A PROPERTY OF MANIFOLDS COMPACTLY EQUIVALENT TO COMPACT MANIFOLDS

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In this paper it is shown that there is a countable collection \( \mathcal{G} = \{G_k\}_{k=1}^\infty \) of connected \( n \)-manifolds such that any manifold \( M \) which is compactly equivalent to a compact manifold is an open monotone union of some \( G_{\alpha(M)} \in \mathcal{G} \).

In [4] it is shown that if \( \mathcal{F} \) is the class consisting of all open 2-manifolds of finite genus, then there is a countable collection \( \mathcal{D} = \{D_k\}_{k=1}^\infty \) of open 2-manifolds with the property that given \( M \in \mathcal{F} \), there exists some \( D_j \in \mathcal{D} \) such that \( M \) is an open monotone union of \( D_j \). By appropriately extending the concept of genus to higher dimensions, one can obtain similar results for a larger class of manifolds.

1. Preliminaries. Unless otherwise specified, all manifolds will be assumed to be connected and \( \text{bd} \, M \) and \( \text{int} \, M \) will denote the boundary and interior respectively of a manifold \( M \). Let \( M \) and \( N \) be \( n \)-manifolds. \( M \) and \( N \) are compactly equivalent, denoted by \( M \sim N \), if given any proper compact set \( K \subset M \) there is an embedding \( i \) of the pair \((K, K \cap \text{bd} \, M)\) into \((N, \text{bd} \, N)\) such that \( i(K \cap \text{bd} \, M) = i(K) \cap \text{bd} \, N \) and given any proper compact set \( L \subset N \) there is an embedding \( j \) of \((L, L \cap \text{bd} \, N)\) into \((M, \text{bd} \, M)\) such that \( j(L \cap \text{bd} \, N) = j(L) \cap \text{bd} \, M \). Clearly compact equivalence is an equivalence relation on the class of all \( n \)-manifolds. Note that a 2-manifold \( M \) without boundary has finite genus if and only if \( M \sim Q \) where \( Q \) is some closed 2-manifold.

Let \( \mathcal{L} \) be the class consisting of all non-compact \( n \)-manifolds \( M \), \( n \geq 2 \) and \( n \neq 4 \), such that \( M \in \mathcal{L} \) if and only if \( M \sim N \), \( N \) a compact manifold. The principal result of this paper is the following:

**Theorem 1.1.** There is a countable collection \( \mathcal{G} = \{G_k\}_{k=1}^\infty \) of manifolds such that given \( M \in \mathcal{L} \) there is some positive integer \( \alpha(M) \) such that \( M \) is an open monotone union of \( G_{\alpha(M)} \).

As usual an \( n \)-manifold \( M \) is called an open monotone union of an \( n \)-manifold \( H \) if \( M = \bigcup_{i=1}^{\alpha} H_i \), where for all \( i, H_i \) is open in \( M \), \( H_i \subset H_{i+1} \) and \( H_i \equiv H \) (\( \equiv \) denotes topological equivalence).

2. Proof of the theorem. If \( M \) is an \( n \)-manifold, let \( I(M) \) rel \( \text{bd} \, M = \{f \mid f \text{ is a homeomorphism of } M \text{ onto itself such that } f \text{ is isotopic to the identity relative to } \text{bd} \, M \} \).
The following lemma gives the existence of a complicated domain which is the basic tool used in the construction of the collection $\mathcal{G}$ mentioned in Theorem 1.1.

**Lemma 2.1.** Let $E$ be an $n$-cell, $n \geq 2$. There exists a proper domain (open connected set) $G$ of $E$, $bd E \subset G$, such that if $U$ is open in $E$ and $K$ is a proper continuum, $bd E \subset K \subset U$, then there exists a $g \in I(E)$ rel $bd E$ such that $K \subset g(G) \subset U$.

*Proof.* This follows immediately from Lemma 3.8 of [5].

**Lemma 2.2.** Let $Q$ be a compact $n$-manifold, $n \geq 2$. There is a proper domain $D$ of $Q$ such that if $U$ is open in $Q$ and contains a residual set $R$ of $Q$, and $K$ is proper continuum in $Q$, $R \subset K \subset U$, then there exists $h \in I(Q)$ rel $bd Q$ such that $K \subset h(D) \subset U$.

*Proof.* Let $E$ be a bicollared $n$-cell, $E \subset \text{int } G$, and let $G$ be a proper domain $G$ of $E$ which satisfies the conditions of Lemma 2.1. We will show that $D = (Q - E) \cup G$ is the required domain. Without loss of generality, we may assume that $U$ is connected. Since $U$ contains a residual set $R$ (see [3] for appropriate definition) there is a bicollared $n$-cell $E'$ and $\alpha \in I(Q)$ rel $bd$ such that $R \subset Q - \text{int } E' \subset U$ and $\alpha(E') = E$. Note that $E'$ and $\alpha$ can be obtained as follows: one easily constructs $\gamma_1, \gamma_2$, and $\gamma_3 \in I(Q)$ rel $bd Q$ such that $\gamma_1$ only moves points inside $E' \cup (\text{collar of } bd E)$ and shrinks $E$ to a very small set, $\gamma_2$ moves $\gamma_1(E)$ into the open $n$-cell $Q - R$, and $\gamma_3$ moves only points inside $Q - R$ and expands $\gamma_2(\gamma_1(E))$ so that $Q - U \subset \gamma_3(\gamma_2(\gamma_1(\text{int } E))) \subset Q - R$. Thus we can set $\alpha^{-1} = \gamma_2 \gamma_3 \gamma_1$ and $E' = \alpha^{-1}(E)$. Let $R \subset K \subset U$, a proper continuum. Without loss of generality, we may assume that $K \cap E'$ is a proper continuum in $E'$ and $bd E' \subset K \cap E'$. Then $K'' = \alpha(K \cap E') = \alpha(K) \cap E$ is a proper continuum in $E$, $U'' = \alpha(U) \cap E = \alpha(U \cap E')$ is open in $E$ and $bd E' \subset K'' \subset U''$. Therefore it follows from Lemma 2.1 that there is a homeomorphism $h \in I(E)$ rel $bd E$ such that $K'' \subset h(G) \subset U''$. Now extend $h$ to all of $Q$ by defining $h(x) = x$, $x \in Q - E$. Then $\alpha(K) \subset h(D) \subset \alpha(U)$ and so $g = \alpha^{-1} h$ is the required homeomorphism.

Since there are only a countable number of topologically distinct compact manifolds [1], Theorem 1.1 follows immediately from the following theorem.

**Theorem 2.3.** Let $Q$ be a compact $n$-manifold, $n > 1$ and $n \neq 4$. There is a domain $D$ of $Q$ such that if $M$ is a non-compact $n$-manifold and $M \sim_{c} Q$, then $M$ is an open monotone union of $D$. 
Proof. Let $D$ be a domain of $Q$ which satisfies Lemma 2.2, and let $L = Q - \text{int} E$, $E$ a bicollared $n$-cell contained in $\text{int} Q$. Let $M$ be a non-compact $n$-manifold such that $M \sim Q$. It is easily seen that $\text{bd} M = \text{bd} Q$ and that there is an embedding $f$ of $(L, \text{bd} Q)$ into $(M, \text{bd} M)$ such that $f(\text{bd} E) = \text{bd} E = L - \text{int}_Q L$ where $\text{int}_Q L$ denotes the point set interior of $L$ relative to $Q$ is a bicollared $(n - 1)$-sphere in $\text{int} M$. Since $M$ is an $n$-manifold, there exists a sequence $(C_i)_{i=1}^\infty$ of continua in $M$ such that $M = \bigcup_{i=1}^\infty C_i$ and for all $i \geq 1$, $f(L) \subset \text{int}_Q C_i \subset C_i \subset \text{int}_M C_{i+1}$. Since $M$ is not compact and $M \sim Q$, for each $i \geq 1$ there is an embedding $h_{i+1}$ of $(C_{i+1}, \text{bd} M)$ into $(Q, \text{bd} Q)$ such that $\text{bd} Q \subset h_{i+1}(f(L)) \subset h_{i+1}(C_i) \subset h_{i+1}(\text{int}_Q C_{i+1})$, where $K_i = h_{i+1}(C_i)$ is a proper continuum in $Q$ and $U_i = h_{i+1}(\text{int}_Q C_{i+1})$ is open in $Q$. Since $n \neq 4$, it follows from [2] that $Q - h_{i+1}(f(\text{int}_Q L))$ is a bicollared $n$-cell and therefore there is a residual set $R$ of $Q$ such that $R \subset K_i \subset U_i$. It follows from Lemma 2.2 that there exists $\alpha_i \in I(Q) \text{ rel } \text{bd} Q$ such that $K_i \subset \alpha_i(D) \subset U_i$. Define $\beta_i: D \rightarrow M$ by $\beta_i(x) = h_{i+1}^{-1}(\alpha_i(x))$. Then $\beta_i$ is an embedding of $(D, \text{bd} Q)$ into $(M, \text{bd} M)$ and $C_i \subset \beta_i(D) \subset \text{int}_M C_{i+1}$. Therefore $M = \bigcup_{i=1}^\infty \beta_i(D)$, where $\beta_i(D)$ is open and $\beta_i(D) \subset \beta_{i+1}(D)$ for all $i \geq 1$. Therefore $M$ is an open monotone union of $D$.

The author would like to thank the referee for his helpful suggestions.

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Received April 5, 1971 and in revised form June 18, 1971.

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