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ON PERMANENTS OF CIRCULANTS

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HENRYK MINC

A recurrence formula is obtained for permanents of circulants of the form $\alpha I_n + \beta P + \gamma P^2$ and explicit formulas are deduced from it. It is shown that for doubly stochastic circulants $\alpha I_n + \beta P + \gamma P^2$ the minimum permanent lies in the interval $(1/2^n, 1/2^{n-1}]$.

1. Introduction. The well-known unresolved conjecture of van der Waerden asserts that in Ω_n , the polyhedron of doubly stochastic $n \times n$ matrices, the permanent function takes its minimum value for the matrix J_n , all of whose entries are $1/n$, i.e.,

$$(1) \quad \min_{A \in \Omega_n} \text{per}(A) = \text{per}(J_n).$$

By a theorem of Birkhoff, Ω_n is a convex polyhedron with the permutation matrices $P_1, \dots, P_{n!}$ as vertices. Thus (1) can be written in the form

$$(2) \quad \min_{\theta} \text{per} \left(\sum_{j=1}^{n!} \theta_j P_j \right) = \text{per} \left(\sum_{j=1}^{n!} \frac{1}{n!} P_j \right),$$

where the minimum is over all nonnegative $(n!)$ -tuples $\theta = (\theta_1, \dots, \theta_{n!})$ satisfying $\sum_{j=1}^{n!} \theta_j = 1$.

Since van der Waerden's conjecture is still unresolved, it is natural to ask whether

$$(3) \quad \min_{\omega} \text{per} \left(\sum_{j=1}^m \omega_j P_j \right) = \text{per} \left(\sum_{j=1}^m \frac{1}{m} P_j \right),$$

for a fixed set of permutation matrices $\{P_1, \dots, P_m\}$, where the minimum is over all nonnegative m -tuples $\omega = (\omega_1, \dots, \omega_m)$ satisfying $\sum_{j=1}^m \omega_j = 1$.

In this paper we study circulants of the form $\alpha I_n + \beta P + \gamma P^2$, where I_n is the $n \times n$ identity matrix and P is the full-cycle permutation matrix with 1's in the positions $(1, 2), (2, 3), \dots, (n-1, n), (n, 1)$. We obtain a recurrence formula and deduce explicit formulas for $\text{per}(\alpha I_n + \beta P + \gamma P^2)$. We then specialize to doubly stochastic circulants of the form $\alpha I_n + \beta P + \gamma P^2$, obtain bounds for the minimum value of the permanent of such circulants, and show that (3) does not hold for the set $\{I_n, P, P^2\}$, $n \geq 5$.

The author is indebted to Dr. David London for drawing his attention to the fact that $\text{per}((1/2)I_n + (1/2)P) < \text{per}((1/3)I_n + (1/3)P + (1/3)P^2)$, for sufficiently large n .

2. Results. We begin with two formulas for the permanent of a tridiagonal matrix of the form

$$(4) \quad \begin{bmatrix} \beta & \gamma & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \alpha & \beta & \gamma & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & \gamma & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \alpha & \beta & \gamma & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \alpha & \beta \end{bmatrix} \cdot$$

Let $F_n(\alpha, \beta, \gamma)$ denote the matrix (4) of order n and let the permanent of $F_n(\alpha, \beta, \gamma)$ be denoted by $f_n(\alpha, \beta, \gamma)$, or simply by f_n . Set $f_0 = 1$, $f_1 = \beta$, and $f_2 = \beta^2 + \alpha\gamma$.

LEMMA 1. *If $n \geq 2$, then*

$$(5) \quad f_n = \beta f_{n-1} + \alpha\gamma f_{n-2}.$$

COROLLARY. *If $n \geq 1$ and $\mu = \sqrt{\beta^2 + 4\alpha\gamma} \neq 0$, then*

$$(6) \quad f_n = \frac{1}{\mu} r_1^{n+1} - \frac{1}{\mu} r_2^{n+1}$$

where $r_1 = (\beta + \mu)/2$ and $r_2 = (\beta - \mu)/2$. *If $\mu = 0$, then*

$$(6') \quad f_n = (n + 1)(\beta/2)^n.$$

(In other words, if the right side of (6) is considered as a polynomial expression in α, β, γ , then (6) holds even in the case $\mu = 0$.)

The lemma is proved easily by expanding the permanent of $F_n(\alpha, \beta, \gamma)$ by the first column. Formula (6) is obtained by solving the difference equation (5) subject to initial conditions.

In the next lemma, formula (5) is used to obtain a relation between the permanent of the circulant $\alpha I_n + \beta P + \gamma P^2$ and permanents of tridiagonal matrices of the form (4).

LEMMA 2. *If $n \geq 3$, then*

$$(7) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) = f_n + \alpha \gamma f_{n-2} + \alpha^n + \gamma^n.$$

Proof. A direct computation shows that the theorem holds for $n = 3$. Assume that $n \geq 4$. Denote the matrix $\alpha I_n + \beta P + \gamma P^2$ by Q_n , and the submatrix of Q_n obtained by deleting rows i_1, i_2 and columns j_1, j_2 by $Q_n(i_1, i_2 | j_1, j_2)$. Expand the permanent of Q_n by the first two columns:

$$\begin{aligned} \text{per}(Q_n) &= \alpha^2 \text{per}(Q_n(1, 2 | 1, 2)) + \beta \gamma \text{per}(Q_n(1, n-1 | 1, 2)) \\ &\quad + (\alpha \gamma + \beta^2) \text{per}(Q_n(1, n | 1, 2)) + \alpha \gamma \text{per}(Q_n(2, n-1 | 1, 2)) \\ &\quad + \alpha \beta \text{per}(Q_n(2, n | 1, 2)) + \gamma^2 \text{per}(Q_n(n-1, n | 1, 2)) \\ &= \alpha^n + \alpha \beta \gamma f_{n-3} + (\alpha \gamma + \beta^2) f_{n-2} + \alpha^2 \gamma^2 f_{n-4} + \alpha \beta \gamma f_{n-3} + \gamma^n \\ &= \beta f_{n-1} + \alpha \gamma f_{n-2} + \alpha \gamma (\beta f_{n-3} + \alpha \gamma f_{n-4}) + \alpha^n + \gamma^n \\ &= f_n + \alpha \gamma f_{n-2} + \alpha^n + \gamma^n. \end{aligned}$$

We now use the preceding result to obtain a recurrence formula for the permanent of $\alpha I_n + \beta P + \gamma P^2$, and then to deduce explicit formulas for these circulants.

THEOREM 1. *If $Q_n = \alpha I_n + \beta P + \gamma P^2$ and $n \geq 5$, then*

$$(8) \quad \begin{aligned} \text{per}(Q_n) &= \beta \text{per}(Q_{n-1}) + \alpha \gamma \text{per}(Q_{n-2}) \\ &\quad + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta). \end{aligned}$$

Proof. We use (7) and (5) to transform the right-hand side of (8) as follows:

$$\begin{aligned} &\beta \text{per}(Q_{n-1}) + \alpha \gamma \text{per}(Q_{n-2}) + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \beta f_{n-1} + \beta \alpha \gamma f_{n-3} + \beta \alpha^{n-1} + \beta \gamma^{n-1} + \alpha \gamma f_{n-2} + \alpha^2 \gamma^2 f_{n-4} + \alpha^{n-1} \gamma \\ &\quad + \alpha \gamma^{n-1} + \alpha^n - \alpha^{n-1} \beta - \alpha^{n-1} \gamma + \gamma^n - \alpha \gamma^{n-1} - \beta \gamma^{n-1} \\ &= (\beta f_{n-1} + \alpha \gamma f_{n-2}) + \alpha \gamma (\beta f_{n-3} + \alpha \gamma f_{n-4}) + \alpha^n + \gamma^n \\ &= f_n + \alpha \gamma f_{n-2} + \alpha^n + \gamma^n \\ &= \text{per}(Q_n). \end{aligned}$$

The difference equation (8) can now be solved subject to the conditions

$$\begin{aligned} \text{per}(Q_3) &= \alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta\gamma \\ \text{per}(Q_4) &= \alpha^4 + \beta^4 + \gamma^4 + 4\alpha\beta^2\gamma + 2\alpha^2\gamma^2 \\ \text{per}(Q_5) &= \alpha^5 + \beta^5 + \gamma^5 + 5\alpha\beta^3\gamma + 5\alpha^2\beta\gamma^2, \quad \text{etc.}, \end{aligned}$$

which are computed directly using a Laplace expansion. We obtain the following explicit formula.

THEOREM 2. *If $n \geq 3$, then*

$$(9) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) = r_1^n + r_2^n + \alpha^n + \gamma^n$$

where r_1 and r_2 are the roots of $x^2 - \beta x - \alpha\gamma = 0$.

Alternatively, formula (9) can be obtained from (7) and (6) if $\mu \neq 0$, or from (7) and (6') in case $\mu = 0$. Thus if $\mu \neq 0$:

$$\begin{aligned} \text{per}(\alpha I_n + \beta P + \gamma P^2) &= f_n + \alpha\gamma f_{n-2} + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}r_1^{n+1} - \frac{1}{\mu}r_2^{n+1} + \frac{\alpha\gamma}{\mu}r_1^{n-1} - \frac{\alpha\gamma}{\mu}r_2^{n-1} \\ &\quad + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}(r_1^{n+1} - r_2^{n+1} - r_1r_2(r_1^{n-1} - r_2^{n-1})) + \alpha^n + \gamma^n \\ &= \frac{1}{\mu}(r_1^n + r_2^n)(r_1 - r_2) + \alpha^n + \gamma^n \\ &= r_1^n + r_2^n + \alpha^n + \gamma^n, \end{aligned}$$

since $\alpha\gamma = -r_1r_2$ and $\mu = r_1 - r_2$. The case $\mu = 0$ is proved similarly.

Formulas (8) and (9) have been obtained in [2] for the special case $\alpha = \beta = \gamma$.

THEOREM 3. *If $n \geq 3$, then*

$$(10) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) = \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{\lfloor n/2 \rfloor} c_t^{(n)} \alpha^t \beta^{n-2t} \gamma^t$$

where $c_t^{(n)} = 2^{-(n-2t-1)} \sum_{k=t}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{k}{t}$.

Proof. Let $r_1 = (\beta + \mu)/2$ and $r_2 = (\beta - \mu)/2$, where $\mu = \sqrt{\beta^2 + 4\alpha\gamma}$. Then by formula (9),

$$\begin{aligned} \text{per}(\alpha I_n + \beta P + \gamma P^2) &= \alpha^n + \gamma^n + \left(\frac{\beta + \mu}{2}\right)^n + \left(\frac{\beta - \mu}{2}\right)^n \\ &= \alpha^n + \gamma^n + 2^{-(n-1)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \beta^{n-2k} (\beta^2 + 4\alpha\gamma)^k \\ &= \alpha^n + \gamma^n + 2^{-(n-1)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \sum_{t=0}^k \binom{k}{t} \beta^{n-2t} (4\alpha\gamma)^t \\ &= \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{\lfloor n/2 \rfloor} \left(\sum_{k=t}^{\lfloor n/2 \rfloor} 2^{-(n-2t-1)} \binom{n}{2k} \binom{k}{t} \right) \alpha^t \beta^{n-2t} \gamma^t. \end{aligned}$$

The following alternative form of formula (10) can be proved by induction:

$$(11) \quad \left\{ \begin{array}{l} \text{per} (\alpha I_n + \beta P + \gamma P^2) = \alpha^n + \beta^n + \gamma^n + \sum_{t=1}^{[n/2]} c_t^{(n)} \alpha^t \beta^{n-2t} \gamma^t, \\ \text{where } c_1^{(n)} = n, c_{n/2}^{(n)} = 2 \text{ in case } n \text{ is even,} \\ \text{and } c_t^{(n)} = c_t^{(n-1)} + c_{t-1}^{(n-2)}, 1 < t < n/2. \end{array} \right.$$

The cases $n = 3$ and 4 can be easily verified. If $Q_n = \alpha I_n + \beta P + \gamma P^2$, $n \geq 5$, then by (8),

$$\begin{aligned} \text{per} (Q_n) &= \beta \text{per} (Q_{n-1}) + \alpha \gamma \text{per} (Q_{n-2}) + \alpha^{n-1}(\alpha - \beta - \gamma) \\ &\quad + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \alpha^{n-1} \beta + \beta^n + \beta \gamma^{n-1} + \sum_{t=1}^{[(n-1)/2]} c_t^{(n-1)} \alpha^t \beta^{n-2t} \gamma^t \\ &\quad + \alpha^{n-1} \gamma + \alpha \beta^{n-2} \gamma + \alpha \gamma^{n-1} + \sum_{s=1}^{[n/2]-1} c_s^{(n-2)} \alpha^{s+1} \beta^{n-2s+2} \gamma^{s+1} \\ &\quad + \alpha^{n-1}(\alpha - \beta - \gamma) + \gamma^{n-1}(\gamma - \alpha - \beta) \\ &= \alpha^n + \beta^n + \gamma^n + \alpha \beta^{n-2} \gamma + c_1^{(n-1)} \alpha \beta^{n-2} \gamma + \sum_{t=2}^{[(n-1)/2]} c_t^{(n-1)} \alpha^t \beta^{n-2t} \gamma^t \\ &\quad + \sum_{t=2}^{[n/2]} c_{t-1}^{(n-2)} \alpha^t \beta^{n-2t} \gamma^t \\ &= \begin{cases} \alpha^n + \beta^n + \gamma^n + (1 + c_1^{(n-1)}) \alpha \beta^{n-2} \gamma + \sum_{t=2}^{[n/2]} (c_t^{(n-1)} + c_{t-1}^{(n-2)}) \alpha^t \beta^{n-2t} \gamma^t, & \text{if } n \text{ is odd,} \\ \alpha^n + \beta^n + \gamma^n + (1 + c_1^{(n-1)}) \alpha \beta^{n-2} \gamma + \sum_{t=2}^{[n/2]-1} (c_t^{(n-1)} + c_{t-1}^{(n-2)}) \alpha^t \beta^{n-2t} \gamma^t \\ \quad + 2\alpha^{n/2} \gamma^{n/2}, & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and formula (11) follows easily.

Formula (11) allows us to construct a table of coefficients $c_t^{(n)}$ in the manner of Pascal's triangle.

n	$c_1^{(n)}$	$c_2^{(n)}$	$c_3^{(n)}$	$c_4^{(n)}$	$c_5^{(n)}$	$c_6^{(n)}$
3	3					
4	4	2				
5	5	5				
6	6	9	2			
7	7	14	7			
8	8	20	16	2		
9	9	27	30	9		
10	10	35	50	25	2	
11	11	44	77	55	11	
12	12	54	112	105	36	2

In the remainder of this paper we assume that $\alpha I_n + \beta P + \gamma P^2$ is doubly stochastic, i.e., that α, β, γ are nonnegative and $\alpha + \beta + \gamma = 1$.

THEOREM 4. *If α, β, γ are nonnegative then*

$$(12) \quad \frac{1}{2^n} < \min_{\alpha+\beta+\gamma=1} (\text{per}(\alpha I_n + \beta P + \gamma P^2)) \leq \frac{1}{2^{n-1}}.$$

Proof. The right inequality in (12) follows immediately from the fact that

$$\text{per}\left(\frac{1}{2}I_n + \frac{1}{2}P\right) = \frac{1}{2^{n-1}}.$$

We prove the left inequality by showing that

$$(13) \quad \text{per}(\alpha I_n + \beta P + \gamma P^2) > \frac{1}{2^n}$$

for any nonnegative α, β, γ satisfying $\alpha + \beta + \gamma = 1$. If any of α, β, γ exceeds $1/2$ then (13) clearly holds, since by (10)

$$\text{per}(\alpha I_n + \beta P + \gamma P^2) \geq \alpha^n + \beta^n + \gamma^n.$$

Suppose that

$$(14) \quad 0 \leq \alpha \leq \frac{1}{2}, 0 \leq \beta \leq \frac{1}{2}, 0 \leq \gamma \leq \frac{1}{2}, \alpha + \beta + \gamma = 1.$$

We assume, without loss of generality, that $\alpha \geq \gamma$, and assert that under these conditions

$$(15) \quad r_1 \geq \frac{1}{2} \quad \text{and} \quad |r_2| \leq \alpha$$

where $r_1 = (1/2)(\beta + \sqrt{\beta^2 + 4\alpha\gamma})$ and $r_2 = (1/2)(\beta - \sqrt{\beta^2 + 4\alpha\gamma})$. We use the method of Lagrange's multipliers to determine the stationary points of the function $r_1 = r_1(\alpha, \beta, \gamma)$. Let

$$F(\alpha, \beta, \gamma) = \frac{1}{2}(\beta + \sqrt{\beta^2 + 4\alpha\gamma}) + \lambda(\alpha + \beta + \gamma - 1).$$

The necessary conditions for a stationary point are

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= \frac{\gamma}{\mu} + \lambda = 0, \\ \frac{\partial F}{\partial \beta} &= \frac{1}{2}\left(1 + \frac{\beta}{\mu}\right) + \lambda = \frac{r_1}{\mu} + \lambda = 0, \end{aligned}$$

$$\frac{\partial F}{\partial \gamma} = \frac{\alpha}{\mu} + \lambda = 0 .$$

Where $\mu = \sqrt{\beta^2 + 4\alpha\gamma}$, i.e., we must have $\alpha = \gamma = r_1$. But then

$$2\alpha = \beta + \sqrt{\beta^2 + 4\alpha^2} ,$$

i.e.,

$$4\alpha^2 - 4\alpha\beta + \beta^2 = \beta^2 + 4\alpha^2 ,$$

which implies that either $\beta = 0$ and $\alpha = \gamma = 1/2$, or $\alpha = \gamma = 0$ and $\beta = 1$. In any case the function $r_1(\alpha, \beta, \gamma)$ has no minimum in the interior of region (14). It is easy to verify that its minimum value on the boundary is $1/2$.

We proceed to the second inequality in (15). Suppose that $|r_2| > \alpha$, i.e., that

$$\sqrt{\beta^2 + 4\alpha\gamma} - \beta > 2\alpha ,$$

or

$$(16) \quad \beta^2 + 4\alpha\gamma > \beta^2 + 4\alpha\beta + 4\alpha^2 .$$

Now α cannot be 0, since $\alpha \geq \gamma$ and $\beta \leq 1/2$. Hence (16) implies that

$$\gamma > \alpha + \beta ,$$

i.e.,

$$\gamma > \frac{1}{2} ,$$

which contradicts (14). Therefore the inequalities (15) hold. Thus for any α, β, γ satisfying (14) we have

$$\begin{aligned} \text{per} (\alpha I_n + \beta P + \gamma P^2) &= r_1^n + r_2^n + \alpha^n + \gamma^n \\ &\geq r_1^n + \gamma^n + (\alpha^n - |r_2|^n) \\ &> r_1^n \\ &\geq \frac{1}{2^n} . \end{aligned}$$

THEOREM 5. *If α, β, γ are nonnegative numbers, $n \geq 5$, then*

$$(17) \quad \min_{\alpha+\beta+\gamma=1} (\text{per} (\alpha I_n + \beta P + \gamma P^2)) < \text{per} \left(\frac{1}{3} I_n + \frac{1}{3} P + \frac{1}{3} P^2 \right) .$$

In other words, the minimum of the permanent function on the convex hull of $I_n, P, P^2, n \geq 5$, is not attained for $\alpha = \beta = \gamma = 1/3$.

Proof. By Theorem 4,

$$\min_{\alpha+\beta+\gamma=1} (\text{per} (\alpha I_n + \beta P + \gamma P^2)) \leq \frac{1}{2^{n-1}}.$$

From (9) we compute

$$\begin{aligned} \text{per} \left(\frac{1}{3} I_n + \frac{1}{3} P + \frac{1}{3} P^2 \right) &= \left(\frac{1 + \sqrt{5}}{6} \right)^n + \left(\frac{1 - \sqrt{5}}{6} \right)^n + \frac{1}{3^n} + \frac{1}{3^n} \\ &> \left(\frac{1 + \sqrt{5}}{6} \right)^n + \frac{1}{3^n}, \end{aligned}$$

which is greater than $1/2^{n-1}$ for $n \geq 10$. It can be checked by computation, that (17) holds for $5 \leq n \leq 9$ as well.

An explicit formula for $\min_{\alpha+\beta+\gamma=1} (\text{per} (\alpha I_n + \beta P + \gamma P^2))$, $\alpha, \beta, \gamma \geq 0$, appears to be out of reach. The available numerical data for $n \leq 18$ seem to indicate that the values of α, β, γ , at which the minimum is attained are the same for $n = 2k - 1$ and $n = 2k$, for any k , but that otherwise they vary with n .

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Stephen Richard Bernfeld, <i>The extendability of solutions of perturbed scalar differential equations</i>	277
James Edwin Brink, <i>Inequalities involving f_p and $f^{(n)}_q$ for f with n zeros</i>	289
Orrin Frink and Robert S. Smith, <i>On the distributivity of the lattice of filters of a groupoid</i>	313
Donald Goldsmith, <i>On the density of certain cohesive basic sequences</i>	323
Charles Lemuel Hagopian, <i>Planar images of decomposable continua</i>	329
W. N. Hudson, <i>A decomposition theorem for biadditive processes</i>	333
W. N. Hudson, <i>Continuity of sample functions of biadditive processes</i>	343
Masako Izumi and Shin-ichi Izumi, <i>Integrability of trigonometric series. II</i>	359
H. M. Ko, <i>Fixed point theorems for point-to-set mappings and the set of fixed points</i>	369
Gregers Louis Krabbe, <i>An algebra of generalized functions on an open interval: two-sided operational calculus</i>	381
Thomas Latimer Kriete, III, <i>Complete non-selfadjointness of almost selfadjoint operators</i>	413
Shiva Narain Lal and Siya Ram, <i>On the absolute Hausdorff summability of a Fourier series</i>	439
Ronald Leslie Lipsman, <i>Representation theory of almost connected groups</i>	453
James R. McLaughlin, <i>Integrated orthonormal series</i>	469
H. Minc, <i>On permanents of circulants</i>	477
Akihiro Okuyama, <i>On a generalization of Σ-spaces</i>	485
Norberto Salinas, <i>Invariant subspaces and operators of class (S)</i>	497
James D. Stafney, <i>The spectrum of certain lower triangular matrices as operators on the l_p spaces</i>	515
Arne Stray, <i>Interpolation by analytic functions</i>	527
Li Pi Su, <i>Rings of analytic functions on any subset of the complex plane</i>	535
R. J. Tondra, <i>A property of manifolds compactly equivalent to compact manifolds</i>	539