RINGS OF ANALYTIC FUNCTIONS ON ANY SUBSET OF THE
COMPLEX PLANE

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We prove that for any two subsets $X, Y$ of $C$, the complex plane, $X$ and $Y$ are conformally homeomorphic if there is an isomorphism between $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ which is the identity on constant functions.

It has been known for some time that the conformal structure of a domain in the complex plane or a Riemann surface is determined by the algebraic structure of certain rings of analytic functions on it. (See [3], [11], [12], [10], [9] and [8].) Iss'sa [5] shows this is also true for a Stein variety of positive dimension.

All functions considered here are complex single-valued.

**DEFINITION 1.** Let $X$ be an arbitrary subset of $C$. A function $f$ on $X$ is said to be analytic at a point $p \in X$ if there is a power series $\sum_{n=0}^{\infty} \alpha_n(z - p)^n$ which converges for $|z - p| < R$, and $f(z) = \sum_{n=0}^{\infty} \alpha_n(z - p)^n$ for all $z \in X$ and $|z - p| < R$, where $R > 0$, and $\alpha_n$ is a complex number for each $n = 0, \ldots$, and $f$ is said to be analytic on $X$ if it is analytic at each point of $X$.

**DEFINITION 2.** Let $X$ and $Y$ be two arbitrary subspaces of $C$. A mapping $\tau$ from $X$ to $Y$ is said to be analytic mapping if $\tau$ is an analytic function on $X$ and has values in $Y$. $\tau$ is said to be a conformal mapping if $\tau$ is analytic, one-to-one, and onto. (See [2, Ch. II. §2].) For any two subsets $X$, $Y$ of $C$, $X$, $Y$ are said to be conformally homeomorphic if there is a one-to-one conformal mapping from $X$ onto $Y$.

Let $X$ be an arbitrary subset of $C$, and $\mathcal{A}(X) = \{f : f$ is analytic on $X\}$. We can then easily show that $\mathcal{A}(X)$ forms a ring with the constant function of value 1 as the identity $u$. By [1, p. 145], if $f \in \mathcal{A}(X)$ and $Z(f) = \{x \in X : f(x) = 0\} = \emptyset$, then $1/f \in \mathcal{A}(X)$.

**LEMMA 3.** For $p \in X$, there is an $f \in M_p = \{f \in \mathcal{A}(X) : f(p) = 0\}$ such that $Z(f) = \{p\}$ and $f$ belongs to no maximal ideal other than $M_p$.

**Proof.** Let $f(z) = z - p$. Then that $f \in M_p$ and $f$ belongs to no other fixed maximal ideal [4, 4.4] is clear. Now, suppose that $M$ is a free maximal ideal [4, 4.1] such that $f \in M$. Since $M$ is free, there is $g \in M$ such that $g(p) \neq 0$. Thus, we have $g(z) = \alpha + \sum_{j=0}^{\infty} \alpha_{k+j} (z - p)^{k+j}$ for $z \in X$ and $|z - p| < R$, for some $R > 0$, $\alpha_0 \neq 0$, $\alpha_k \neq 0$ and $k \geq 1$. 535
Hence $a_0^e = g(z) - (z - p)^{k-1} \cdot f(z) \cdot h(z)$ for some $h \in \mathfrak{H}(X)$. Now $f, g \in M$ which is an ideal, $a_0 \in M$. This is impossible as $a_0 \neq 0$. Hence, the assertion holds.

**Lemma 4.** If $\Phi$ is an isomorphism from $\mathfrak{H}(X)$ onto $\mathfrak{H}(Y)$, then $\Phi(M_p)$ is a fixed maximal ideal.

**Proof.** That $\Phi(M_p)$ is a maximal ideal is clear. From Lemma 3, there is an $f_0 \in M_p$ such that $Z(f_0) = \{p\}$, and $f_0$ belongs to no other maximal ideal. Consider $Z(\Phi(f_0))$. If $Z(\Phi(f_0)) = \emptyset$, then $\Phi(f_0)$ is a unit so that $\Phi(M_p)$ is the whole ring, $\mathfrak{H}(X)$. This is impossible. Hence, $Z(\Phi(f_0)) \neq \emptyset$. But if $Z(\Phi(f_0))$ contains more than one point, say $q_1$ and $q_2$, then $\Phi(f_0) \in M_{q_1}$ and $M_{q_2}$ so that $f_0$ would belong to at least two maximal ideals which is again impossible. Hence $Z(\Phi(f_0)) = \{q\}$ for some $q \in Y$. Hence $\Phi(M_p) = M_q$ is fixed ideal.

**Theorem 5.** Let $X$ and $Y$ be two subsets of $C$, and $\Phi$ be an isomorphism from $\mathfrak{H}(Y)$ onto $\mathfrak{H}(X)$ such that it is the identity on the constant functions. Then $\Phi$ induces a mapping $\tau : X \to Y$, defined by $\Phi(g) = g \circ \tau$, and $\tau$ is a conformal mapping of $X$ onto $Y$.

**Proof.** Define $\tau$ to be a mapping from $X$ to $Y$ as follows: $\tau(p) = \cap Z[\Phi^{-1}(M_p)]$. By hypothesis $\Phi^{-1}$ is an isomorphism of $\mathfrak{H}(X)$ onto $\mathfrak{H}(Y)$. By Lemma 4, $\Phi^{-1}(M_p)$ is a fixed maximal ideal in $\mathfrak{H}(Y)$. Thus, $\tau$ is a single-valued mapping. Evidently, $M_{\tau(p)} = \Phi^{-1}(M_p)$, and $\tau$ is one-to-one and onto. Now, for each $g \in \mathfrak{H}(Y)$, and $p \in X$, let $\Phi(g)(p) = \alpha$. Then $\Phi(g) - \alpha \in M_p$, $g - \Phi^{-1}(\alpha) \in M_{\tau(p)}$, so that $g(\tau(p)) = \Phi^{-1}(\alpha)(\tau(p)) = \alpha = \Phi(g)(p)$. Hence $\Phi(g) = g \circ \tau$. Similarly, $\Phi^{-1}(f) = f \circ \tau^{-1}$, where $\tau^{-1} : Y \to X$ with $\tau^{-1}(q) = \cap Z[\Phi(M_q)]$. If we choose $g(w) = w$ on $Y$, and $f(z) = z$ on $X$, then $\tau(p) = g \circ \tau(p)$, and $\tau^{-1}(q) = f \circ \tau^{-1}(q)$ are analytic. Hence, $\tau$ is a conformal mapping.

**Corollary 6.** Let $X$ and $Y$ be two subsets of $C$, and $\Phi$ be an isomorphism of $\mathfrak{H}(X)$ onto $\mathfrak{H}(Y)$ which is the identity on real constant functions. Then $X$ and $Y$ can be decomposed respectively into $X_1 \cup X_2$ and $Y_1 \cup Y_2$ such that the sets $X_1, X_2$ are open and disjoint in $X$ and similarly for $Y_1$ and $Y_2$, in such a way that $X_1$ is conformal with $Y_1$, and $X_2$ is anti-conformal with $Y_2$, where some of $X_1, X_2, Y_1$ and $Y_2$ could be empty.

Note that a set is anti-conformal with another set if it is conformal with its complex conjugate.

* $\alpha_0$ stands for the constant function of value $\alpha_0$. 
Proof. As in Theorem 5, the mapping \( \tau \) defined by \( \tau(p) = \cap Z[\Phi^{-1}(M_\alpha)] \) is one-to-one and onto. We know that \( (\Phi(i))^2 = \Phi(-1) = -1 \), hence \( \Phi(i) = i, -i \) or \( i \) on one clopen subset of \( X \), say \( X_1 \), and \( -i \) on \( X_2 = \overline{X - X_1} \), (which is then a clopen subset). We will set \( X_1 = X \) and \( X_2 = X \), respectively, according as \( \Phi(i) = i \) and \( \Phi(i) = -i \). Therefore, \( \Phi(\alpha) = \alpha \) on \( X_1 \), and \( \overline{\alpha} \) on \( X_2 \) for any constant \( \alpha \). Then, by an argument similar to that used in Theorem 5, we can show that \( \Phi(g) = g \circ \tau \) on \( X_1 \), and \( \overline{g \circ \tau} \) on \( X_2 \); and \( \Phi^{-1}(f) = f \circ \tau^{-1} \) on \( X_1 \) and \( \overline{f \circ \tau^{-1}} \) on \( X_2 \), for any \( g \in \mathcal{A}(Y) \) and \( f \in \mathcal{A}(X) \). Hence the assertion holds.

REMARK. In Theorem 5, the condition that \( \Phi \) is the identity on the constant functions cannot be omitted. Consider \( X = \{p\}, Y = \{q\} \). Then \( \mathcal{A}(X) = C = \mathcal{A}(Y) \). We know that there is an isomorphism of \( C \) to \( C \) other than \( z \rightarrow z \) and \( z \rightarrow \overline{z} \) (see [7, Remark on p. 119]). Define \( \Phi: \mathcal{A}(X) \rightarrow \mathcal{A}(Y) \) in the obvious way. Then \( \Phi(\alpha) \neq \alpha \) for some \( \alpha \in \mathcal{A}(Y) \). On the other hand, \( \alpha \circ \tau(p) = \alpha \). Hence, \( \Phi(\alpha) \neq \alpha \circ \tau \).

However, L. Bers shows that if \( X \) and \( Y \) are domains with boundary points, then every isomorphism of \( \mathcal{A}(Y) \) onto \( \mathcal{A}(X) \) induces a mapping which is either conformal or anti-conformal. (See [3].) Nevertheless, Royden [10], and Ozawa and Mizumoto [9] assumed that the given isomorphism preserves the constant functions. Recently, Nakai [8]** shows that if \( X \) and \( Y \) are open Riemann surfaces and \( \Phi \) is such that \( \Phi(i) = i \) (or \( -i \)), then \( \Phi \) induces a conformal (or conjugate-conformal, resp.) mapping. Iss'ssa [5]** shows that if \( X \) and \( Y \) are Stein varieties of positive dimensions, then \( \Phi \) induces a unique conformal or a unique conjugate-conformal mapping.

THEOREM 7. Let \( X \) and \( Y \) be two subsets of \( C \), and \( \tau \) be a conformal mapping of \( X \) onto \( Y \). Then the induced mapping \( \tau' \), defined by \( \tau'(g) = g \circ \tau \), is an isomorphism of \( \mathcal{A}(Y) \) onto \( \mathcal{A}(X) \) leaving the constant function unchanged.

Proof. Use the Weirstrass' double-series theorem in [6] to show the composition of \( g \circ \tau \in \mathcal{A}(X) \) for any \( g \in \mathcal{A}(Y) \). The others are obvious.

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** The author wishes to express her thanks to the referee for bringing her attention to these two articles.


Received April 14, 1971.

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