

Pacific Journal of Mathematics

INTEGRATED ORTHONORMAL SERIES

JAMES R. MCLAUGHLIN

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Throughout this paper the author defines

$$F_\alpha(t) = \sum_{m=1}^{\infty} |\Phi_m(t)|^\alpha = \sum_{m=1}^{\infty} \left| \int_a^t \varphi_m(x) dx \right|^\alpha$$

where $0 < \alpha \leq 2$, $a \leq t \leq b$, and $\{\varphi_m\}$ is a sequence in $L^1[a, b]$, usually orthonormal. In this paper, $F_\alpha(t)$ is studied for the Haar, Walsh, trigonometric, and general orthonormal sequences. For instance, it is proved that for the Haar system $F_\alpha(t)$ satisfies a Lipschitz condition of order $\alpha/2$ in $[0, 1]$ and that this result is best possible for any complete orthonormal sequence. An application is also given regarding the absolute convergence of Walsh series.

Previously, Bosanquet and Kestelman essentially proved [3, p. 91]

THEOREM A. *Let $\{\varphi_m\}$ be orthonormal. Then the Fourier coefficients of every absolutely continuous function are absolutely convergent if and only if $F_1(t) \in L^\infty[a, b]$.*

Also, applying Parseval's equality to the characteristic function of $[a, t]$, we obtain

THEOREM B. *Let $\{\varphi_m\}$ be orthonormal. Then $\{\varphi_m\}$ is complete in $L^2[a, b]$ if and only if $F_2(t) = t - a$, $a \leq t \leq b$.*

For certain systems, such as the Haar system, the following extension of Theorem A is possible.

THEOREM 1. *Assume $\{\varphi_m\}$ is orthonormal, $\Phi_m(t)$ has constant sign on $[a, b]$ for each $m = 1, 2, \dots$, and $\sum |\Phi_m(b)| < \infty$. Then the Fourier coefficients of every absolutely continuous function $f(t)$, such that $f'(t) \in L^p$, are absolutely convergent if and only if $F_1(t) \in L^q$, $1 \leq p \leq \infty$, $p^{-1} + q^{-1} = 1$.*

Proof. Necessity. Integrating by parts we obtain

$$\int_a^b f'(t) \sum_{m=1}^{\infty} |\Phi_m(t)| dt$$

exists for every $f' \in L^p$. Hence, $F_1(t) \in L^q$ [7, p. 166].

Sufficiency. By Hölder's inequality

$$\sum_{m=1}^N \left| \int_a^b f'(t) \Phi_m(t) dt \right| \leq \int_a^b |f'(t)| \sum_1^N |\Phi_m(t)| dt \leq \|f'\|_p \|F_1\|_q.$$

If an orthonormal sequence $\{\varphi_m\}$ is not complete we still obtain $F_2(t)$ continuous since the “completed” series converges to a continuous function and hence (i.e. by Dini’s theorem) the convergence must be uniform. In fact, we have

THEOREM 2. *If $\{\varphi_m\}$ is orthonormal, then $F_2(t) \in \text{Lip}(1/2)$.*

Proof. Let $x, y \in [a, b]$. Using Bessel’s inequality, we obtain

$$\begin{aligned} |F_2(x) - F_2(y)| &= \left| \sum_{m=1}^{\infty} [\Phi_m(x)]^2 - [\Phi_m(y)]^2 \right| \\ &\leq \sum_{m=1}^{\infty} |\Phi_m(x) - \Phi_m(y)| \{ |\Phi_m(x)| + |\Phi_m(y)| \} \\ &\leq \left\{ \sum_{m=1}^{\infty} [\Phi_m(x) - \Phi_m(y)]^2 \sum_{m=1}^{\infty} [\Phi_m(x)]^2 \right\}^{1/2} \\ &\quad + \left\{ \sum_{m=1}^{\infty} [\Phi_m(x) - \Phi_m(y)]^2 \sum_{m=1}^{\infty} [\Phi_m(y)]^2 \right\}^{1/2} \\ &\leq 2|b - a|^{1/2} |x - y|^{1/2}. \end{aligned}$$

REMARK 1. This result is best possible in the following sense: For every $\epsilon > 0$ if we set $\varphi_1(x) = (1 - x)^{(\epsilon-1)/2}$, $0 \leq x < 1$, then $\varphi_1 \in L^2[0, 1]$ but $[\Phi_1(t)]^2 \notin \text{Lip}(1/2 + \epsilon)$.

REMARK 2. It would be interesting to know if $F_2(t)$ is absolutely continuous and if $F_2'(t) \in L^2$ for any orthonormal sequence $\{\varphi_m\}$.

THEOREM 3. *For any complete orthonormal system $\{\varphi_m\}$, $F_\alpha(t) \in \text{Lip}(\alpha/2 + \epsilon)$ for any $\epsilon > 0$.*

Proof. Let $t \in [a, b]$. By Parseval’s equality

$$[F_\alpha(t)]^{1/\alpha} \geq [F_2(t)]^{1/2} = (t - a)^{1/2}, \quad 0 < \alpha \leq 2,$$

since for any nonnegative sequence $\{a_m\}$, $[\sum a_m^\alpha]^{1/\alpha}$ is a non-increasing function of α for $\alpha > 0$.

We will now determine which Lipschitz class $F_\alpha(t)$ belongs to for the Haar, Walsh, and trigonometric systems.

DEFINITION. If $0 < \alpha \leq 1$, set

$$N_\alpha(f) = \sup |f(x) - f(y)| |x - y|^{-\alpha} \quad \text{for } x \neq y \quad \text{and } x, y \in [a, b].$$

LEMMA 1. *Let $\alpha > 0$ and $0 < \alpha - \beta \leq 1$. If*

$$\sum_{m=1}^n N_\alpha(f_m) = O(n^\beta)$$

and

$$\sum_{m=n}^\infty \|f_m\|_\infty = O(n^{\beta-\alpha}),$$

then

$$f(t) = \sum_{m=1}^\infty f_m(t) \in \text{Lip}(\alpha - \beta).$$

Proof. Let $2^{-n-1} < h \leq 2^{-n}$. Then

$$\begin{aligned} |f(t+h) - f(t)| &\leq \sum_{m=1}^\infty |f_m(t+h) - f_m(t)| = \sum_{m=1}^{2^n} + \sum_{m=2^{n+1}}^\infty = P + Q. \\ P &= O\left[h^\alpha \sum_{m=1}^{2^n} N_\alpha(f_m)\right] = O(h^{\alpha-\beta}), \\ Q &= O\left[\sum_{m=2^{n+1}}^\infty \|f_m\|_\infty\right] = O(h^{\alpha-\beta}). \end{aligned}$$

LEMMA 2. (a) If $\sum_{m=2^{n+1}}^{2^{n+1}} |a_m| m^\alpha = O(2^{n\beta})$, then

$$\sum_{m=n}^\infty |a_m| = O(n^{\beta-\alpha}), \beta - \alpha < 0.$$

(b) If $\sum_{m=2^{n+1}}^{2^{n+1}} |a_m| = O(2^{n\beta})$, then $\sum_{m=1}^n |a_m| m^\alpha = O(n^{\alpha+\beta})$, $\alpha + \beta > 0$.

Proof. Straightforward.

LEMMA 3. Let $0 < \gamma \leq 1$ and suppose $f \in \text{Lip } \gamma$.

- (a) If $0 < \alpha \leq 1$, $|f|^\alpha \in \text{Lip}(\alpha\gamma)$.
- (b) If $\alpha > 1$, $|f|^\alpha \in \text{Lip } \gamma$.

Proof. We may assume $f(t) \geq 0$ because

$$||f(t+h)| - |f(t)|| \leq |f(t+h) - f(t)|.$$

Part (a). Since $|x+y|^\alpha \leq |x|^\alpha + |y|^\alpha$, $0 < \alpha \leq 1$, we obtain

$$|f^\alpha(t+h) - f^\alpha(t)| \leq |f(t+h) - f(t)|^\alpha = O(h^{\alpha\gamma}).$$

Part (b). Since $|x^\alpha - y^\alpha| \leq |\alpha t^{\alpha-1}|_\infty |x - y|$, $\alpha \geq 1$, it follows that

$$|f^\alpha(t+h) - f^\alpha(t)| \leq \|\alpha f^{\alpha-1}(t)\|_\infty |f(t+h) - f(t)| = O(h^\gamma).$$

THEOREM 4. Let $0 < \gamma \leq 1$ and assume $f \in \text{Lip } \gamma$ and is of period $b - a$.

(a) If $0 < \alpha \leq 1$, $0 < \alpha\gamma - \delta \leq 1$, and

$$\sum_{m=1}^n |a_m| m^{\alpha\gamma} = O(n^\delta),$$

then

$$f_\alpha(t) = \sum_{m=1}^\infty a_m |f(mt)|^\alpha \in \text{Lip}(\alpha\gamma - \delta) .$$

(b) If $\alpha > 1, 0 < \gamma - \delta \leq 1$, and

$$\sum_{m=1}^n |a_m| m^\gamma = O(n^\delta) ,$$

then

$$f_\alpha(t) = \sum_{m=1}^\infty a_m |f(mt)|^\alpha \in \text{Lip}(\gamma - \delta) .$$

Proof. Part (a). By hypothesis and Lemma 3 (a)

$$\sum_{m=1}^n N_{\alpha\gamma} [a_m |f(mt)|^\alpha] = O\left(\sum_1^n |a_m| m^{\alpha\gamma}\right) = O(n^\gamma) .$$

Also, by Lemma 2 (a), if $0 < \alpha\gamma - \delta$, then

$$\sum_{m=n}^\infty \| |a_m| |f(mt)|^\alpha \|_\infty = O\left(\sum_n^\infty |a_m|\right) = O(n^{\delta-\alpha\gamma})$$

and so our result follows by Lemma 1.

Part (b). By hypothesis and Lemma 3(b)

$$\sum_{m=1}^n N_\gamma [a_m |f(mt)|^\alpha] = O\left(\sum_1^n |a_m| m^\gamma\right) = O(n^\delta) .$$

Also, by Lemma 2 (a), if $0 < \gamma - \delta$, then

$$\sum_{m=n}^\infty \| |a_m| |f(mt)|^\alpha \| = O\left(\sum_n^\infty |a_m|\right) = O(n^{\delta-\gamma}) ,$$

and so our result again follows from Lemma 1.

THEOREM 5. Let $0 < \alpha \leq 2$ and assume $\varphi \in L^\infty[a, b]$, $\varphi_m(x) = \varphi(mx)$, and $\Phi_1(t)$ is of period $b - a$. If

$$\sum_{m=1}^n |b_m| = O(n^\beta), 0 < \alpha - \beta < 1 ,$$

then

$$G_\alpha(t) = \sum_{m=1}^\infty b_m |\Phi_m(t)|^\alpha \in \text{Lip}(\alpha - \beta) .$$

Proof. $\Phi_m(t) = m^{-1}\Phi_1(mt)$ and so

$$G_\alpha(t) = \sum_{m=1}^{\infty} b_m m^{-\alpha} |\Phi_1(mt)|^\alpha .$$

Now let $\gamma = 1$ and $a_m = b_m m^{-\alpha}$ in Theorem 4. Then, if $0 < \alpha \leq 1$, our result follows by Theorem 4 (a) with $\delta = \beta$.

If $\alpha > 1$ and $\alpha - \beta < 1$, then by Lemma 2 (b)

$$\sum_{m=1}^n |a_m| m^1 = \sum_{m=1}^n |b_m| m^{1-\alpha} = O(n^{\beta-\alpha+1}) .$$

Thus, utilizing Theorem 4 (b) with $\delta = \beta - \alpha + 1$, we obtain

$$G_\alpha(t) \in \text{Lip} [1 - (\beta - \alpha + 1)] = \text{Lip} (\alpha - \beta) .$$

COROLLARY 1. (a) $\sum_{m=1}^{\infty} \left| \int_0^t \sin mx \, dx \right|^\alpha \in \text{Lip} (\alpha - 1)$, $1 < \alpha < 2$, on $[0, 2\pi]$.

(b) If $1 < \alpha < 2$ and $\{w_m(x)\}$ and $\{r_m(x)\} = \{r_1(2^{m-1}x)\}$ denote the Walsh and Rademacher functions (defined in [1]), then

$$\sum_{m=0}^{\infty} \left| \int_0^t w_m(x) dx \right|^\alpha = t^\alpha + \sum_{m=1}^{\infty} 2^{m-1} \left| \int_0^t r_m(x) dx \right|^\alpha \in \text{Lip} (\alpha - 1) \text{ on } [0, 1] ,$$

since $\left| \int_0^t w_m(x) dx \right| = \left| \int_0^t r_k(x) dx \right|$ for $2^{k-1} \leq m < 2^k$, $k = 1, 2, \dots$, as can be easily seen directly.

(c) If $0 < \alpha < 2$ and $\{h_m\}$ denotes the Haar system (defined in [1]), then

$$\sum_{m=0}^{\infty} \left| \int_0^t h_m(x) dx \right|^\alpha = t^\alpha + \sum_{m=1}^{\infty} 2^{(m-1)\alpha/2} \left| \int_0^t r_m(x) dx \right|^\alpha \in \text{Lip} (\alpha/2) \text{ on } [0, 1] ,$$

since $\sum_{m=2^{k-1}}^{2^k-1} \left| \int_0^t h_m(x) dx \right| = 2^{(k-1)\alpha/2} \left| \int_0^t r_k(x) dx \right|$ for $k = 1, 2, \dots$.

REMARK 3. For the Haar system $F_1(t)$ has no finite derivative anywhere [5, p. 279].

THEOREM 6. Let $0 < \|\varphi\|_1 < \infty$, $\varphi_m(x) = \varphi(mx)$, and assume $\Phi_1(t)$ is of period $b - a$.

(a) $\sum |a_m| m^{-\alpha} < \infty$ if and only if $\sum |a_m| |\Phi_m(t)|^\alpha \in L^1[a, b]$.

(b) If $\sum |a_m| m^{-\alpha} = \infty$, then $\sum |a_m| |\Phi_m(t)|^\alpha = \infty$ almost everywhere.

Proof. Part (a). Since $\Phi_m(t) = m^{-1}\Phi_1(mt)$, we obtain

$$\int_a^b |\Phi_m(t)|^\alpha dt = m^{-\alpha} \int_a^b |\Phi_1(mt)|^\alpha dt = m^{-\alpha} \int_a^b |\Phi_1(t)|^\alpha dt .$$

Part (b). Applying Fejer's Lemma [7, p. 49], we obtain for every set E of positive measure

$$\lim \int_E |\Phi(mt)|^\alpha dt = \frac{\mu(E)}{b-a} \int_a^b |\Phi_1(t)|^\alpha dt > 0 \quad \text{as } m \rightarrow \infty ,$$

and so by a theorem of Orlicz [1, p. 327]

$$\sum |a_m| m^{-\alpha} |\Phi_1(mt)|^\alpha = \sum |a_m| |\Phi_m(t)|^\alpha = \infty$$

almost everywhere.

COROLLARY 2. *There exists an absolutely continuous function whose Walsh-Fourier series is absolutely divergent.*

Proof. For the Walsh system $F_1(t) \notin L^\infty$ by Theorem 6 and so the result follows from Theorem A.

It now seems appropriate to prove

THEOREM 7. *Let*

$$\omega^2(\delta, f) = \sup_{0 < h \leq \delta} \left\{ \int_0^1 [f(x+h) - f(x)]^2 dx \right\}^{1/2} .$$

If $\sum 2^{n/2} \omega^2(2^{-n}, f) < \infty$, then the Walsh-Fourier series of f converges absolutely.

Proof. Let $\{c_n\}$ denote the Walsh-Fourier coefficients of f and let $x \dot{+} y = \sum_{n=1}^\infty |x_n - y_n| 2^{-n}$ where $x = \sum x_n 2^{-n}$ and $y = \sum y_n 2^{-n}$ are the binary expansions of x and y (where for dyadic rationals we choose the finite expansion). N. Fine proved [4, p. 395]

$$\sum_{k=2^{2^n-1}}^{2^{2^n}-1} c_k^2 \leq \int_0^1 [f(x \dot{+} 2^{-n}) - f(x)]^2 dx .$$

Also, by definition of $\dot{+}$, we obtain

$$\begin{aligned} & \int_0^1 [f(x \dot{+} 2^{-n}) - f(x)]^2 dx \\ &= \int_{E_0} [f(x + 2^{-n}) - f(x)]^2 dx + \int_{E_1} [f(x - 2^{-n}) - f(x)]^2 dx \\ &= 2 \int_{E_0} [f(x + 2^{-n}) - f(x)]^2 dx \end{aligned}$$

where $E_p = \{x \in [0, 1]: x_n = p\}$ for $p = 0, 1$. Hence,

$$\sum_{k=2^{2^n-1}}^{2^{2^n}-1} c_k^2 \leq 2[\omega^2(2^{-n}, f)]^2 ,$$

and so by Schwarz's inequality

$$\sum_{k=2^{2^n-1}}^{2^{2^n}-1} |c_k| \leq \left(\sum_{k=2^{2^n-1}}^{2^{2^n}-1} c_k^2 \right)^{1/2} \left(\sum_{k=2^{2^n-1}}^{2^{2^n}-1} 1 \right)^{1/2} \leq \omega^2(2^{-n}, f) 2^{n/2} .$$

REMARK 4. Previously N. Fine [4, p.394] and N. Vilenkin [6, p.32] proved that if $f \in \text{Lip } \alpha$, $\alpha > 1/2$, then the Walsh-Fourier series of f converges absolutely. By Theorem 7 it follows that all of the sufficiency theorems on absolute convergence for trigonometric series [2, p.154-161] in terms of modulus of continuity carry over completely for the Walsh system.

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