

# Pacific Journal of Mathematics

**ON PERMANENTS OF CIRCULANTS**

H. MINC

## ON PERMANENTS OF CIRCULANTS

HENRYK MINC

**A recurrence formula is obtained for permanents of circulants of the form  $\alpha I_n + \beta P + \gamma P^2$  and explicit formulas are deduced from it. It is shown that for doubly stochastic circulants  $\alpha I_n + \beta P + \gamma P^2$  the minimum permanent lies in the interval  $(1/2^n, 1/2^{n-1}]$ .**

**1. Introduction.** The well-known unresolved conjecture of van der Waerden asserts that in  $\Omega_n$ , the polyhedron of doubly stochastic  $n \times n$  matrices, the permanent function takes its minimum value for the matrix  $J_n$ , all of whose entries are  $1/n$ , i.e.,

$$(1) \quad \min_{A \in \Omega_n} \text{per}(A) = \text{per}(J_n).$$

By a theorem of Birkhoff,  $\Omega_n$  is a convex polyhedron with the permutation matrices  $P_1, \dots, P_{n!}$  as vertices. Thus (1) can be written in the form

$$(2) \quad \min_{\theta} \text{per} \left( \sum_{j=1}^{n!} \theta_j P_j \right) = \text{per} \left( \sum_{j=1}^{n!} \frac{1}{n!} P_j \right),$$

where the minimum is over all nonnegative  $(n!)$ -tuples  $\theta = (\theta_1, \dots, \theta_{n!})$  satisfying  $\sum_{j=1}^{n!} \theta_j = 1$ .

Since van der Waerden's conjecture is still unresolved, it is natural to ask whether

$$(3) \quad \min_{\omega} \text{per} \left( \sum_{j=1}^m \omega_j P_j \right) = \text{per} \left( \sum_{j=1}^m \frac{1}{m} P_j \right),$$

for a fixed set of permutation matrices  $\{P_1, \dots, P_m\}$ , where the minimum is over all nonnegative  $m$ -tuples  $\omega = (\omega_1, \dots, \omega_m)$  satisfying  $\sum_{j=1}^m \omega_j = 1$ .

In this paper we study circulants of the form  $\alpha I_n + \beta P + \gamma P^2$ , where  $I_n$  is the  $n \times n$  identity matrix and  $P$  is the full-cycle permutation matrix with 1's in the positions  $(1, 2), (2, 3), \dots, (n-1, n), (n, 1)$ . We obtain a recurrence formula and deduce explicit formulas for  $\text{per}(\alpha I_n + \beta P + \gamma P^2)$ . We then specialize to doubly stochastic circulants of the form  $\alpha I_n + \beta P + \gamma P^2$ , obtain bounds for the minimum value of the permanent of such circulants, and show that (3) does not hold for the set  $\{I_n, P, P^2\}$ ,  $n \geq 5$ .

The author is indebted to Dr. David London for drawing his attention to the fact that  $\text{per}((1/2)I_n + (1/2)P) < \text{per}((1/3)I_n + (1/3)P + (1/3)P^2)$ , for sufficiently large  $n$ .

2. Results. We begin with two formulas for the permanent of a tridiagonal matrix of the form

$$(4) \quad \begin{pmatrix} \beta & \gamma & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & \gamma & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \alpha & \beta & \gamma & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \alpha & \beta \end{pmatrix}.$$

Let  $F_n(\alpha, \beta, \gamma)$  denote the matrix (4) of order  $n$  and let the permanent of  $F_n(\alpha, \beta, \gamma)$  be denoted by  $f_n(\alpha, \beta, \gamma)$ , or simply by  $f_n$ . Set  $f_0 = 1$ ,  $f_1 = \beta$ , and  $f_2 = \beta^2 + \alpha\gamma$ .

LEMMA 1. *If  $n \geq 2$ , then*

$$(5) \quad f_n = \beta f_{n-1} + \alpha\gamma f_{n-2}.$$

COROLLARY. *If  $n \geq 1$  and  $\mu = \sqrt{\beta^2 + 4\alpha\gamma} \neq 0$ , then*

$$(6) \quad f_n = \frac{1}{\mu} r_1^{n+1} - \frac{1}{\mu} r_2^{n+1}$$

where  $r_1 = (\beta + \mu)/2$  and  $r_2 = (\beta - \mu)/2$ . *If  $\mu = 0$ , then*

$$(6') \quad f_n = (n + 1)(\beta/2)^n.$$

(In other words, if the right side of (6) is considered as a polynomial expression in  $\alpha, \beta, \gamma$ , then (6) holds even in the case  $\mu = 0$ .)

The lemma is proved easily by expanding the permanent of  $F_n(\alpha, \beta, \gamma)$  by the first column. Formula (6) is obtained by solving the difference equation (5) subject to initial conditions.

In the next lemma, formula (5) is used to obtain a relation between the permanent of the circulant  $\alpha I_n + \beta P + \gamma P^2$  and permanents of tridiagonal matrices of the form (4).

LEMMA 2. *If  $n \geq 3$ , then*

$$(7) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) = f_n + \alpha \gamma f_{n-2} + \alpha^n + \gamma^n .$$

*Proof.* A direct computation shows that the theorem holds for  $n = 3$ . Assume that  $n \geq 4$ . Denote the matrix  $\alpha I_n + \beta P + \gamma P^2$  by  $Q_n$ , and the submatrix of  $Q_n$  obtained by deleting rows  $i_1, i_2$  and columns  $j_1, j_2$  by  $Q_n(i_1, i_2 | j_1, j_2)$ . Expand the permanent of  $Q_n$  by the first two columns:

$$\begin{aligned} \text{per}(Q_n) &= \alpha^2 \text{per}(Q_n(1, 2 | 1, 2)) + \beta \gamma \text{per}(Q_n(1, n - 1 | 1, 2)) \\ &\quad + (\alpha \gamma + \beta^2) \text{per}(Q_n(1, n | 1, 2)) + \alpha \gamma \text{per}(Q_n(2, n - 1 | 1, 2)) \\ &\quad + \alpha \beta \text{per}(Q_n(2, n | 1, 2)) + \gamma^2 \text{per}(Q_n(n - 1, n | 1, 2)) \\ &= \alpha^n + \alpha \beta \gamma f_{n-3} + (\alpha \gamma + \beta^2) f_{n-2} + \alpha^2 \gamma^2 f_{n-4} + \alpha \beta \gamma f_{n-3} + \gamma^n \\ &= \beta f_{n-1} + \alpha \gamma f_{n-2} + \alpha \gamma (\beta f_{n-3} + \alpha \gamma f_{n-4}) + \alpha^n + \gamma^n \\ &= f_n + \alpha \gamma f_{n-2} + \alpha^n + \gamma^n . \end{aligned}$$

We now use the preceding result to obtain a recurrence formula for the permanent of  $\alpha I_n + \beta P + \gamma P^2$ , and then to deduce explicit formulas for these circulants.

**THEOREM 1.** *If  $Q_n = \alpha I_n + \beta P + \gamma P^2$  and  $n \geq 5$ , then*

$$(8) \quad \begin{aligned} \text{per}(Q_n) &= \beta \text{per}(Q_{n-1}) + \alpha \gamma \text{per}(Q_{n-2}) \\ &\quad + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) . \end{aligned}$$

*Proof.* We use (7) and (5) to transform the right-hand side of (8) as follows:

$$\begin{aligned} &\beta \text{per}(Q_{n-1}) + \alpha \gamma \text{per}(Q_{n-2}) + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \beta f_{n-1} + \beta \alpha \gamma f_{n-3} + \beta \alpha^{n-1} + \beta \gamma^{n-1} + \alpha \gamma f_{n-2} + \alpha^2 \gamma^2 f_{n-4} + \alpha^{n-1} \gamma \\ &\quad + \alpha \gamma^{n-1} + \alpha^n - \alpha^{n-1} \beta - \alpha^{n-1} \gamma + \gamma^n - \alpha \gamma^{n-1} - \beta \gamma^{n-1} \\ &= (\beta f_{n-1} + \alpha \gamma f_{n-2}) + \alpha \gamma (\beta f_{n-3} + \alpha \gamma f_{n-4}) + \alpha^n + \gamma^n \\ &= f_n + \alpha \gamma f_{n-2} + \alpha^n + \gamma^n \\ &= \text{per}(Q_n) . \end{aligned}$$

The difference equation (8) can now be solved subject to the conditions

$$\begin{aligned} \text{per}(Q_3) &= \alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta\gamma \\ \text{per}(Q_4) &= \alpha^4 + \beta^4 + \gamma^4 + 4\alpha\beta^2\gamma + 2\alpha^2\gamma^2 \\ \text{per}(Q_5) &= \alpha^5 + \beta^5 + \gamma^5 + 5\alpha\beta^3\gamma + 5\alpha^2\beta\gamma^2, \quad \text{etc.}, \end{aligned}$$

which are computed directly using a Laplace expansion. We obtain the following explicit formula.

**THEOREM 2.** *If  $n \geq 3$ , then*

$$(9) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) = r_1^n + r_2^n + \alpha^n + \gamma^n$$

where  $r_1$  and  $r_2$  are the roots of  $x^2 - \beta x - \alpha\gamma = 0$ .

Alternatively, formula (9) can be obtained from (7) and (6) if  $\mu \neq 0$ , or from (7) and (6') in case  $\mu = 0$ . Thus if  $\mu \neq 0$ :

$$\begin{aligned} \text{per}(\alpha I_n + \beta P + \gamma P^2) &= f_n + \alpha\gamma f_{n-2} + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}r_1^{n+1} - \frac{1}{\mu}r_2^{n+1} + \frac{\alpha\gamma}{\mu}r_1^{n-1} - \frac{\alpha\gamma}{\mu}r_2^{n-1} \\ &\quad + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}(r_1^{n+1} - r_2^{n+1} - r_1r_2(r_1^{n-1} - r_2^{n-1})) + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}(r_1^n + r_2^n)(r_1 - r_2) + \alpha^n + \gamma^n \\ &= r_1^n + r_2^n + \alpha^n + \gamma^n, \end{aligned}$$

since  $\alpha\gamma = -r_1r_2$  and  $\mu = r_1 - r_2$ . The case  $\mu = 0$  is proved similarly.

Formulas (8) and (9) have been obtained in [2] for the special case  $\alpha = \beta = \gamma$ .

**THEOREM 3.** *If  $n \geq 3$ , then*

$$(10) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) = \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{[n/2]} c_t^{(n)} \alpha^t \beta^{n-2t} \gamma^t$$

where  $c_t^{(n)} = 2^{-(n-2t-1)} \sum_{k=t}^{[n/2]} \binom{n}{2k} \binom{k}{t}$ .

*Proof.* Let  $r_1 = (\beta + \mu)/2$  and  $r_2 = (\beta - \mu)/2$ , where  $\mu = \sqrt{\beta^2 + 4\alpha\gamma}$ . Then by formula (9),

$$\begin{aligned} \text{per}(\alpha I_n + \beta P + \gamma P^2) &= \alpha^n + \gamma^n + \left(\frac{\beta + \mu}{2}\right)^n + \left(\frac{\beta - \mu}{2}\right)^n \\ &= \alpha^n + \gamma^n + 2^{-(n-1)} \sum_{k=0}^{[n/2]} \binom{n}{2k} \beta^{n-2k} (\beta^2 + 4\alpha\gamma)^k \\ &= \alpha^n + \gamma^n + 2^{-(n-1)} \sum_{k=0}^{[n/2]} \binom{n}{2k} \sum_{t=0}^k \binom{k}{t} \beta^{n-2t} (4\alpha\gamma)^t \\ &= \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{[n/2]} \left(\sum_{k=t}^{[n/2]} 2^{-(n-2t-1)} \binom{n}{2k} \binom{k}{t}\right) \alpha^t \beta^{n-2t} \gamma^t. \end{aligned}$$

The following alternative form of formula (10) can be proved by induction:

$$(11) \quad \begin{cases} \text{per}(\alpha I_n + \beta P + \gamma P^2) = \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{\lfloor n/2 \rfloor} c_t^{(n)} \alpha^t \beta^{n-2t} \gamma^t, \\ \text{where } c_1^{(n)} = n, c_{n/2}^{(n)} = 2 \text{ in case } n \text{ is even,} \\ \text{and } c_t^{(n)} = c_{t-1}^{(n-1)} + c_{t-1}^{(n-2)}, 1 < t < n/2. \end{cases}$$

The cases  $n = 3$  and  $4$  can be easily verified. If  $Q_n = \alpha I_n + \beta P + \gamma P^2$ ,  $n \geq 5$ , then by (8),

$$\begin{aligned} \text{per}(Q_n) &= \beta \text{per}(Q_{n-1}) + \alpha\gamma \text{per}(Q_{n-2}) + \alpha^{n-1}(\alpha - \beta - \gamma) \\ &\quad + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \alpha^{n-1}\beta + \beta^n + \beta\gamma^{n-1} + \sum_{t=1}^{\lfloor (n-1)/2 \rfloor} c_t^{(n-1)} \alpha^t \beta^{n-2t} \gamma^t \\ &\quad + \alpha^{n-1}\gamma + \alpha\beta^{n-2}\gamma + \alpha\gamma^{n-1} + \sum_{s=1}^{\lfloor n/2 \rfloor - 1} c_s^{(n-2)} \alpha^{s+1} \beta^{n-2s+2} \gamma^{s+1} \\ &\quad + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \alpha^n + \beta^n + \gamma^n + \alpha\beta^{n-2}\gamma + c_1^{(n-1)}\alpha\beta^{n-2}\gamma + \sum_{t=2}^{\lfloor (n-1)/2 \rfloor} c_t^{(n-1)} \alpha^t \beta^{n-2t} \gamma^t \\ &\quad + \sum_{t=2}^{\lfloor n/2 \rfloor} c_{t-1}^{(n-2)} \alpha^t \beta^{n-2t} \gamma^t \\ &= \begin{cases} \alpha^n + \beta^n + \gamma^n + (1 + c_1^{(n-1)})\alpha\beta^{n-2}\gamma + \sum_{t=2}^{\lfloor n/2 \rfloor} (c_t^{(n-1)} + c_{t-1}^{(n-2)})\alpha^t \beta^{n-2t} \gamma^t, & \text{if } n \text{ is odd,} \\ \alpha^n + \beta^n + \gamma^n + (1 + c_1^{(n-1)})\alpha\beta^{n-2}\gamma + \sum_{t=2}^{\lfloor n/2 \rfloor - 1} (c_t^{(n-1)} + c_{t-1}^{(n-2)})\alpha^t \beta^{n-2t} \gamma^t \\ \quad + 2\alpha^{n/2}\gamma^{n/2}, & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and formula (11) follows easily.

Formula (11) allows us to construct a table of coefficients  $c_t^{(n)}$  in the manner of Pascal's triangle.

$n$	$c_1^{(n)}$	$c_2^{(n)}$	$c_3^{(n)}$	$c_4^{(n)}$	$c_5^{(n)}$	$c_6^{(n)}$
3	3					
4	4	2				
5	5	5				
6	6	9	2			
7	7	14	7			
8	8	20	16	2		
9	9	27	30	9		
10	10	35	50	25	2	
11	11	44	77	55	11	
12	12	54	112	105	36	2

In the remainder of this paper we assume that  $\alpha I_n + \beta P + \gamma P^2$  is doubly stochastic, i.e., that  $\alpha, \beta, \gamma$  are nonnegative and  $\alpha + \beta + \gamma = 1$ .

**THEOREM 4.** *If  $\alpha, \beta, \gamma$  are nonnegative then*

$$(12) \quad \frac{1}{2^n} < \min_{\alpha+\beta+\gamma=1} (\text{per}(\alpha I_n + \beta P + \gamma P^2)) \leq \frac{1}{2^{n-1}}.$$

*Proof.* The right inequality in (12) follows immediately from the fact that

$$\text{per}\left(\frac{1}{2}I_n + \frac{1}{2}P\right) = \frac{1}{2^{n-1}}.$$

We prove the left inequality by showing that

$$(13) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) > \frac{1}{2^n}$$

for any nonnegative  $\alpha, \beta, \gamma$  satisfying  $\alpha + \beta + \gamma = 1$ . If any of  $\alpha, \beta, \gamma$  exceeds  $1/2$  then (13) clearly holds, since by (10)

$$\text{per}(\alpha I_n + \beta P + \gamma P^2) \geq \alpha^n + \beta^n + \gamma^n.$$

Suppose that

$$(14) \quad 0 \leq \alpha \leq \frac{1}{2}, 0 \leq \beta \leq \frac{1}{2}, 0 \leq \gamma \leq \frac{1}{2}, \alpha + \beta + \gamma = 1.$$

We assume, without loss of generality, that  $\alpha \geq \gamma$ , and assert that under these conditions

$$(15) \quad r_1 \geq \frac{1}{2} \quad \text{and} \quad |r_2| \leq \alpha$$

where  $r_1 = (1/2)(\beta + \sqrt{\beta^2 + 4\alpha\gamma})$  and  $r_2 = (1/2)(\beta - \sqrt{\beta^2 + 4\alpha\gamma})$ . We use the method of Lagrange's multipliers to determine the stationary points of the function  $r_1 = r_1(\alpha, \beta, \gamma)$ . Let

$$F(\alpha, \beta, \gamma) = \frac{1}{2}(\beta + \sqrt{\beta^2 + 4\alpha\gamma}) + \lambda(\alpha + \beta + \gamma - 1).$$

The necessary conditions for a stationary point are

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= \frac{\gamma}{\mu} + \lambda = 0, \\ \frac{\partial F}{\partial \beta} &= \frac{1}{2}\left(1 + \frac{\beta}{\mu}\right) + \lambda = \frac{r_1}{\mu} + \lambda = 0, \end{aligned}$$

$$\frac{\partial F}{\partial \gamma} = \frac{\alpha}{\mu} + \lambda = 0 .$$

Where  $\mu = \sqrt{\beta^2 + 4\alpha\gamma}$ , i.e., we must have  $\alpha = \gamma = r_1$ . But then

$$2\alpha = \beta + \sqrt{\beta^2 + 4\alpha^2} ,$$

i.e.,

$$4\alpha^2 - 4\alpha\beta + \beta^2 = \beta^2 + 4\alpha^2 ,$$

which implies that either  $\beta = 0$  and  $\alpha = \gamma = 1/2$ , or  $\alpha = \gamma = 0$  and  $\beta = 1$ . In any case the function  $r_1(\alpha, \beta, \gamma)$  has no minimum in the interior of region (14). It is easy to verify that its minimum value on the boundary is  $1/2$ .

We proceed to the second inequality in (15). Suppose that  $|r_2| > \alpha$ , i.e., that

$$\sqrt{\beta^2 + 4\alpha\gamma} - \beta > 2\alpha ,$$

or

$$(16) \quad \beta^2 + 4\alpha\gamma > \beta^2 + 4\alpha\beta + 4\alpha^2 .$$

Now  $\alpha$  cannot be 0, since  $\alpha \geq \gamma$  and  $\beta \leq 1/2$ . Hence (16) implies that

$$\gamma > \alpha + \beta ,$$

i.e.,

$$\gamma > \frac{1}{2} ,$$

which contradicts (14). Therefore the inequalities (15) hold. Thus for any  $\alpha, \beta, \gamma$  satisfying (14) we have

$$\begin{aligned} \text{per}(\alpha I_n + \beta P + \gamma P^2) &= r_1^n + r_2^n + \alpha^n + \gamma^n \\ &\geq r_1^n + \gamma^n + (\alpha^n - |r_2|^n) \\ &> r_1^n \\ &\geq \frac{1}{2^n} . \end{aligned}$$

**THEOREM 5.** *If  $\alpha, \beta, \gamma$  are nonnegative numbers,  $n \geq 5$ , then*

$$(17) \quad \min_{\alpha+\beta+\gamma=1} (\text{per}(\alpha I_n + \beta P + \gamma P^2)) < \text{per}\left(\frac{1}{3}I_n + \frac{1}{3}P + \frac{1}{3}P^2\right) .$$

*In other words, the minimum of the permanent function on the convex hull of  $I_n, P, P^2$ ,  $n \geq 5$ , is not attained for  $\alpha = \beta = \gamma = 1/3$ .*



*Proof.* By Theorem 4,

$$\min_{\alpha+\beta+\gamma=1} (\text{per} (\alpha I_n + \beta P + \gamma P^2)) \leq \frac{1}{2^{n-1}}.$$

From (9) we compute

$$\begin{aligned} \text{per} \left( \frac{1}{3} I_n + \frac{1}{3} P + \frac{1}{3} P^2 \right) &= \left( \frac{1 + \sqrt{5}}{6} \right)^n + \left( \frac{1 - \sqrt{5}}{6} \right)^n + \frac{1}{3^n} + \frac{1}{3^n} \\ &> \left( \frac{1 + \sqrt{5}}{6} \right)^n + \frac{1}{3^n}, \end{aligned}$$

which is greater than  $1/2^{n-1}$  for  $n \geq 10$ . It can be checked by computation, that (17) holds for  $5 \leq n \leq 9$  as well.

An explicit formula for  $\min_{\alpha+\beta+\gamma=1} (\text{per} (\alpha I_n + \beta P + \gamma P^2))$ ,  $\alpha, \beta, \gamma \geq 0$ , appears to be out of reach. The available numerical data for  $n \leq 18$  seem to indicate that the values of  $\alpha, \beta, \gamma$ , at which the minimum is attained are the same for  $n = 2k - 1$  and  $n = 2k$ , for any  $k$ , but that otherwise they vary with  $n$ .

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