0 \) and \( 0 < \alpha - \beta \leq 1 \). If
\[ \sum_{m=1}^{\infty} N_\alpha(f_m) = O(n^\delta) \]

and
\[ \sum_{m=1}^{\infty} \|f_m\|_\infty = O(n^{\delta-\alpha}) , \]

then
\[ f(t) = \sum_{m=1}^{\infty} f_m(t) \in \text{Lip} (\alpha - \beta) . \]

**Proof.** Let \( 2^{-\delta-1} < h \leq 2^{-\delta} \). Then
\[
|f(t + h) - f(t)| \leq \sum_{m=1}^{\infty} |f_m(t + h) - f_m(t)| = \sum_{m=1}^{2^{\delta}} + \sum_{m=2^{\delta}+1}^{\infty} = P + Q .
\]

\[
P = O \left( h^\alpha \sum_{m=1}^{2^{\delta}} N_\alpha(f_m) \right) = O(h^{\alpha-\beta}) ,
\]

\[
Q = O \left( \sum_{m=2^{\delta}+1}^{\infty} \|f_m\|_\infty \right) = O(h^{\alpha-\beta}) .
\]

**Lemma 2.** (a) If \( \sum_{m=1}^{2^{\delta}+1} |a_m| n^\alpha = O(2^{\alpha \delta}) \), then
\[ \sum_{m=1}^{n} |a_m| = O(n^{\delta-\alpha}), \beta - \alpha < 0 . \]

(b) If \( \sum_{m=2^{\delta}+1}^{n} |a_m| = O(2^{\alpha \delta}) \), then \( \sum_{m=1}^{n} |a_m| n^\alpha = O(n^{\alpha+\beta}), \alpha + \beta > 0 . \)

**Proof.** Straightforward.

**Lemma 3.** Let \( 0 < \gamma \leq 1 \) and suppose \( f \in \text{Lip} \gamma \).

(a) If \( 0 < \alpha \leq 1, \|f\|^\alpha \in \text{Lip} (\alpha \gamma) \).

(b) If \( \alpha > 1, \|f\|^\alpha \in \text{Lip} \gamma \).

**Proof.** We may assume \( f(t) \geq 0 \) because
\[ |f(t + h) - f(t)| \leq |f(t + h) - f(t)| . \]

**Part (a).** Since \( |x + y|^\alpha \leq |x|^\alpha + |y|^\alpha, 0 < \alpha \leq 1 \), we obtain
\[ |f^\alpha(t + h) - f^\alpha(t)| \leq |f(t + h) - f(t)|^\alpha = O(h^{\alpha \gamma}) . \]

**Part (b).** Since \( |x^\alpha - y^\alpha| \leq \|\alpha t^{\alpha-1}\|_\infty |x - y|, \alpha \geq 1 \), it follows that
\[ |f^\alpha(t + h) - f^\alpha(t)| \leq \|\alpha f^{\alpha-1}(t)\|_\infty |f(t + h) - f(t)| = O(h^\gamma) . \]

**Theorem 4.** Let \( 0 < \gamma \leq 1 \) and assume \( f \in \text{Lip} \gamma \) and is of period \( b - a \).

(a) If \( 0 < \alpha \leq 1, 0 < \alpha \gamma - \delta \leq 1 \), and
\[ \sum_{m=1}^{n} |a_m| n^\gamma \in O(n^\delta) , \]
then

\[ f_\alpha(t) = \sum_{m=1}^{\infty} a_m |f(mt)|^\alpha \in \text{Lip} \left( \alpha \gamma - \delta \right) . \]

(b) If \( \alpha > 1, 0 < \gamma - \delta \leq 1, \) and

\[ \sum_{m=1}^{\infty} |a_m|^m = O(n^\delta), \]

then

\[ f_\alpha(t) = \sum_{m=1}^{\infty} a_m |f(mt)|^\alpha \in \text{Lip} \left( \gamma - \delta \right) . \]

**Proof.** Part (a). By hypothesis and Lemma 3 (a)

\[ \sum_{m=1}^{n} N_{\alpha \gamma} [a_m |f(mt)|^\alpha] = O \left( \sum_{m=1}^{n} |a_m|^m \right) = O(n^\delta) . \]

Also, by Lemma 2 (a), if \( 0 < \alpha \gamma - \delta, \) then

\[ \sum_{m=1}^{\infty} \| a_m |f(mt)|^\alpha \|_\infty = O \left( \sum_{m=1}^{\infty} |a_m|^m \right) = O(n^{\delta - \gamma}) \]

and so our result follows by Lemma 1.

Part (b). By hypothesis and Lemma 3 (b)

\[ \sum_{m=1}^{n} N_\gamma [a_m |f(mt)|^\alpha] = O \left( \sum_{m=1}^{n} |a_m|^m \right) = O(n^\delta) . \]

Also, by Lemma 2 (a), if \( 0 < \gamma - \delta, \) then

\[ \sum_{m=1}^{\infty} \| a_m |f(mt)|^\alpha \|_\infty = O \left( \sum_{m=1}^{\infty} |a_m|^m \right) = O(n^{\delta - \gamma}) , \]

and so our result again follows from Lemma 1.

**Theorem 5.** Let \( 0 < \alpha \leq 2 \) and assume \( \varphi \in L^\alpha[a, b], \varphi_m(x) = \varphi(mx), \) and \( \Phi(t) \) is of period \( b - a. \) If

\[ \sum_{m=1}^{n} |b_m| = O(n^\delta), 0 < \alpha - \beta < 1 , \]

then

\[ G_\alpha(t) = \sum_{m=1}^{\infty} b_m |\Phi_m(t)|^\alpha \in \text{Lip} \left( \alpha - \beta \right) . \]

**Proof.** \( \Phi_m(t) = m^{-1}\Phi(mt) \) and so
Now let $\gamma = 1$ and $a_m = b_m m^{-\alpha}$ in Theorem 4. Then, if $0 < \alpha \leq 1$, our result follows by Theorem 4 (a) with $\delta = \beta$.

If $\alpha > 1$ and $\alpha - \beta < 1$, then by Lemma 2 (b)

$$\sum_{m=1}^{n} |a_m| m^\beta = \sum_{m=1}^{n} |b_m| m^{1-\alpha} = O(n^{\delta-\alpha+1}) .$$

Thus, utilizing Theorem 4 (b) with $\delta = \beta - \alpha + 1$, we obtain

$$G_a(t) \in \text{Lip } [1 - (\beta - \alpha + 1)] = \text{Lip } (\alpha - \beta) .$$

**Corollary 1.**

(a) $\sum_{m=1}^{\infty} \left| \int_{0}^{t} \sin mx \, dx \right|^\alpha \in \text{Lip } (\alpha - 1), 1 < \alpha < 2$, on $[0, 2\pi]$.

(b) If $1 < \alpha < 2$ and $\{w_m(x)\}$ and $\{r_m(x)\} = \{r_1(2m^{-1}x)\}$ denote the Walsh and Rademacher functions (defined in [1]), then

$$\sum_{m=1}^{\infty} \left| \int_{0}^{t} w_m(x) \, dx \right|^\alpha = t^\alpha + \sum_{m=1}^{\infty} 2^{m-1} \left| \int_{0}^{t} r_m(x) \, dx \right|^\alpha \in \text{Lip } (\alpha - 1) \text{ on } [0, 1],$$

since $\left| \int_{0}^{t} w_m(x) \, dx \right| = \left| \int_{0}^{t} r_k(x) \, dx \right|$ for $2^{k-1} \leq m < 2^k, k = 1, 2, \cdots$, as can be easily seen directly.

(c) If $0 < \alpha < 2$ and $\{h_m\}$ denotes the Haar system (defined in [1]), then

$$\sum_{m=1}^{\infty} \left| \int_{0}^{t} h_m(x) \, dx \right|^\alpha = t^\alpha + \sum_{m=1}^{\infty} 2^{(m-1)\alpha/2} \left| \int_{0}^{t} r_m(x) \, dx \right|^\alpha \in \text{Lip } (\alpha/2) \text{ on } [0, 1],$$

since $\sum_{m=2^{k-1}}^{2^k-1} \left| \int_{0}^{t} h_m(x) \, dx \right| = 2^{(k-1)\alpha/2} \left| \int_{0}^{t} r_k(x) \, dx \right|$ for $k = 1, 2, \cdots$.

**Remark 3.** For the Haar system $F_i(t)$ has no finite derivative anywhere [5, p. 279].

**Theorem 6.** Let $0 < \| \varphi \|_1 < \infty$, $\varphi_m(x) = \varphi(mx)$, and assume $\Phi(t)$ is of period $b - a$.

(a) $\sum |a_m| m^{-\alpha} < \infty$ if and only if $\sum |\varphi_m(x)|^\alpha \in L[a, b]$.

(b) If $\sum |a_m| m^{-\alpha} = \infty$, then $\sum |\varphi_m(x)|^\alpha = \infty$ almost everywhere.

**Proof.** Part (a). Since $\Phi_m(t) = m^{-\alpha} \Phi(mt)$, we obtain

$$\int_{a}^{b} |\Phi_m(t)|^\alpha \, dt = m^{-\alpha} \int_{a}^{b} |\Phi(mt)|^\alpha \, dt = m^{-\alpha} \int_{a}^{b} |\Phi(t)|^\alpha \, dt .$$

Part (b). Applying Fejer's Lemma [7, p. 49], we obtain for every set $E$ of positive measure
\[
\lim_{m \to \infty} \int_{E} |\Phi(mt)|^{\alpha} dt = \frac{t_{E}(E)}{b - a} \int_{a}^{b} |\Phi_{1}(t)|^{\alpha} dt > 0 \text{ as } m \to \infty,
\]
and so by a theorem of Orlicz [1, p. 327]
\[
\sum |a_{m}| m^{-\alpha} |\Phi_{1}(mt)|^{\alpha} = \sum |a_{m}| |\Phi_{n}(t)|^{\alpha} = \infty
\]
almost everywhere.

**Corollary 2.** There exists an absolutely continuous function whose Walsh-Fourier series is absolutely divergent.

**Proof.** For the Walsh system \(F_{1}(t) \in L^{\infty}\) by Theorem 6 and so the result follows from Theorem A.

It now seems appropriate to prove

**Theorem 7.** Let
\[
\omega^{2}(\delta, f) = \sup_{0 < h < \delta} \left\{ \int_{0}^{1} [f(x + h) - f(x)]^{2} dx \right\}^{1/2}.
\]
If \(\sum 2^{n/2} \omega^{2}(2^{-n}, f) < \infty\), then the Walsh-Fourier series of \(f\) converges absolutely.

**Proof.** Let \(\{c_{n}\}\) denote the Walsh-Fourier coefficients of \(f\) and let \(x + y = \sum_{n=1}^{\infty} |x_{n} - y_{n}| 2^{-n}\) where \(x = \sum x_{n} 2^{-n}\) and \(y = \sum y_{n} 2^{-n}\) are the binary expansions of \(x\) and \(y\) (where for dyadic rationals we choose the finite expansion). N. Fine proved [4, p. 395]
\[
\sum_{k=2^{n-1}}^{2^{n}-1} c_{k}^{2} \leq \int_{0}^{1} [f(x + 2^{-n}) - f(x)]^{2} dx.
\]
Also, by definition of \(\oplus\), we obtain
\[
\int_{0}^{1} [f(x + 2^{-n}) - f(x)]^{2} dx = \int_{E_{0}} [f(x + 2^{-n}) - f(x)]^{2} dx + \int_{E_{1}} [f(x - 2^{-n}) - f(x)]^{2} dx = 2 \int_{E_{0}} [f(x + 2^{-n}) - f(x)]^{2} dx
\]
where \(E_{p} = \{x \in [0, 1]: x_{n} = p\}\) for \(p = 0, 1\). Hence,
\[
\sum_{k=2^{n-1}}^{2^{n}-1} c_{k}^{2} \leq 2[\omega^{2}(2^{-n}, f)]^{2},
\]
and so by Schwarz's inequality
\[
\sum_{k=2^{n-1}}^{2^{n}-1} |c_{k}| \leq \left( \sum_{k=2^{n-1}}^{2^{n}-1} c_{k}^{2} \right)^{1/2} \left( \sum_{k=2^{n-1}}^{2^{n}-1} 1 \right)^{1/2} \leq \omega^{2}(2^{-n}, f) 2^{n/2}.
\]
REMARK 4. Previously N. Fine [4, p. 394] and N. Vilenkin [6, p. 32] proved that if \( f \in \text{Lip} \alpha, \alpha > 1/2 \), then the Walsh-Fourier series of \( f \) converges absolutely. By Theorem 7 it follows that all of the sufficiency theorems on absolute convergence for trigonometric series [2, p. 154–161] in terms of modulus of continuity carry over completely for the Walsh system.

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