RINGS OF ANALYTIC FUNCTIONS ON ANY SUBSET OF THE COMPLEX PLANE

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We prove that for any two subsets $X, Y$ of $\mathbb{C}$, the complex plane, $X$ and $Y$ are conformally homeomorphic if there is an isomorphism between $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ which is the identity on constant functions.

It has been known for some time that the conformal structure of a domain in the complex plane or a Riemann surface is determined by the algebraic structure of certain rings of analytic functions on it. (See [3], [11], [12], [10], [9] and [8].) Iss’sa [5] shows this is also true for a Stein variety of positive dimension.

All functions considered here are complex single-valued.

**Definition 1.** Let $X$ be an arbitrary subset of $\mathbb{C}$. A function $f$ on $X$ is said to be analytic at a point $p \in X$ if there is a power series $\sum_{n=0}^{\infty} \alpha_n(z - p)^n$ which converges for $|z - p| < R$, and $f(z) = \sum_{n=0}^{\infty} \alpha_n(z - p)^n$ for all $z \in X$ and $|z - p| < R$, where $R > 0$, and $\alpha_n$ is a complex number for each $n = 0, \ldots$, and $f$ is said to be analytic on $X$ if it is analytic at each point of $X$.

**Definition 2.** Let $X$ and $Y$ be two arbitrary subspaces of $\mathbb{C}$. A mapping $\tau$ from $X$ to $Y$ is said to be analytic mapping if $\tau$ is an analytic function on $X$ and has values in $Y$. $\tau$ is said to be a conformal mapping if $\tau$ is analytic, one-to-one, and onto. (See [2, Ch. II. §2].) For any two subsets $X, Y$ of $\mathbb{C}$, $X, Y$ are said to be conformally homeomorphic if there is a one-to-one conformal mapping from $X$ onto $Y$.

Let $X$ be an arbitrary subset of $\mathbb{C}$, and $\mathcal{A}(X) = \{f: f$ is analytic on $X\}$. We can then easily show that $\mathcal{A}(X)$ forms a ring with the constant function of value 1 as the identity $u$. By [1, p. 145], if $f \in \mathcal{A}(X)$ and $Z(f) = \{x \in X: f(x) = 0\} = \emptyset$, then $1/f \in \mathcal{A}(X)$.

**Lemma 3.** For $p \in X$, there is an $f \in M_p = \{f \in \mathcal{A}(X): f(p) = 0\}$ such that $Z(f) = \{p\}$ and $f$ belongs to no maximal ideal other than $M_p$.

**Proof.** Let $f(z) = z - p$. Then that $f \in M_p$ and $f$ belongs to no other fixed maximal ideal [4, 4.4] is clear. Now, suppose that $M$ is a free maximal ideal [4, 4.1] such that $f \in M$. Since $M$ is free, there is $g \in M$ such that $g(p) \neq 0$. Thus, we have $g(z) = \alpha + \sum_{j=0}^{k} \alpha_{k+j}(z - p)^{k+j}$ for $z \in X$ and $|z - p| < R$, for some $R > 0$, $\alpha_0 \neq 0$, $\alpha_k \neq 0$ and $k \geq 1$.
Hence \( \alpha^n = g(z) - (z - p)^{n-1} \cdot f(z) \cdot h(z) \) for some \( h \in \mathbb{A}(X) \). Now \( f, g \in M \) which is an ideal, \( \alpha \in M \). This is impossible as \( \alpha \neq 0 \). Hence, the assertion holds.

**Lemma 4.** If \( \Phi \) is an isomorphism from \( \mathbb{A}(X) \) onto \( \mathbb{A}(Y) \), then \( \Phi(M_p) \) is a fixed maximal ideal.

**Proof.** That \( \Phi(M_p) \) is a maximal ideal is clear. From Lemma 3, there is an \( f_0 \in M_p \) such that \( Z(f_0) = \{ p \} \), and \( f_0 \) belongs to no other maximal ideal. Consider \( Z(\Phi(f_0)) \). If \( Z(\Phi(f_0)) = \varnothing \), then \( \Phi(f_0) \) is a unit so that \( \Phi(M_p) \) is the whole ring, \( \mathbb{A}(X) \). This is impossible. Hence, \( Z(\Phi(f_0)) \neq \varnothing \). But if \( Z(\Phi(f_0)) \) contains more than one point, say \( q_1 \) and \( q_2 \), then \( \Phi(f_0) \in M_{q_1} \) and \( M_{q_2} \) so that \( f_0 \) would belong to at least two maximal ideals which is again impossible. Hence \( Z(\Phi(f_0)) = \{ q \} \) for some \( q \in Y \). Hence \( \Phi(M_p) = M_q \) is fixed ideal.

**Theorem 5.** Let \( X \) and \( Y \) be two subsets of \( C \), and \( \Phi \) be an isomorphism from \( \mathbb{A}(Y) \) onto \( \mathbb{A}(X) \) such that it is the identity on the constant functions. Then \( \Phi \) induces a mapping \( \tau: X \rightarrow Y \), defined by \( \Phi(g) = g \circ \tau \), and \( \tau \) is a conformal mapping of \( X \) onto \( Y \).

**Proof.** Define \( \tau \) to be a mapping from \( X \) to \( Y \) as follows: \( \tau(p) = \cap Z(\Phi^{-1}(M_p)) \). By hypothesis \( \Phi^{-1} \) is an isomorphism of \( \mathbb{A}(X) \) onto \( \mathbb{A}(Y) \). By Lemma 4, \( \Phi^{-1}(M_p) \) is a fixed maximal ideal in \( \mathbb{A}(Y) \). Thus, \( \tau \) is a single-valued mapping. Evidently, \( M_{\tau(p)} = \Phi^{-1}(M_p) \), and \( \tau \) is one-to-one and onto. Now, for each \( g \in \mathbb{A}(Y) \), and \( p \in X \), let \( \Phi(g)(p) = \alpha \). Then \( \Phi(g) - \alpha \in M_p \), \( g - \Phi^{-1}(\alpha) \in M_{\tau(p)} \), so that \( g(\tau(p)) = \Phi^{-1}(\alpha)(\tau(p)) = \alpha = \Phi(g)(p) \). Hence \( \Phi(g) = g \circ \tau \). Similarly, \( \Phi^{-1}(f) = f \circ \tau^{-1} \), where \( \tau^{-1}: Y \rightarrow X \) with \( \tau^{-1}(q) = \cap Z(\Phi(M_q)) \). If we choose \( g(w) = w \) on \( Y \), and \( f(z) = z \) on \( X \), then \( \tau(p) = g \circ \tau(p) \), and \( \tau^{-1}(q) = f \circ \tau^{-1}(q) \) are analytic. Hence, \( \tau \) is a conformal mapping.

**Corollary 6.** Let \( X \) and \( Y \) be two subsets of \( C \), and \( \Phi \) be an isomorphism of \( \mathbb{A}(X) \) onto \( \mathbb{A}(Y) \) which is the identity on real constant functions. Then \( X \) and \( Y \) can be decomposed respectively into \( X_1 \cup X_2 \) and \( Y_1 \cup Y_2 \) such that the sets \( X_1 \), \( X_2 \) are open and disjoint in \( X \) and similarly for \( Y_1 \) and \( Y_2 \), in such a way that \( X_1 \) is conformal with \( Y_1 \), and \( X_2 \) is anti-conformal with \( Y_2 \), where some of \( X_1 \), \( X_2 \), \( Y_1 \) and \( Y_2 \) could be empty.

Note that a set is anti-conformal with another set if it is conformal with its complex conjugate.

* \( \alpha_0 \) stands for the constant function of value \( \alpha_0 \).
Proof. As in Theorem 5, the mapping \( \tau \) defined by \( \tau(p) = \cap Z[\Phi^{-1}(M_p)] \) is one-to-one and onto. We know that \( (\Phi(i))^3 = \Phi(-1) = -1 \), hence \( \Phi(i) = i, -i \) or \( i \) on one clopen subset of \( X \), say \( X_1 \), and \( -i \) on \( X_2 = \overline{X} - X_1 \), (which is then a clopen subset). We will set \( X_1 = X \) and \( X_2 = X \), respectively, according as \( \Phi(i) = i \) and \( \Phi(i) = -i \). Therefore, \( \Phi(\alpha) = \alpha \) on \( X_1 \), and \( \overline{\alpha} \) on \( X_2 \) for any constant \( \alpha \). Then, by an argument similar to that used in Theorem 5, we can show that \( \Phi(g) = g \circ \tau \) on \( X_1 \), and \( g \circ \tau \) on \( X_2 \); and \( \Phi^{-1}(f) = f \circ \tau^{-1} \) on \( X_1 \) and \( \overline{f} \circ \tau^{-1} \) on \( X_2 \), for any \( g \in \mathcal{A}(Y) \) and \( f \in \mathcal{A}(X) \). Hence the assertion holds.

REMARK. In Theorem 5, the condition that \( \Phi \) is the identity on the constant functions cannot be omitted. Consider \( X = \{p\}, Y = \{q\} \). Then \( \mathcal{A}(X) = C = \mathcal{A}(Y) \). We know that there is an isomorphism of \( C \) to \( C \) other than \( z \rightarrow z \) and \( z \rightarrow \overline{z} \) (see [7, Remark on p. 119]). Define \( \Phi: \mathcal{A}(X) \rightarrow \mathcal{A}(Y) \) in the obvious way. Then \( \Phi(\alpha) \neq \alpha \) for some \( \alpha \in \mathcal{A}(Y) \). On the other hand, \( \alpha \circ \tau(p) = \alpha \). Hence, \( \Phi(\alpha) \neq \alpha \circ \tau \).

However, L. Bers shows that if \( X \) and \( Y \) are domains with boundary points, then every isomorphism of \( \mathcal{A}(Y) \) onto \( \mathcal{A}(X) \) induces a mapping which is either conformal or anti-conformal. (See [3].) Nevertheless, Royden [10], and Ozawa and Mizumoto [9] assumed that the given isomorphism preserves the constant functions. Recently, Nakai [8]** shows that if \( X \) and \( Y \) are open Riemann surfaces and \( \Phi \) is such that \( \Phi(i) = i \) (or \( -i \)), then \( \Phi \) induces a conformal (or conjugate-conformal, resp.) mapping. Iss'sa [5]** shows that if \( X \) and \( Y \) are Stein varieties of positive dimensions, then \( \Phi \) induces a unique conformal or a unique conjugate-conformal mapping.

THEOREM 7. Let \( X \) and \( Y \) be two subsets of \( C \), and \( \tau \) be a conformal mapping of \( X \) onto \( Y \). Then the induced mapping \( \tau' \), defined by \( \tau'(g) = g \circ \tau \), is an isomorphism of \( \mathcal{A}(Y) \) onto \( \mathcal{A}(X) \) leaving the constant function unchanged.

Proof. Use the Weierstrass’ double-series theorem in [6] to show the composition of \( g \circ \tau \in \mathcal{A}(X) \) for any \( g \in \mathcal{A}(Y) \). The others are obvious.

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