Pacific Journal of Mathematics

ON A GENERALIZATION OF Σ-SPACES

AKIHIRO OKUYAMA

Vol. 42, No. 2 February 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 2, 1972

ON A GENERALIZATION OF Σ-SPACES

AKIHIRO OKUYAMA

In order to simultaneously generalize the class of M-spaces and σ-spaces, K. Nagami introduced Σ-spaces. Subsequently, E. Michael defined a class of $\Sigma^\#$-spaces. In this paper we will discuss the class of Σ^*-spaces which lies between Σ-spaces and $\Sigma^\#$-spaces and which contains all images of Σ-spaces under closed continuous maps.

1. Introduction. Recently K. Nagami [6] has investigated a new class of spaces, called Σ-spaces, containing two different classes of generalized metric spaces; i.e. the class of M-spaces (cf. [4]) as well as the class of σ-spaces (cf. [5], [7]).

If $\mathscr{K}$ is a cover of a space X, then a cover $\mathscr{A}$ is called a *(mod $\mathscr{K}$)-network* for X if, whenever $K \subset U$ with $K \in \mathscr{K}$ and U open in X, then $K \subset A \subset U$ for some $A \in \mathscr{A}$. According to K. Nagami [6], X is a *Σ-space* if it has a σ-locally finite closed (mod $\mathscr{K}$)-network for some cover $\mathscr{K}$ of X by countably compact sets.

E. Michael [2] has pointed out that the image of a paracompact, T_2 Σ-space under a closed continuous map need not be a Σ-space and also that replacing "σ-locally finite" by "σ-closure-preserving" in the definition of a Σ-space leads to a strictly larger class of spaces, which are called *$\Sigma^\#$-spaces*.

We say that a space X is a *Σ^*-space* if it satisfies the definition of a Σ-space with "σ-locally finite" weakened to "σ-hereditarily closure-preserving", where we say that a collection $\mathscr{A} = \{A_\lambda : \lambda \in \Lambda\}$ is *hereditarily closure-preserving* if any collection $\{B_\lambda : \lambda \in \Lambda\}$ with $B_\lambda \subset A_\lambda$ is closure-preserving (cf. [3]).

Clearly, every Σ-space is a Σ^*-space and every Σ^*-space is a $\Sigma^\#$-space. Since the image of a locally finite closed cover of the domain under a closed continuous onto map is a hereditarily closure-preserving closed cover of the range, we can easily see that the image of a Σ-space by a closed continuous map is always a Σ^*-space. As a matter of fact, E. Michael [2] has pointed out that a paracompact, T_2 Σ^*-space need not be a Σ-space, in general. Hence this fact arouses our interest in studying Σ^*-spaces comparing with Σ-spaces as well as $\Sigma^\#$-spaces.

In this paper we will investigate some relationship between above spaces and obtain the following results:

(A) Any image of a Σ^*-space under a closed continuous map is a Σ^*-space.

(B) Any inverse image of a $\Sigma^\#$-space by a perfect map (i.e. a

closed continuous map whose fibre at each point is compact) is a $\Sigma^{\sharp}$-space, while this is not true for a Σ^*-space.

(C) Every Lindelöf, T_2, Σ^*-space is a Σ-space, while this is not true for a $\Sigma^{\sharp}$-space.

(D) A Σ^*-space X is a Σ-space if every open set of X is an F_σ.

(E) For a paracompact, T_2 space X the following conditions are equivalent:

(1) X is a Σ-space.

(2) $X \times I$ is a Σ-space, where I denotes the unit closed interval with usual topology.

(3) $X \times I$ is a Σ^*-space.

According to the first half of (B), the product of a $\Sigma^{\sharp}$-space with I is a $\Sigma^{\sharp}$-space. On the other hand, as noted above there exists a paracompact, T_2, Σ^*-, non Σ-space. Hence statement (E) shows that the product of a paracompact, T_2, Σ^*-, non Σ-space X with I is a $\Sigma^{\sharp}$-, non Σ^*-space. Since the projection from $X \times I$ to I is perfect, this is an example for the later half of (B). Also, this shows that the class of $\Sigma^{\sharp}$-spaces is strictly larger than the class of Σ^*-spaces.

Concerning (D), it raises the following question:

Is (D) true for $\Sigma^{\sharp}$-spaces?

§2 is concerned with hereditarily closure-preserving closed covers of a countably compact, T_2 space, a Lindelöf, T_2 space and a T_2 space whose open sets are F_σ's. As an immediate consequence of 2.1 and 2.3 we have the simple facts that every hereditarily closure-preserving closed cover of a countably compact, T_2 space (resp. a Lindelöf, T_2 space) has a finite (resp. a countable) subcover. In §3 we will prove main results.

We will use the following notations in §2 and §3:

For a cover $\mathscr{F}$ of a space X and a point x of X we put

$$C(x, \mathscr{F}) = \cap \{F: x \in F \in \mathscr{F}\} ,$$

and for a sequence $\{\mathscr{F}_n: n = 1, 2, \cdots\}$ of covers of X and a point x of X we put

$$C(x) = \bigcap_{n=1}^{\infty} C(x, \mathscr{F}_n) .$$

Throughout this paper we assume that all spaces are T_2 and all maps are continuous.

2. Some properties of a hereditarily closure-preserving closed cover.

THEOREM 2.1. *Let $\mathscr{F} = \{F_\lambda: \lambda \in \Lambda\}$ be a hereditarily closure-preserving closed cover of a space X and C a countably compact set*

of X. Then $\mathscr{F}$ is locally finite at almost all points of C; i.e. *there exist $x_1, \cdots, x_n$ in C such that $\mathscr{F}$ is locally finite at any $x \in C - \{x_1, \cdots, x_n\}$, and only finitely many members of $\mathscr{F}$ meet $C - \{x_1, \cdots, x_n\}$.*

Proof. On the contrary, suppose $\mathscr{F}$ is not locally finite at infinitely many points of C. Since any closure-preserving, point-finite collection of closed sets is locally finite, $\mathscr{F}$ is not point-finite at infinitely many points of C. Then we can choose, step by step, countably many points $x_1, x_2 \cdots$ in C and countably many $\lambda_1, \lambda_2, \cdots$ in Λ such that $x_n \in F_{\lambda_n}$ for $n = 1, 2, \cdots$. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{x_1, x_2, \cdots\}$ must be discrete in X. On the other hand, since C is countably compact, $\{x_1, x_2, \cdots\}$ must have a cluster point in C. This is a contradiction. Hence $\mathscr{F}$ is locally finite at all points of C but finitely many points $x_1, \cdots, x_n$.

To complete the proof of 2.1, assume that $D = C - \{x_1, \cdots, x_n\}$ is infinite. If infinitely many members of $\mathscr{F}$ meet D, then we can again obtain a sequence $\{p_1, p_2, \cdots\}$ in D and a sequence $\{F_{\lambda_1}, F_{\lambda_2}, \cdots\}$ in $\mathscr{F}$ with $p_i \in F_{\lambda_i}$ for $i = 1, 2, \cdots$ by noting that $\mathscr{F}$ is point-finite at any point of D. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{p_1, p_2, \cdots\}$ must be discrete in X, therefore, in C, which is a contradiction. Hence only finitely many members of $\mathscr{F}$ meet D. This completes the proof.

As an immediate corollary of 2.1 we have:

COROLLARY 2.2. *Every hereditarily closure-preserving closed cover of a countably compact space contains a finite subcover.*

REMARK. 2.2 does not necessarily hold for a closure-preserving closed cover even if a space is compact and metrizable; for example, let $X = \{1/n: n = 1, 2, \cdots\} \cup \{0\}$ be a subspace of real line and put $\mathscr{F} = \{\{0, 1/n\}: n = 1, 2, \cdots\}$. Then X is a compact, metric space and $\mathscr{F}$ is a closure-preserving closed cover of X, but we cannot choose any finite subcover.

THEOREM 2.3. *Let $\mathscr{F} = \{F_\lambda: \lambda \in \Lambda\}$ be a hereditarily closure-preserving closed cover of a Lindelöf space X. Then the set*

$$X_0 = \{x \in X: \Lambda(x) = \{\lambda \in \Lambda: x \in F_\lambda\} \text{ is uncountable}\}$$

is countable, and the set

$$\Lambda' = \{\lambda \in \Lambda: F_\lambda \cap (X - X_0) \neq \emptyset\}$$

is countable if $X - X_0$ is uncountable.

Proof. On the contrary, suppose X_0 is uncountable. Then X_0 contains a subset $\{x_\alpha: \alpha < \omega_1\}$, where ω_1 denotes the least uncountable ordinal. For each $\alpha < \omega_1$, by transfinite induction we can obtain x_α in X_0 and a $\lambda_\alpha \in \Lambda(x_\alpha)$ with $x_\alpha \in F_{\lambda_\alpha}$ and such that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$ and $\lambda_\alpha \neq \lambda_\beta$, because for each $x \in X_0$ $\Lambda(x)$ is uncountable. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{x_\alpha: \alpha < \omega_1\}$ must be discrete in X. This contradicts the assumption that X is Lindelöf, and hence the first half of 2.3 is proved.

To complete the proof, again suppose Λ' is uncountale. From the definition of X_0, $\mathscr{F}$ must be point-countable at any $x \in X - X_0$. If $X - X_0$ is uncountable, by transfinite induction, we can choose an uncountable set $\{x_\alpha: \alpha < \omega_1\}$ in $X - X_0$ and a corresponding set $\{\lambda_\alpha: \alpha < \omega_1\}$ with $x_\alpha \in F_{\lambda_\alpha}$ for each $\alpha < \omega_1$ and so that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$ as well as $\lambda_\alpha \neq \lambda_\beta$. Since $\mathscr{F}$ is hereditarily closure-preserving, $\{x_\alpha: \alpha < \omega_1\}$ must be an uncountable discrete set in X, which contradicts the assumption that X is Lindelöf. Therefore $X - X_0$ is countable, and hence the proof is completed.

As an immediate consequence of 2.3 we have:

COROLLARY 2.4. *Every hereditarily closure-preserving closed cover of a Lindelöf space contains a countable subcover.*

REMARK. Example 3.4 in next section shows that 2.4 does not necessarily hold for a closure-preserving closed cover.

LEMMA 2.5. *Let $\mathscr{F}$ be a closure-preserving closed cover of a space X. Then the set*

$$X_1 = \{x \in X: C(x, \mathscr{F}) = \{x\}\}$$

is discrete in X.

Proof. Let $y \in X$ be an arbitrary point and

$$U = X - \cup \{F \in \mathscr{F}: y \notin F\} .$$

Then U is an open neighborhood of y, because $\mathscr{F}$ is a closure-preserving closed cover. If $x \in U \cap X_1$, then we have

$$\phi \neq U \cap C(x, \mathscr{F}) = (X - \cup \{F \in \mathscr{F}: y \notin F\}) \cap (\cap \{F \in \mathscr{F}: x \in F\})$$

and hence $C(y, \mathscr{F}) \subset C(x, \mathscr{F})$. Since $x \in X_1$, $C(x, \mathscr{F}) = \{x\}$ and thus we have $y = x$. This means that U contains at most one point of X_1, which completes the proof.

THEOREM 2.6. *Let X be a space each of whose open sets is an*

F_σ, and let $\mathscr{F}$ be a closure-preserving closed cover of X. Then the set

$$X_n = \{x \in X: |C(x, \mathscr{F})| = n\}$$

is σ-discrete in X for $n = 1, 2, \cdots$, where we denote by $|A|$ the cardinality of A.

Proof. We shall prove 2.6 by induction on n. By 2.5 X_1 is discrete in X. Assume that X_n is σ-discrete in X for any $n \leqq k$. We shall show that X_{k+1} is also σ-discrete.

First note that $X - \bigcup_{n=1}^k X_n$ is open in X. Let y be any point of $X - \bigcup_{n=1}^k X_n$ and let $U = X - \cup\{F \in \mathscr{F}: y \notin F\}$. Then U is an open neighborhood of y. If $x \in X$ belongs to U, we have $C(y, \mathscr{F}) \subset C(x, \mathscr{F})$. Since y does not belong to $\bigcup_{n=1}^k X_n$, $C(y, \mathscr{F})$ contains at least $k+1$ points of X and thus $C(x, \mathscr{F})$ also contains at least $k+1$ points. In other words, $x \notin \bigcup_{n=1}^k X_n$. This shows that $U \cap (\bigcup_{n=1}^k X_n) = \emptyset$ and hence $X - \bigcup_{n=1}^k X_n$ is open in X.

According to hypothesis, $X - \bigcup_{n=1}^k X_n$ is an F_σ; i.e. $X - \bigcup_{n=1}^k X_n = \bigcup_{i=1}^\infty Y_i$, where each Y_i is closed in X and $Y_i \subset Y_{i+1}$ for $i = 1, 2, \cdots$. Since $X_{k+1} \subset \bigcup_{i=1}^\infty Y_i$, it suffices to show that $Z_i = X_{k+1} \cap Y_i$ is discrete in X for $i = 1, 2, \cdots$.

Let $y \in X$ be an arbitrary point and i fixed. If $y \notin Y_i$, then $X - Y_i$ is clearly the desired neighborhood of y. If $y \in Y_i$, put $U = X - \cup\{F \in \mathscr{F}: y \notin F\}$. Then $x \in U \cap Z_i$ implies $C(y, \mathscr{F}) \subset C(x, \mathscr{F})$ and $|C(x, \mathscr{F})| = k+1$. Since y belongs to Y_i, y does not belong to any X_n with $n \leqq k$: i.e. $|C(y, \mathscr{F})| > k$. Hence we have $C(y, \mathscr{F}) = C(x, \mathscr{F})$. This means that x must be in $C(y, \mathscr{F})$ which is finite. Consequently, U contains at most $k+1$ points of Z_i. Since X is T_1, we obtain the desired neighborhood of y by deleting finitely many points from U. Therefore Z_i is discrete in X. This completes the proof.

3. Some relations. Let f be a closed map from a space X onto a space Y and $\mathscr{F}$ a hereditarily closure-preserving closed cover of X. Then $f(\mathscr{F})$ is also a hereditarily closure-preserving closed cover of Y. Since the image of any countably compact space by a map is countably compact, we have the following:

THEOREM 3.1. *Any image of a Σ^*-space under a closed map is a Σ^*-space.*

Let f be a perfect map from X onto Y and $\mathscr{A}$ a (mod $\mathscr{K}$)-network for Y. Then we can easily see that $f^{-1}(\mathscr{A})$ is a (mod $f^{-1}(\mathscr{K})$)-network for X. Since the inverse image of any countably compact space by a perfect map is countably compact, we have the following:

THEOREM 3.2. *Any inverse image of a $\Sigma^\sharp$-space by a perfect map is a $\Sigma^\sharp$-space.*

THEOREM 3.3. *Every Lindelöf Σ^*-space is a Σ-space.*

Proof. Let X be a Lindelöf Σ^*-space having a σ-hereditarily closure-preserving closed (mod $\mathscr{K}$)-network $\mathscr{F}$ for some cover $\mathscr{K}$ of X by countably compact sets. Without loss of generality, we can denote $\mathscr{F}$ by $\bigcup_{n=1}^{\infty}\mathscr{F}_n$ such that each $\mathscr{F}_n$ is a hereditarily closure-preserving closed cover of X. Put $\mathscr{F}_n = \{F_\lambda: \lambda \in \Lambda_n\}$ for $n = 1, 2, \cdots$.

By 2.3, for each n the set

$$X_n = \{x \in X: \Lambda(x) = \{\lambda \in \Lambda_n: x \in F_\lambda\} \text{ is uncountable}\}$$

is countable. If $X - X_n$ is countable for some n, then X is countable. Since X is T_2, X is clearly a Σ-space; more precisely, it is a cosmic space (cf. [1]). If $X - X_n$ is uncountable for $n = 1, 2, \cdots$, then again by 2.3,

$$\Lambda'_n = \{\lambda \in \Lambda_n: F_\lambda \cap (X - X_n) \neq \emptyset\}$$

is countable for $n = 1, 2, \cdots$. Put $\mathscr{H}_n = \{\{x\}: x \in X_n\} \cup \{F_\lambda: \lambda \in \Lambda'_n\}$ for $n = 1, 2, \cdots$. Then each $\mathscr{H}_n$ is countable and, therefore, $\mathscr{H} = \bigcup_{n=1}^{\infty}\mathscr{H}_n$ is still countable. Since each $\mathscr{H}_n$ covers X, $\mathscr{H}$ covers X and thus $\mathscr{H}$ is a σ-locally finite closed cover of X. Furthermore, if we put $\mathscr{K}' = \{\{x\}: x \in \bigcup_{n=1}^{\infty} X_n\} \cup \{K \in \mathscr{K}: K \cap (X - X_n) \neq \emptyset \text{ for some } n\}$, then $\mathscr{K}'$ is a cover of X by countably compact sets. It is easy to see that $\mathscr{H}$ is a (mod $\mathscr{K}'$)-network, and hence X is a Σ-space.

EXAMPLE 3.4. We shall show that in general a Lindelöf $\Sigma^\sharp$-space need not be a Σ-space.

Let $X = \{x_\alpha: \alpha \in A\} \cup \{p\}$ be an uncountable set with a special point p. We define the topology for X as follows: each $\{x_\alpha\}$ is open; V is an open set containing p iff $X - V$ is countable. Then we can easily see that X is a regular, Lindelöf (T_2) space.

Now, put $\mathscr{F} = \{\{p, x_\alpha\}: \alpha \in A\}$. Then $\mathscr{F}$ is a closure-preserving closed cover of X, because any subset of X missing p is open. If we put $\mathscr{K} = \mathscr{F}$, then $\mathscr{K}$ is a cover of X by countably compact sets such that $\mathscr{F}$ is a (mod $\mathscr{K}$)-network for X; i.e. X is a $\Sigma^\sharp$-space.

Next, we shall show that X is not a Σ-space. On the contrary, suppose X is a Σ-space. Then there exists a σ-locally finite closed cover $\mathscr{H} = \bigcup_{n=1}^{\infty}\mathscr{H}_n$ of X which is a (mod $\mathscr{K}$)-network for some cover $\mathscr{K}$ by countably compact sets. We can assume without loss of generality that $\{\mathscr{H}_n: n = 1, 2, \cdots\}$ is an increasing sequence of locally finite closed covers of X and that each $\mathscr{H}_n$ is closed under

finite intersections. Furthermore, in case of a Σ-space we can put $\mathscr{K} = \{C(x): x \in X\}$, where $C(x) = \bigcap_{n=1}^{\infty} C(x, \mathscr{H}_n)$ as noted in the introduction. Since X is Lindelöf, each $\mathscr{H}_n$ is countable. From the definition of the topology for X any member of $\mathscr{H}$ missing p is a countable set. Therefore $X' = X - \cup\{H \in \mathscr{H}: p \notin H\}$ is an uncountable closed subspace of X, which is a Σ-space having $\mathscr{H}/X' = \{H \cap X': H \in \mathscr{H}\}$ as a σ-locally finite (mod $\mathscr{K}/X'$)-network. Consequently, we could have assumed from the beginning that each $\mathscr{H}_n$ is finite and each member of $\mathscr{H}$ contains p. For each $x \in X$ and n, let $H(x, n)$ be the smallest (as a subset) member of $\mathscr{H}_n$ containing x. $H(x, n)$ exists because $\mathscr{H}_n$ is closed under finite intersections. Since the compact sets of X are exactly the finite sets, $C(x) = \bigcap_{n=1}^{\infty} H(x, n)$ must be finite for each $x \in X$. Furthermore, for each $x \in X$ there is an n_x such that $H(x, n_x)$ is finite. To see this, suppose not. Then there is an increasing sequence $n_1 < n_2 < \cdots$ with $H(x, n_{i+1}) \subsetneqq H(x, n_i)$ for $i = 1, 2, \cdots$. Now pick a point $x_i \in H(x, n_i) - H(x, n_{i+1})$ which is distinct from p and x. Then $F = \{x_i: i = 1, 2, \cdots\}$ is a closed set in X with $F \cap C(x) = \emptyset$ but $F \cap H(x, n) \neq \emptyset$ for all n. This contradicts the fact that $\mathscr{H}$ forms a network around $C(x)$. Hence there exists such an n_x. We denote by $n(x)$ the smallest n_x for which $H(x, n_x)$ is finite. Put

$$L_n = \{x \in X: n(x) \leqq n\} \quad \text{for} \quad n = 1, 2, \cdots .$$

Then $\{L_n: n = 1, 2, \cdots\}$ is an increasing cover of X. Since X is uncountable, there exists an n_0 such that L_{n_0} is an uncountable set containing p. Clearly L_{n_0} is closed in X and hence it is a Σ-space having $\mathscr{H} | L_{n_0}$ as a (mod $\mathscr{K} | L_{n_0}$)-network. But $\bigcup_{i=1}^{n_0} \mathscr{H}_i$ is finite and for each $x \in L_{n_0}$ there exists an $H(x, n(x))$ with $n(x) \leqq n_0$. This means that L_{n_0} must be finite, which is a contradiction. Thus X is not a Σ-space.

LEMMA 3.5. *If X is a Σ^*-space* (resp. *a $\Sigma^\#$-space*), *then X has a sequence $\{\mathscr{F}_n: n = 1, 2, \cdots\}$ of hereditarily closure-preserving* (resp. *closure-preserving*) *closed covers of X such that any sequence $\{x_n: n = 1, 2, \cdots\}$ with $x_n \in C(x, \mathscr{F}_n)$ for some $x \in X$ has a cluster point. In particular, X is a Σ-space iff X has a sequence $\{\mathscr{F}_n: n = 1, 2, \cdots\}$ of locally finite closed covers of X such that any sequence $\{x_n: n = 1, 2, \cdots\}$ with $x_n \in C(x, \mathscr{F}_n)$ for some $x \in X$ has a cluster point.*

Proof. Since all cases are proved similarly, we shall prove for a $\Sigma^\#$-space, only. Let X be a $\Sigma^\#$-space having a σ-closure-preserving closed (mod $\mathscr{K}$)-network $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ for a cover $\mathscr{K}$ of X by countably compact sets, where we can assume that each $\mathscr{H}_n$ is a

closure-preserving closed cover of X. Put $\mathscr{F}_n = \bigcup_{k \leqq n} \mathscr{H}_k$ for $n = 1, 2, \cdots$. Now we shall show that $\{\mathscr{F}_n: n = 1, 2, \cdots\}$ satisfies the required condition. On the contrary, suppose not. Then there exists a discrete sequence $\{x_n: n = 1, 2, \cdots\}$ with $x_n \in C(x, \mathscr{F}_n)$ for some $x \in X$. Since $\mathscr{K}$ covers X, there is a $K \in \mathscr{K}$ containing x. Since $\{x_n: n = 1, 2, \cdots\}$ is discrete, there exists an n_0 such as $\{x_n: n \geqq n_0\} \cap K = \emptyset$. Then $G = X - \{x_n: n \geqq n_0\}$ is an open set containing K and thus, by the assumption, there exists an $F \in \mathscr{F}_m$ for some m with $K \subset F \subset G$. Hence we have $x_i \in C(x, \mathscr{F}_i) \subset C(x, \mathscr{F}_m) \subset F \subset G$ for any i with $m < i$ as well as $n_0 < i$, which is a contradiction.

The 'if' part in the later half is easily seen noting that any $C(x, \mathscr{F}_n)$ could have been a member of $\mathscr{F}_n$.

THEOREM 3.6. *Let X be a Σ^*-space for which every open set is an F_σ. Then X is a Σ-space.*

Proof. Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ be a σ-hereditarily closure-preserving closed (mod $\mathscr{K}$)-network for a cover $\mathscr{K}$ by countably compact sets. We can assume that each $\mathscr{F}_n$ covers X and that $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for $n = 1, 2, \cdots$. Put

$$X' = \{x \in X: |C(x, \mathscr{F}_n)| \text{ is finite for some } n\} .$$

Then X' is σ-discrete in X by 2.6. Denote X' by $\bigcup_{n=1}^{\infty} P_n$, where each P_n is discrete in X and we can assume $P_n \subset P_{n+1}$ for $n = 1, 2, \cdots$.

We shall show that each $\mathscr{F}_n$ is locally finite at any $x \in X - X'$. On the contrary, suppose some $\mathscr{F}_{n_0}$ is not locally finite at some $x \in X - X'$. Since $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ and since each $\mathscr{F}_n$ is closure-preserving, $\Lambda'_n = \{\lambda \in \Lambda_n: x \in F_\lambda\}$ must be infinite for all $n \geqq n_0$. Since $x \notin X'$, $C(x, \mathscr{F}_n)$ is infinite for all $n \geqq n_0$. We can choose a point $x_n \in C(x, \mathscr{F}_n)$ and a $\lambda_n \in \Lambda'_{n_0}$ with $x_n \in F_{\lambda_n}$ for each $n \geqq n_0$ and such that $n \neq m$ implies $x_n \neq x_m$ as well as $\lambda_n \neq \lambda_m$. By 3.5 $\{x_n: n = n_0, n_0 + 1, \cdots\}$ has a cluster point. On the other hand, it must be discrete, because each $\{x_n\} \subset F_{\lambda_n} \in \mathscr{F}_{n_0}$ and $\mathscr{F}_{n_0}$ is hereditarily closure-preserving. This contradiction shows that each $\mathscr{F}_n$ is locally finite at any $x \in X - X'$.

Next, put

$$Y_n = \{x \in X: \mathscr{F}_n \text{ is locally finite at } x\}, \qquad n = 1, 2, \cdots .$$

Then each Y_n is open in X and therefore an F_σ. Denote Y_n by $\bigcup_{m=1}^{\infty} Q_{nm}$, where each Q_{nm} is closed in X and $Q_{nm} \subset Q_{nm+1}$ for $m, n = 1, 2, \cdots$. Further, as was seen above, we have $X - X' \subset Y_n$ for $n = 1, 2, \cdots$.

Finally, put

$$\mathscr{F}_{nm} = \{F_\lambda \cap Q_{nm}: \lambda \in \Lambda_n\} \cup \{X\} \quad \text{for} \quad n, m = 1, 2, \cdots,$$
$$\mathscr{H}_n = \{\{x\}: x \in P_n\} \cup \{X\} \quad \text{for} \quad n = 1, 2, \cdots.$$

Then each $\mathscr{F}_{nm}$ as well as $\mathscr{H}_n$ is locally finite closed cover of X. In order that X be a Σ-space, it suffices to show that the sequence $\{\mathscr{F}_{nm}: n, m = 1, 2, \cdots\} \cup \{\mathscr{H}_n: n = 1, 2, \cdots\} = \{\mathscr{G}_i: i = 1, 2, \cdots\}$ satisfies the condition in 3.5. Let $x \in X$ be any point and $\{x_i: i = 1, 2, \cdots\}$ a sequence with $x_i \in C(x, \mathscr{G}_i)$. If $x \in X'$, then $x \in P_k$ for some k, and since $\{P_n: n = 1, 2, \cdots\}$ is increasing, we have $C(x, \mathscr{H}_n) = \{x\} \in \mathscr{H}_n$ for all $n \geqq k$. Hence $\{x_i: i = 1, 2, \cdots\}$ has a cluster point x. If $x \notin X'$, then $x \in Y_n$ for $n = 1, 2, \cdots$ and hence, for each n, there exists a k_n with $x \in Q_{nk_n}$. Thus, for any n we have $C(x, \mathscr{F}_{nk_n}) \subset C(x, \mathscr{F}_n)$. On the other hand, by 3.5 any sequence $\{p_n: n = 1, 2, \cdots\}$ with $p_n \in C(x, \mathscr{F}_n)$ has a cluster point. Hence $\{x_i: i = 1, 2, \cdots\}$ must have a cluster point. This shows by 3.5 that X is a Σ-space.

THEOREM 3.7. *Let X be a paracompact space. Then the following conditions are equivalent.*

(1) *X is a Σ-space.*

(2) *$X \times I$ is a Σ-space.*

(3) *$X \times I$ is a Σ^*-space.*

Proof. Since the property of being a paracompact Σ-space is countably productive (cf. [6]), we have (1) $\Rightarrow$ (2). From the definition clearly (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ be a σ-hereditarily closure-preserving (mod $\mathscr{K}$)-network for some cover $\mathscr{K}$ of $X \times I$ by countably compact sets. We assume that $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for $n = 1, 2, \cdots$.

At first we shall construct by induction on n a collection $\{V(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n; n = 1, 2, \cdots\}$ of open sets of X and a corresponding collection

$$\{I(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n; n = 1, 2, \cdots\}$$

of subsets of I satisfying the following conditions:

(i) $\{V(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ is a locally finite open cover of X for $n = 1, 2, \cdots$.

(ii) $V(\alpha_1, \cdots, \alpha_n, \alpha_{n+1}) \subset V(\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \in A_1, \cdots, \alpha_n \in A_n$, $\alpha_{n+1} \in A_{n+1}$; $n = 1, 2, \cdots$.

(iii) If $V(\alpha_1, \cdots, \alpha_n)$ is nonempty, then $I(\alpha_1, \cdots, \alpha_n)$ is a closed interval.

(iv) $I(\alpha_1, \cdots, \alpha_n, \alpha_{n+1}) \subset I(\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \in A_1, \cdots, \alpha_n \in A_n, \alpha_{n+1} \in A_{n+1}$; $n = 1, 2, \cdots$.

(v) $\overline{V(\alpha_1, \cdots, \alpha_n)} \times I(\alpha_1, \cdots, \alpha_n)$ meets only finitely many members of $\mathscr{F}_n$ for $\alpha_1 \in A_1, \cdots, \alpha_n \in A_n$; $n = 1, 2, \cdots$.

Assume that such collections are constructed for all $n \leqq k$ and

consider $n = k + 1$.

Fix $\alpha_1 \in A_1, \cdots, \alpha_k \in A_k$ with $V(\alpha_1, \cdots, \alpha_k) \neq \emptyset$. For any point $x \in \overline{V(\alpha_1, \cdots, \alpha_k)}$, since $\{x\} \times I(\alpha_1, \cdots, \alpha_k)$ is compact and $\mathscr{F}_{k+1}$ is hereditarily closure-preserving, by 2.1 $\mathscr{F}_{k+1}$ is locally finite at all but finitely many points of $\{x\} \times I(\alpha_1, \cdots, \alpha_k)$. Let $\{p_1, \cdots, p_m\}$ be those points of $\{x\} \times I(\alpha_1, \cdots, \alpha_k)$ at which $\mathscr{F}_{k+1}$ is not locally finite. Let I_x be a closed subinterval of $I(\alpha_1, \cdots, \alpha_k)$ missing $p_1, \cdots, p_m$. Since $\{x\} \times I_x$ is compact, there exists an open neighborhood U_x(in $\overline{V(\alpha_1, \cdots \alpha_k)}$) of x such that

$$\bar{U}_x \times I_x \subset X \times I - \cup\{F \in \mathscr{F}_{k+1}: F \cap (\{x\} \times I_x) = \emptyset\} .$$

Since $\overline{V(\alpha_1, \cdots, \alpha_k)}$ is paracompact, there is a locally finite open cover $\{V_\lambda: \lambda \in \Lambda(\alpha_1, \cdots, \alpha_k)\}$ of $\overline{V(\alpha_1, \cdots, \alpha_k)}$ which refines $\{U_x: x \in \overline{V(\alpha_1, \cdots, \alpha_k)}\}$. Let

$$\varphi: \Lambda(\alpha_1, \cdots, \alpha_k) \longrightarrow \overline{V(\alpha_1, \cdots, \alpha_k)} \subset X$$

be a function which satisfies $V_\lambda \subset U_{\varphi(\lambda)}$ for $\lambda \in \Lambda(\alpha_1, \cdots, \alpha_k)$.

Now varying $\alpha_1 \in A_1, \cdots, \alpha_k \in A_k$, put

$$A_{k+1} = \cup\{\Lambda(\alpha_1, \cdots, \alpha_k): \alpha_1 \in A_1, \cdots, \alpha_k \in A_k\}$$

and

$$V(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) = \left\{\begin{matrix} V(\alpha_1, \cdots, \alpha_k) \cap V_{\alpha_{k+1}} \text{ if } V(\alpha_1, \cdots, \alpha_k) \neq \emptyset \text{ and } \alpha_{k+1} \in \Lambda(\alpha_1, \cdots, \alpha_k) \\ \emptyset \qquad \text{otherwise} \end{matrix}\right\} .$$

Furthermore, if $V(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) \neq \emptyset$, then from the definition we have $V(\alpha_1, \cdots, \alpha_k) \neq \emptyset$ and $\alpha_{k+1} \in \Lambda(\alpha_1, \cdots, \alpha_k)$. By inductive hypothesis $I(\alpha_1, \cdots, \alpha_k)$ is not empty. Hence we put $I(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) = I_{\varphi(\alpha_{k+1})}$, which is not empty. Otherwise we put $I(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) = \emptyset$. Then we can easily see that $\{V(\alpha_1, \cdots, \alpha_{k+1}): \alpha_1 \in A_1, \cdots, \alpha_{k+1} \in A_{k+1}\}$ and $\{I(\alpha_1, \cdots, \alpha_{k+1}): \alpha_1 \in A_1, \cdots, \alpha_{k+1} \in A_{k+1}\}$ satisfy all required conditions (i)—(v).

Consequently, for each n we can construct $\{V(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ and $\{I(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots \alpha_n \in A_n\}$ satisfying (i)—(v).

Next, put

$$Y_n = \cup\{\overline{V(\alpha_1, \cdots, \alpha_n)} \times I(\alpha_1, \cdots, \alpha_n): \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$$

and

$$Y = \bigcap_{n=1}^{\infty} Y_n .$$

Since $\{\overline{V(\alpha_1, \cdots, \alpha_n)}: \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ is locally finite in X, Y_n is

closed in $X \times I$ and thus Y is closed in $X \times I$. Also by (v) the collection

$$\mathscr{H}_n = \mathscr{F}_n | Y = \{F \cap Y : F \in \mathscr{F}_n\}$$

is a locally finite closed cover of Y for $n = 1, 2, \cdots$.

Now we show that Y is a Σ-space. For this purpose it suffices to show that $\{\mathscr{H}_n : n = 1, 2, \cdots\}$ satisfies the condition in 3.5. Let $y \in Y$ be any point and $\{y_n : n = 1, 2, \cdots\}$ any sequence with $y_n \in C(y, \mathscr{H}_n)$. Since $C(y, \mathscr{H}_n) \subset C(y, \mathscr{F}_n)$ for each n and since $X \times I$ is a Σ^*-space, by 3.5 $\{y_n : n, = 1, 2, \cdots\}$ has a cluster point in $X \times I$. Since Y is closed in $X \times I$, $\{y_n : n = 1, 2, \cdots\}$ must have a cluster point in Y, which shows by 3.5 that Y is a Σ-space.

Finally, let π be the restriction to Y of the projection from $X \times I$ onto Y. Since the projection is perfect and since Y is closed in $X \times I$, π is perfect. It remains to show that π is onto, because a Σ-space is preserved by a perfect map (cf. [6]). Let x be any point of X. Since $\{V(\alpha_1, \cdots, \alpha_n) : \alpha_1 \in A_1, \cdots, \alpha_n \in A_n\}$ covers X for $n = 1, 2, \cdots$, by (ii) we can choose a point $(\alpha_1, \alpha_2, \cdots)$ in $A_1 \times A_2 \times \cdots$ with $x \in V(\alpha_1, \cdots, \alpha_n)$ for $n = 1, 2, \cdots$. Since each $V(\alpha_1, \cdots, \alpha_n)$ is nonempty, by (iv) $\{I(\alpha_1, \cdots, \alpha_n) : n = 1, 2, \cdots\}$ is a decreasing sequence of nonempty closed intervals. Hence $\bigcap_{n=1}^{\infty} I(\alpha_1, \cdots, \alpha_n) \neq \emptyset$. Pick a point q in this intersection. Then (x, q) belongs to $\overline{V(\alpha_1, \cdots, \alpha_n)} \times I(\alpha_1, \cdots, \alpha_n) \subset Y_n$ for $n = 1, 2, , \cdots$ and thus belongs to Y. Clearly $\pi((x, q)) = x$. This shows that π is onto and hence X is a Σ-space, which completes the proof.

REFERENCES

1. E. Michael, *$\aleph_0$-spaces*, J. Math. and Mech., **15** (1966), 983-1002.
2. ———, *On Nagami's Σ-spaces and some related matters*, Proc. of the Washington State University Conference on General Topology, 13-19.
3. E. Michael and F. Slaughter, Jr., *Σ-spaces with a point-countable separating open cover are σ-spaces*, to appear.
4. K. Morita, *Products of normal spaces with metric spaces*, Math., Ann. **154** (1964), 365-382.
5. K. Nagami, *σ-spaces and product spaces*, Math. Ann., **181** (1969), 109-118.
6. ———, *Σ-spaces*, Fund. Math., **65** (1969), 196-192.
7. A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku, sect. A **9** (1967), 236-254.

Received April 16, 1971 and in revised form October 8, 1971. This research was accomplished during author's stay in University of Pittsburgh as a visiting professor and was supported in part by NSF Grant GP 29401.

OSAKA UNIVERSITY OF EDUCATION

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 42, No. 2 February, 1972