Pacific Journal of Mathematics

RINGS OF ANALYTIC FUNCTIONS ON ANY SUBSET OF THE COMPLEX PLANE

LI PI SU

Vol. 42, No. 2 February 1972

PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 2, 1972

RINGS ON ANALYTIC FUNCTIONS ON ANY SUBSET OF THE COMPLEX PLANE

LI PI SU

We prove that for any two subsets X, Y of C, the complex plane, X and Y are conformally homeomorphic if there is an isomorphism between $\mathfrak{A}(X)$ and $\mathfrak{A}(Y)$ which is the identity on constant functions.

It has been known for some time that the conformal structure of a domain in the complex plane or a Riemann surface is determined by the algebraic structure of certain rings of analytic functions on it. (See [3], [11], [12], [10], [9] and [8].) Iss'sa [5] shows this is also true for a Stein variety of positive dimension.

All functions considered here are complex single-valued.

DEFINITION 1. Let X be an arbitrary subset of C. A function f on X is said to be analytic at a point $p \in X$ if there is a power series $\sum_{n=0}^{\infty} \alpha_n(z-p)^n$ which converges for $|z-p| < R$, and $f(z) = \sum_{n=0}^{\infty} \alpha_n(z-p)^n$ for all $z \in X$ and $|z-p| < R$, where $R > 0$, and α_n is a complex number for each $n = 0, \cdots$, and f is said to be analytic on X if it is analytic at each point of X.

DEFINITION 2. Let X and Y be two arbitrary subspaces of C. A mapping τ from X to Y is said to be analytic mapping if τ is an analytic function on X and has values in Y. τ is said to be a conformal mapping if τ is analytic, one-to-one, and onto. (See [2, Ch. II. §2].) For any two subsets X, Y of C, X, Y are said to be conformally homeomorphic if there is a one-to-one conformal mapping from X onto Y.

Let X be an arbitrary subset of C, and $\mathfrak{A}(X) = \{f : f \text{ is analytic on } X\}$. We can then easily show that $\mathfrak{A}(X)$ forms a ring with the constant function of value 1 as the identity u. By [1, p. 145], if $f \in \mathfrak{A}(X)$ and $Z(f) = \{x \in X : f(x) = 0\} = \varnothing$, then $1/f \in \mathfrak{A}(X)$.

LEMMA 3. *For $p \in X$, there is an $f \in M_p = \{f \in \mathfrak{A}(X) : f(p) = 0\}$ such that $Z(f) = \{p\}$ and f belongs to no maximal ideal other than M_p.*

Proof. Let $f(z) = z - p$. Then that $f \in M_p$ and f belongs to no other fixed maximal ideal [4, 4.4] is clear. Now, suppose that M is a free maximal ideal [4, 4.1] such that $f \in M$. Since M is free, there is $g \in M$ such that $g(p) \neq 0$. Thus, we have $g(z) = \alpha + \sum_{j=0}^{\infty} \alpha_{k+j}(z-p)^{k+j}$ for $z \in X$ and $|z-p| < R$, for some $R > 0$, $\alpha_0 \neq 0$, $\alpha_k \neq 0$ and $k \geqq 1$.

Hence $\underline{\alpha_0}^* = g(z) - (z - p)^{k-1} \cdot f(z) \cdot h(z)$ for some $h \in \mathfrak{A}(X)$. Now f, $g \in M$ which is an ideal, $\underline{\alpha_0} \in M$. This is impossible as $\alpha_0 \neq 0$. Hence, the assertion holds.

LEMMA 4. *If Φ is an isomorphism from $\mathfrak{A}(X)$ onto $\mathfrak{A}(Y)$, then $\Phi(M_p)$ is a fixed maximal ideal.*

Proof. That $\Phi(M_p)$ is a maximal ideal is clear. From Lemma 3, there is an $f_0 \in M_p$ such that $Z(f_0) = \{p\}$, and f_0 belongs to no other maximal ideal. Consider $Z(\Phi(f_0))$. If $Z(\Phi(f_0)) = \varnothing$, then $\Phi(f_0)$ is a unit so that $\Phi(M_p)$ is the whole ring, $\mathfrak{A}(X)$. This is impossible. Hence, $Z(\Phi(f_0)) \neq \varnothing$. But if $Z(\Phi(f_0))$ contains more than one point, say q_1 and q_2, then $\Phi(f_0) \in M_{q_1}$ and M_{q_2} so that f_0 would belong to at least two maximal ideals which is again impossible. Hence $Z(\Phi(f_0)) = \{q\}$ for some $q \in Y$. Hence $\Phi(M_p) = M_q$ is fixed ideal.

THEOREM 5. *Let X and Y be two subsets of $\boldsymbol{C}$, and Φ be an isomorphism from $\mathfrak{A}(Y)$ onto $\mathfrak{A}(X)$ such that it is the identity on the constant functions. Then Φ induces a mapping $\tau: X \to Y$, defined by $\Phi(g) = g \circ \tau$, and τ is a conformal mapping of X onto Y.*

Proof. Define τ to be a mapping from X to Y as follows: $\tau(p) = \cap\, Z[\Phi^{-1}(M_p)]$. By hypothesis Φ^{-1} is an isomorphism of $\mathfrak{A}(X)$ onto $\mathfrak{A}(Y)$. By Lemma 4, $\Phi^{-1}(M_p)$ is a fixed maximal ideal in $\mathfrak{A}(Y)$. Thus, τ is a single-valued mapping. Evidently, $M_{\tau(p)} = \Phi^{-1}(M_p)$, and τ is one-to-one and onto. Now, for each $g \in \mathfrak{A}(Y)$, and $p \in X$, let $\Phi(g)(p) = \alpha$. Then $\Phi(g) - \underline{\alpha} \in M_p$, $g - \Phi^{-1}(\underline{\alpha}) \in M_{\tau(p)}$, so that $g(\tau(p)) = \Phi^{-1}(\underline{\alpha})(\tau(p)) = \alpha = \Phi(g)(p)$. Hence $\Phi(g) = g \circ \tau$. Similarly, $\Phi^{-1}(f) = f \circ \tau^{-1}$, where τ^{-1}; $Y \to X$ with $\tau^{-1}(q) = \cap\, Z[\Phi(M_q)]$. If we choose $g(w) = w$ on Y, and $f(z) = z$ on X, then $\tau(p) = g \circ \tau(p)$, and $\tau^{-1}(q) = f \circ \tau^{-1}(q)$ are analytic. Hence, τ is a conformal mapping.

COROLLARY 6. *Let X and Y be two subsets of $\boldsymbol{C}$, and Φ be an isomorphism of $\mathfrak{A}(X)$ onto $\mathfrak{A}(Y)$ which is the identity on real constant functions. Then X and Y can be decomposed respectively into $X_1 \cup X_2$ and $Y_1 \cup Y_2$ such that the sets X_1, X_2 are open and disjoint in X and similarly for Y_1 and Y_2, in such a way that X_1 is conformal with Y_1, and X_2 is anti-conformal with Y_2, where some of X_1, X_2, Y_1 and Y_2 could be empty.*

Note that a set is anti-conformal with another set if it is conformal with its complex conjugate.

* $\underline{\alpha_0}$ stands for the constant function of value α_0.

Proof. As in Theorem 5, the mapping τ defined by $\tau(p) = \cap Z[\Phi^{-1}(M_p)]$ is one-to-one and onto. We know that $(\Phi(\underline{i}))^2 = \Phi(-\underline{1}) = -\underline{1}$, hence $\Phi(\underline{i}) = \underline{i}$, $-\underline{i}$ or $\underline{i}$ on one clopen subset of X, say X_1, and $-i$ on $X_2 = X - X_1$, (which is then a clopen subset). We will set $X_1 = X$ and $X_2 = X$, respectively, according as $\Phi(\underline{i}) = i$ and $\Phi(\underline{i}) = -i$. Therefore, $\Phi(\underline{\alpha}) = \underline{\alpha}$ on X_1, and $\overline{\alpha}$ on X_2 for any constant α. Then, by an argument similar to that used in Theorem 5, we can show that $\Phi(g) = g \circ \tau$ on X_1, and $\overline{g \circ \tau}$ on X_2; and $\Phi^{-1}(f) = f \circ \tau^{-1}$ on X_1 and $\overline{f \circ \tau^{-1}}$ on X_2, for any $g \in \mathfrak{A}(Y)$ and $f \in \mathfrak{A}(X)$. Hence the assertion holds.

REMARK. In Theorem 5, the condition that Φ is the identity on the constant functions can not be omitted. Consider $X = \{p\}$, $Y = \{q\}$. Then $\mathfrak{A}(X) = \boldsymbol{C} = \mathfrak{A}(Y)$. We know that there is an isomorphism of $\boldsymbol{C}$ to $\boldsymbol{C}$ other than $z \to z$ and $z \to \bar{z}$ (see [7, Remark on p. 119]). Define Φ: $\mathfrak{A}(X) \to \mathfrak{A}(Y)$ in the obvious way. Then $\Phi(\alpha) \neq \alpha$ for some $\alpha \in \mathfrak{A}(Y)$. On the other hand, $\alpha \circ \tau(p) = \alpha$. Hence, $\Phi(\alpha) \neq \alpha \circ \tau$.

However, L. Bers shows that if X and Y are domains with boundary points, then every isomorphism of $\mathfrak{A}(Y)$ onto $\mathfrak{A}(X)$ induces a mapping which is either conformal or anti-conformal. (See [3].) Nevertheless, Royden [10], and Ozawa and Mizumoto [9] assumed that the given isomorphism preserves the constant functions. Recently, Nakai [8]** shows that if X and Y are open Riemann surfaces and Φ is such that $\Phi(i) = i$ (or $-i$), then Φ induces a conformal (or conjugate-conformal, resp.) mapping. Iss'sa [5]** shows that if X and Y are Stein varieties of positive dimensions, then Φ induces a unique conformal or a unique conjugate-conformal mapping.

THEOREM 7. *Let X and Y be two subsets of $\boldsymbol{C}$, and τ be a conformal mapping of X onto Y. Then the induced mapping τ', defined by $\tau'(g) = g \circ \tau$, is an isomorphism of $\mathfrak{A}(Y)$ onto $\mathfrak{A}(X)$ leaving the constant function unchanged.*

Proof. Use the Weirstrass' double-series theorem in [6] to show the composition of $g \circ \tau \in \mathfrak{A}(X)$ for any $g \in \mathfrak{A}(Y)$. The others are obvious.

REFERENCES

1. L. V. Ahlfors, *Complex Analysis*, McGraw-Hill Book Company, Inc., New York, 1953.
2. L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press, New Jersey, 1960.
3. L. Bers, *On rings of analytic functions*, Bull. Amer. Math. Soc., **54** (1948), 311-315.
4. L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. Van Nostrand Company,

** The author wishes to express her thanks to the referee for bringing her attention to these two articles.

Inc., New Jersey, 1960.
5. H. Iss'sa, *On the meromorphic function field of a Stein variety*, Ann. of Math., **83** (1966), 34-46.
6. K. Knopp, *Theory of Functions*, Dover Publications, Inc., New York, 1945.
7. J. P. Meyer, G. D. Mostor and J. H. Sampson, *Fundamental Structures of Algebra*, McGraw-Hill Book Company, Toronto, 1963.
8. M. Nakai, *On rings of analytic functions on Riemann surfaces*, Proc. Japan Acad., **39** (1963), 79-84.
9. M. Ozawa and H. Mizumoto, *On rings of analytic functions*, Japan J. Math., **29** (1959), 114-117.
10. H. L. Royden, *Rings of analytic and meromorphic functions*, Trans. Amer. Math. Soc., **83** (1956), 269-276.
11. W. Rudin, *Some theorems on bounded analytic functions*, Trans. Amer. Math. Soc., **78** (1955), 333-342.
12. ———, *An algebraic characterization of conformal equivalence*, Bull. Amer. Math. Soc., **61** (1955), 543.

Received April 14, 1971.

THE UNIVERSITY OF OKLAHOMA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 42, No. 2 February, 1972